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Moment Properties of Transmuted Power Function Distribution Based on Order Statistics

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Abstract

In this paper, we have obtained the exact expressions for the single and product moments in conjunction with some recurrence relations from transmuted power function (TPF) distribution based on order statistics. We have also obtained the expressions for L-moments. Further, we compute the first four moments and other statistical properties for the TPF distribution numerically.

Keywords: Single moments, L-moments, product moments, recurrence relations.

1. Introduction

Ordered random variables can be seen applicable in various practical scenarios. Ordered random variables have been the centre of attention for the years among several researchers due to their applicability in many areas, for example, we might be interested in arranging the prices of commodities or in arranging the list of students with respect to their grades in the final examination. Another use of ordered random variables can be seen in games when dealing with records, and these variables occur as a natural choice when dealing with extremes like floods, earthquakes, etc. The arrangement of data in increasing or decreasing order produces the order statistics, which can further be utilized to understand the characteristics of the data such as maximum or minimum values of the range of the data etc.

The subject of order statistics is one of the important fields in statistics. The traditional object of a statistician is a group of n independent random variables $X_1, X_2, \ldots, X_n$ and when they are arranged in ascending order of magnitude such that $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$, $X_{r:n}$ or $X_{(r)}$ is called the r^{th} order statistic in a sample of size n. It becomes evident, in this situation, that the well-developed theory of ordered random variables presents an important tool for treatment with observations $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ and different statistics based on $X_{r:n}$. Order statistic deal with the properties and applications of these ordered random variables and the functions relating to them (David and Nagaraja 2003). The marginal and product moments based on the order statistics are important in various fields for example in statistical inference, actuarial science engineering and quality control etc. Various developments in the field of order statistics and related topics have been studied by Arnold et al. (1992), Joshi and Balakrishnan (1982), Khan et a1. (1983), Malik (1966, 1967), and Balakrishnan and Aggarwala (1996). Recent literature includes Genç (2012), Jodrá (2013), Nagaraja (2013), Saran et al. (2014), Kumar et al. (2017), Kumar and Dey (2017), and Çetinkaya and Genç (2018). The computation of moments of order statistics is not an easy task for complicated distributions. For this reason, the recursive computational methods are easier to apply in this sense. The identities and moment relations are important for computing higher-order moments since they reduce the amount of time of calculations and exhibits a general form.

The probability density function (pdf) of the r^{th} order statistic is given by

$$f_{r:n}(x) = C_{n,r}[F(x)]^{r-1} f(x)[1-F(x)]^{n-r}, \quad -\infty < x < \infty, \tag{1}$$

where $C_{n,r} = \dfrac{n!}{(r-1)!(n-r)!} = r\dbinom{n}{r}.$

The joint pdf $X_{(r)}$ and $X_{(s)}$ order statistics is given by

$$\begin{aligned} f(x,y) = {} & \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-1} \\ & \times[1-F(y)]^{n-s} f(y), \quad x<y. \end{aligned} \tag{2}$$

The exponential distribution is frequently preferred over mathematically more complex distributions, such as the lognormal and the Weibull distribution among others. But most of the researchers favor the preference of usage of power function distribution over other distributions because of its applicability to obtain failure rates and reliability figures quickly. It is, therefore, proposed that the power function distribution should be considered as a simple alternative which, in some circumstances, may manifest a better fit for failure data and provides more appropriate information regarding reliability and hazard rates.

The generalization of power function distribution using the quadratic rank transmutation map introduced by Shaw and Buckley (2007), later probed by Haq et al. (2016) and explored a new generalized distribution called TPF distribution was explored by Haq et al. (2016).

A continuous random variable X is said to follow TPF distribution with pdf

$$f(x) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\left[1+\theta-2\theta\left(\frac{x}{\beta}\right)^{\alpha}\right], \; 0<x<\beta, \; \beta>0, \; \theta>0 \text{ and } \alpha>0, \tag{3}$$

and the corresponding cdf

$$F(x) = \left(\frac{x}{\beta}\right)^{\alpha}\left[1+\theta-\theta\left(\frac{x}{\beta}\right)^{\alpha}\right], 0<x<\beta, \; \beta>0, \; \theta>0, \text{ and } \alpha>0, \tag{4}$$

where β is a scale parameter and α, θ are shape parameters.

Using (3) and (4), we get

$$f(x) = \alpha x^{-1}F(x) - \frac{\alpha\theta}{\beta^{2\alpha}}x^{2\alpha-1}. \tag{5}$$

The rest of the paper is arranged as follows: In Section 2, explicit expression, as well as recurrence relations of single moments, are derived and the first four moments are computed along with the variances. In Section 3, we have obtained L-moments for the above mentioned distribution. Further, a numerical study is conducted for L-moments to show the applicability of the derived results. Section 4 deals with relations for product moments. Conclusion is presented in Section 5.

2. Single Moments

In this section, we first derive a closed form expression for the single moments of the r^{th} order statistics.

Theorem 2.1 *Let* $X_1, X_2, \ldots, X_n$ *be a random sample having the size* n *from the TPF distribution and* $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ *denote the corresponding order statistics, then for* $r = 1,2,\ldots,n,\ \beta > 0,\ \theta > 0$ *and* $\alpha > 0$.

$$E[X_{r:n}^{i-b}] = \frac{\beta^{i-b} n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} \sum_{v=0}^{u+r-1} (-1)^u \binom{n-r}{u} \binom{u+r-1}{v} \theