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Interval Estimation of the Overlapping Coefficient of Two Multivariate Normal Distributions

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Abstract

This paper introduces the use of a generalized pivotal statistic for the interval estimation of the overlapping coefficient between two multivariate normal distributions. Percentile bootstrap and the delta methods are also considered. Simulation results are reported to compare the performance of these methods in terms of expected lengths and coverage probabilities of the confidence intervals. The value of overlapping coefficient is the major deciding factor affecting the performance of the confidence intervals. For small values of ρ, the generalized pivotal quantity (GPQ) method provides satisfactory performance. Illustrative examples are also presented.

Keywords: Generalized pivotal statistic, generalized p-value, delta method, bootstrap calibration.

1. Introduction

Various similarity or distance measures are available in the literature to assess the closeness of two populations. The overlapping coefficient (OVL) is one such measure, and is defined as the common area under two probability distributions. In the literature one can find different versions of the OVL, see, for example, Matusita (1955, 1966), Morisita (1959) and Macarthur and Levins (1967). Good and Smith (1987) studied two general measures which include Matusita's measure, MacArthur-Levins' measure, and Morisita's measure as special cases. Mulekar and Mishra (2000) simulated the sampling distribution of estimators of the overlap measures for two normal distributions with equal means and obtained approximate expressions for the bias and variance of the OVL estimators. Al-Saidy et al. (2005) considered the problem of drawing inference about three overlap measures for Weibull distributions with the same shape parameter. Recently, Jose and Thomas (2018) considered the interval estimation of the OVL of two Pareto distributions.

In the present article, we shall focus on Matusita's measure of OVL. If $f_1(.)$ and $f_2(.)$ are two probability density functions, then Matusita's measure of OVL is defined as follows

$$\rho = \int \sqrt{f_1(x) f_2(x)} dx. \tag{1}$$

If the variable under consideration is discrete, the integral is to be replaced with summation. The value of ρ ranges from 0 to 1, where a value 0 indicates that there is no similarity and a value 1 shows that the two populations are identical. Minami et al. (2000) considered ML and REML estimation of Matusita's measure for bivariate normal samples. Lu et al. (1989) considered the multivariate measures of similarity to assess the similarity in resources used by two species. They derived an expression for Matusita's measure (and also for MacArthur-Levins' measure and Morisita's measure) under multivariate normality. For general multivariate normal samples with unknown mean vectors and covariance matrices there seems to be no literature available that discusses inference on Matusita's measure, especially the interval estimation. In this study we explore different methods for the interval estimation of Matusita's measure for two multivariate normal distributions. The methodologies discussed are the generalized pivotal quantity (GPQ) approach, percentile bootstrap method, and the delta method. A bootstrap calibration approach is also considered for the delta method. The GPQ approach is due to Weerahandi (1993) and has found numerous applications in a variety of inference problems; see the books by Weerahandi (1994, 2004), the book by Krishnamoorthy and Mathew (2008), and the articles by Krishnamoorthy and Mathew (2003, 2004), Mathew and Web (2012) and Jose and Thomas (2019). The different methods are introduced in the article, and their performance is assessed using simulations. Illustrative examples are also given.

2. Confidence Intervals for the OVL

For two p-variate normal populations $N_p(\mu_i, \Sigma_i), i = 1,2$ the expression for Matusita's measure of OVL ρ is

$$\rho = \frac{2^{\frac{p}{2}} |\Sigma_1|^{\frac{1}{4}} |\Sigma_1|^{\frac{1}{4}}}{|\Sigma_1 + \Sigma_2|^{\frac{1}{2}}} \exp\left\{-\frac{1}{4}(\mu_1 - \mu_2)'(\Sigma_1 + \Sigma_2)^{-1}(\mu_1 - \mu_2)\right\}. \tag{2}$$

As can be seen from (2), and as noted in Lu et al. (1989), ρ in (2) can be written as the product of the following two factors:

$$\mathrm{Q} = \frac{2^{\frac{p}{2}} |\Sigma_1|^{\frac{1}{4}} |\Sigma_1|^{\frac{1}{4}}}{|\Sigma_1 + \Sigma_2|^{\frac{1}{2}}} \text{ and } R = \exp\left\{-\frac{1}{4}(\mu_1 - \mu_2)'(\Sigma_1 + \Sigma_2)^{-1}(\mu_1 - \mu_2)\right\}.$$

The factor Q measures the difference between the covariance matrices and the other factor R measures the difference between the means.

2.1. The GPQ method

The GPQ methodology was introduced by Weerahandi (1993); see also Weerahandi (1994, 2004) for numerous examples and applications. Some applications of the GPQ methodology in the multivariate context are given in Gamage (1997), Gamage et al. (2004) and Lin et al. (2007). We shall now explain the construction of a GPQ for the OVL ρ.

Let $X_{i\alpha}, \alpha = 1,2,\ldots,N_i$, be random sample of sizes N_i from $N_p\left(\mu_{i,} \Sigma_i\right)$ for $i = 1,2$. If $\bar{X}_i$ denotes the corresponding sample mean then $\bar{X}_i \sim N_p\left(\mu_{i,} \frac{\Sigma_i}{N_i}\right)$ for $i = 1,2$. Furthermore,

$$\mathrm{Z}_i = \left(\frac{\Sigma_i}{N_i}\right)^{-\frac{1}{2}} \left(\bar{X}_i - \mu_i\right) \sim N_p(0, \mathrm{I}),$$

$$\mathrm{U}_i = \frac{1}{N_i}\sum_{\alpha=1}^{N_i}\left(X_{i\alpha}-\bar{X}_i\right)\left(X_{i\alpha}-\bar{X}_i\right)' \sim W_p\left(\mathrm{N}_i-1,\frac{\Sigma_i}{N_i}\right),\ i=1,2.$$

Let $\bar{x}_i$ and u_i denote the observed values of $\bar{X}_i$ and U_i, respectively for $i=1,2.$ Now let us define the matrix R_i as follows

$$R_i=\left(u_i^{-1/2}\frac{\Sigma_i}{N_i}u_i^{-1/2}\right)^{-\frac{1}{2}}\left(u_i^{-1/2}U_iu_i^{-1/2}\right)\left(u_i^{-1/2}\frac{\Sigma_i}{N_i}u_i^{-1/2}\right)^{-\frac{1}{2}} \sim W_p\left(N_i-1,I_p\right)\ \text{for}\ i=1,2.$$

GPQs for the parameters μ_i and Σ_i, denoted by T_{μ_i} and T_{Σ_i}, respectively, are given by

$$T_{\mu_i}=\bar{x}_i-\left(u_i^{1/2}R_i^{-1}u_i^{1/2}\right)^{\frac{1}{2}}Z_i, \tag{3}$$

and

$$T_{\Sigma_i}=N_i\left(u_i^{1/2}R_i^{-1}u_i^{1/2}\right), i=1,2. \tag{4}$$

We refer to Gamage et al. (2004) for more details. For any function of the parameters μ_i and Σ_i, a GPQ can be obtained as the corresponding function of T_{μ_i} and T_{Σ_i}. In particular, a GPQ can be obtained for the OVL ρ. The percentiles of the GPQ so obtained give confidence limits for ρ.

2.2. Percentile bootstrap method

In the percentile bootstrap method, we shall first obtain sample estimates of the parameters μ_i and Σ_i such that $\bar{X}_i \sim N_p\left(\mu_{i,}\frac{\Sigma_i}{N_i}\right)$ and $U_i \sim W_p\left(N_i-1,\frac{\Sigma_i}{N_i}\right)$, $i=1,2.$ Note that $\hat{\Sigma}_i=\frac{N_i}{N_i-1}U_i$ is an unbiased estimator of Σ_i. We can now generate a parametric bootstrap sample and obtain the estimates of μ_i and Σ_i, say $\bar{X}_i^*$ and U_i^* as

$$\bar{X}_i^* \sim N_p\left(\bar{X}_i,\frac{\hat{\Sigma}_i}{N_i}\right),\ U_i^* \sim W_p\left(N_i-1,\frac{\hat{\Sigma}_i}{N_i}\right),\ i=1,2.$$

Consider B parametric bootstrap samples so generated and denote the sample estimators of $\mu_{1,}\mu_{2,}\ \Sigma_1$ and Σ_2 by $\bar{X}_1^{*\mathrm{b}},\bar{X}_2^{*\mathrm{b}},U_1^{*\mathrm{b}}$ and $U_2^{*\mathrm{b}}$, respectively. Let $\hat{\Sigma}_i^{*b}=\frac{N_i}{N_i-1}U_i^{*\mathrm{b}}$ be the unbiased estimator of Σ_i using the $b-$th bootstrap sample for $b=1,2,...,B.$ Let $\hat{\rho}^*(b)$ denote the value of ρ evaluated at $\mu_{\ i}=\bar{X}_i^{*\mathrm{b}}$ and $\Sigma_i=\hat{\Sigma}_i^{*b}, i=1,2.$ The percentiles of $\hat{\rho}^*(b),\ b=1,2,...,B$ so obtained provide confidence limits for ρ.

2.3. Delta method

Let $\hat{\rho}$ denotes the estimator of ρ obtained by replacing the parameters with the corresponding unbiased estimators in the expression for ρ given in (2). The asymptotic variance of $\hat{\rho}$ is derived in Minami and Shimizu (1999). In order to give the expressions for the asymptotic variance, let us introduce

$$\alpha_i = tr\left\{\left(\Sigma_1+\Sigma_2\right)^{-1}\Sigma_i\right\},$$

$$\alpha_i^* = tr\left\{\left(\left(\Sigma_1+\Sigma_2\right)^{-1}\Sigma_i\right)^2\right\},$$

$$\beta_i = \left(\mu_1-\mu_2\right)'\left(\Sigma_1+\Sigma_2\right)^{-1}\Sigma_i\left(\Sigma_1+\Sigma_2\right)^{-1}\left(\mu_1-\mu_2\right),$$

$$\beta_i^* = \left(\mu_1-\mu_2\right)'\left(\left(\Sigma_1+\Sigma_2\right)^{-1}\Sigma_i\right)^2\left(\Sigma_1+\Sigma_2\right)^{-1}\left(\mu_1-\mu_2\right), i=1,2.$$

Then the expression for asymptotic variance of $\hat{\rho}$ is given by

$$Var(\hat{\rho}) = \frac{\rho^2}{8}\sum_{i=1}^{2}\left\{\frac{2\beta_i}{n_i}+\frac{1}{n_i-1}\left(p-4\alpha_i+4\alpha_i^*+2\beta_i+\beta_i^2-4\beta_i^*\right)\right\}.$$

The asymptotic variance of $\ln(\hat{\rho})$ is derived to be

$$Var(\ln(\hat{\rho})) = \frac{1}{\rho^2}Var(\hat{\rho}).$$

Using estimated variances and the asymptotic normality of $\hat{\rho}$ and $\ln(\hat{\rho})$, large sample confidence intervals can be constructed for ρ.

3. Simulation Study

In order to assess the performance of the proposed confidence intervals, we shall now report the results of some simulation studies for the bivariate and trivariate normal distributions. Table 1 and Table 2 give the estimated coverage probabilities of the 95% confidence intervals obtained using the GPQ and the percentile bootstrap methods. Numerical results in both the tables are based on 5,000 simulated samples and are computed using the R software (R Core Team 2018). For each simulated sample, 5,000 values of the GPQ are generated in order to compute the generalized confidence limits. For the bootstrap method, 5,000 parametric bootstrap samples are generated. The coverage probabilities and expected lengths of the confidence intervals resulting from the asymptotic normality of $\ln(\hat{\rho})$ are given in Tables 3 and 4. They also include the confidence interval resulting from the application of a bootstrap calibration to the above asymptotic confidence interval; see Efron and Tibshirani (1993, Chapter 18) for a description of bootstrap calibration. Results are not given when the asymptotic normality of $\hat{\rho}$ is used, since the coverage probabilities turned out to be very unsatisfactory. The sample sizes used are specified in the tables.

For the bivariate normal case, the following parameter values were chosen for the simulation:

(i) $\mu_1 = \begin{bmatrix}0\\0\end{bmatrix}, \mu_2 = \begin{bmatrix}4\\0\end{bmatrix}, \Sigma_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \Sigma_2 = \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix},$ yielding $\rho = 0.0914$.

(ii) $\mu_1 = \begin{bmatrix}0\\0\end{bmatrix}, \mu_2 = \begin{bmatrix}1\\1\end{bmatrix}, \Sigma_1 = \begin{bmatrix}10 & 6\\6 & 9\end{bmatrix}, \Sigma_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$ yielding $\rho = 0.6114$.

(iii) $\mu_1 = \begin{bmatrix}3\\-2\end{bmatrix}, \mu_2 = \begin{bmatrix}3\\1\end{bmatrix}, \Sigma_1 = \begin{bmatrix}14 & -8\\-8 & 18\end{bmatrix}, \Sigma_2 = \begin{bmatrix}10 & 1\\1 & 17\end{bmatrix},$ yielding $\rho = 0.8868$.

Table 1 Coverage probabilities and expected lengths of the 95% confidence intervals for under bivariate normality: GPQ and bootstrap percentile methods

ρ	(n_1,n_2)	GPQ		Percentile Bootstrap	
		Coverage	Length	Coverage	Length
0.0904	(20,20)	0.9576	0.2359	0.8416	0.1610
	(20,30)	0.9492	0.2202	0.8524	0.1448
	(50,50)	0.9518	0.1439	0.9066	0.1225
	(50,100)	0.9486	0.1808	0.9092	0.1132
	(100,100)	0.9508	0.1011	0.9260	0.0927
0.6114	(20,20)	0.9008	0.2788	0.8694	0.2780
	(20,30)	0.9148	0.2553	0.8464	0.2516
	(50,50)	0.9411	0.1777	0.9112	0.1771
	(50,100)	0.9394	0.1535	0.9054	0.1531
	(100,100)	0.9420	0.1260	0.9358	0.1257
0.8868	(20,20)	0.7802	0.2881	0.7085	0.3045
	(20,30)	0.8082	0.2614	0.7378	0.2722
	(50,50)	0.8874	0.1742	0.8494	0.1790
	(50,100)	0.9032	0.1495	0.8734	0.1522
	(100,100)	0.9194	0.1208	0.8971	0.1226

In the trivariate normal case, here are the parameter choices:

(i) $\mu_1 = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, \mu_2 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$ yielding $\rho = 0.1136$

(ii) $\mu_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mu_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 8 & 3 \\ -5 & 3 & 7 \end{bmatrix},$ yielding $\rho = 0.4531$

(iii) $\mu_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mu_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$ yielding $\rho = 0.8617$.

The conclusions emerging from the numerical results are as follows. For small values of ρ, the GPQ method appears quite satisfactory. For large values of ρ, there is no clear winner. However, the bootstrap calibration applied to the delta method appears to provide reasonable results. The percentile bootstrap appears to be really unsatisfactory. For small values of ρ, the delta method does provide satisfactory coverage, but the interval has expected length more than one, which is unacceptable. The overall conclusion is that no method can be singled out for the interval estimation of ρ. Furthermore, the magnitude of ρ (and not the sample sizes) is the major deciding factor affecting the performance of the confidence intervals.

Table 2 Coverage probabilities and expected lengths of 95% confidence intervals for ρ under trivariate normality: GPQ and bootstrap percentile methods

ρ	(n_1, n_2)	GPQ		Percentile Bootstrap	
		Coverage	Length	Coverage	Length
0.1136	(20,20)	0.9436	0.2295	0.7222	0.1538
	(20,30)	0.9522	0.2062	0.7614	0.1478
	(50,50)	0.9458	0.1572	0.8626	0.1342
	(50,100)	0.9528	0.1265	0.8846	0.1119
	(100,100)	0.9448	0.1142	0.9032	0.1052
0.4531	(20,20)	0.8438	0.2710	0.6968	0.2731
	(20,30)	0.8801	0.2502	0.6804	0.2446
	(50,50)	0.9411	0.1826	0.8712	0.1842
	(50,100)	0.9271	0.1574	0.8446	0.1556
	(100,100)	0.9326	0.1314	0.9108	0.1319
0.8617	(20,20)	0.3494	0.3070	0.2628	0.3180
	(20,30)	0.4558	0.2755	0.3108	0.2877
	(50,50)	0.6892	0.1850	0.6494	0.1881
	(50,100)	0.7791	0.1557	0.6710	0.1612
	(100,100)	0.8256	0.1267	0.7908	0.1279

Table 3 Coverage probabilities and expected lengths of 95% confidence intervals for $\ln(\hat{\rho})$ under bivariate normality: Delta method and its bootstrap calibration

$\ln(\rho)$	(n_1, n_2)	Delta Method		Bootstrap Calibration	
		Coverage	Length	Coverage	Length
$\ln(\rho) = -2.3925$	(20,20)	0.9699	2.0552	0.9437	3.4703
$(\rho = 0.0914)$	(20,30)	0.9670	1.9619	0.9413	3.1361
	(50,50)	0.9700	1.5717	0.9453	1.4790
	(50,100)	0.9654	1.5453	0.9438	1.2062
	(100,100)	0.9676	0.8818	0.9503	1.1852
$\ln(\rho) = -0.4920$	(20,20)	0.9401	0.4975	0.9353	0.4716
$(\rho = 0.6114)$	(20,30)	0.9254	0.3527	0.9309	0.4209
	(50,50)	0.9569	0.3022	0.9413	0.2804
	(50,100)	0.9479	0.2198	0.9377	0.2345
	(100,100)	0.9646	0.2089	0.9483	0.1854
$\ln(\rho) = -0.12014$	(20,20)	0.9488	0.2894	0.9397	0.2991
$(\rho = 0.8868)$	(20,30)	0.9513	0.2804	0.9331	0.2929
	(50,50)	0.9658	0.2208	0.9447	0.2317
	(50,100)	0.9658	0.1544	0.9478	0.1632
	(100,100)	0.9676	0.1337	0.9493	0.1262

Table 4 Coverage probabilities and expected lengths of 95% confidence intervals for $\ln(\hat{\rho})$ under trivariate normality: Delta method and its bootstrap calibration

$\ln(\rho)$	(n_1, n_2)	Delta Method		Bootstrap Calibration	
		Coverage	Length	Coverage	Length
$\ln(\rho) = -2.1754$	(20,20)	0.9473	2.2920	0.9451	2.1420
$(\rho = 0.1136)$	(20,30)	0.9531	1.7686	0.9679	0.1478
	(50,50)	0.9641	1.8421	0.9539	0.1342
	(50,100)	0.9643	1.1011	0.9554	0.1119
	(100,100)	0.9666	1.0010	0.9544	0.1052
$\ln(\rho) = -0.7916$	(20,20)	0.9092	0.8608	0.9298	0.9091
$(\rho = 0.4531)$	(20,30)	0.8974	0.6215	0.9319	0.8840
	(50,50)	0.9531	0.3645	0.9503	0.5126
	(50,100)	0.9388	0.3300	0.9512	0.4254
	(100,100)	0.9606	0.3126	0.9510	0.4032
$\ln(\rho) = -0.1488$	(20,20)	0.8535	0.3677	0.9188	0.3214
$(\rho = 0.8617)$	(20,30)	0.8456	0.3532	0.9089	0.2846
	(50,50)	0.9207	0.2551	0.9334	0.2234
	(50,100)	0.9470	0.2167	0.9480	0.2074
	(100,100)	0.9467	0.1520	0.9578	0.1806

4. Examples

To illustrate the practical utility of the present study, let us see the following examples.

4.1. Example 1

The data used for this example is taken from Rencher and Christensen (2012) and it consists of two variables, namely, the head length and head breadth of siblings. Let X_1 be the 'Head length' and 'Head breadth' of the first son' and X_2 be those of the second son. The data consist of 25 observations on (X_1, X_2), in each sample. The sample mean vectors and dispersion matrices are given below

$$\bar{X}_1 = \begin{bmatrix} 185.72 \\ 151.16 \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} 183.22 \\ 149.36 \end{bmatrix}, \quad \hat{\Sigma}_1 = \begin{bmatrix} 102.83 & 59.62 \\ 59.62 & 51.86 \end{bmatrix}, \quad \hat{\Sigma}_2 = \begin{bmatrix} 97.98 & 51.71 \\ 51.71 & 46.24 \end{bmatrix}.$$

Based on the data, the estimate of ρ is $\hat{\rho}$ = 0.9661. The 95% confidence intervals for ρ are as follows:

GPQ: (0.7969, 0.9805)
Percentile bootstrap: (0.7434, 0.9947)
Delta method: (0.8705, 1.0005).

It is clear that there is considerable overlap between the two distributions. The delta method based interval given above is based on $\ln(\hat{\rho})$, along with a bootstrap calibration. We note that the upper limit of the corresponding interval just exceeds one. This is consistent with the numerical results reported earlier. Between the other two intervals, the GPQ-based interval is shorter.

4.2. Example 2

The data for this example is taken from Beall (1945), and is on the scores on four psychological tests: pictorial inconsistencies, tool recognition, paper form board, and vocabulary, on 32 men and 32 women. The respective sample mean vectors and sample dispersion matrices are as follows:

$$\bar{X}_1 = \begin{bmatrix} 15.97 \\ 15.91 \\ 27.19 \\ 22.75 \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} 12.34 \\ 13.91 \\ 16.66 \\ 21.81 \end{bmatrix},$$

$$\hat{\Sigma}_1 = \begin{bmatrix} 5.19 & 4.55 & 6.52 & 5.25 \\ 4.55 & 13.18 & 6.76 & 6.27 \\ 6.52 & 6.76 & 28.67 & 14.47 \\ 5.25 & 6.27 & 14.47 & 16.66 \end{bmatrix}, \quad \hat{\Sigma}_2 = \begin{bmatrix} 9.14 & 7.55 & 4.86 & 4.15 \\ 7.55 & 18.6 & 10.22 & 5.44 \\ 4.86 & 10.22 & 30.04 & 13.49 \\ 4.15 & 5.44 & 13.49 & 28.00 \end{bmatrix}.$$

Based on the above data, we have $\hat{\rho}$ = 0.4188. The 95% confidence intervals for ρ are as follows:

GPQ:	(0.2432, 0.5206)
Percentile bootstrap:	(0.2944, 0.6015)
Delta method:	(0.3107, 1.1714),

It is clear that the overlap between the two groups is significant. The delta method based interval, based on $\ln(\hat{\rho})$, once again has an upper limit that exceeds one. Furthermore, the GPQ-based interval is the shortest.

5. Conclusions

The present investigation is on the interval estimation of the overlap between two multivariate normal populations. Three methods are investigated, and their performance is assessed using estimated coverage probabilities and expected lengths. The main conclusion is that no single method can be recommended for the interval estimation of the overlap. This conclusion is further reinforced by two examples. In short, caution should be exercised on the choice of the methodology to be used for this interval estimation problem.

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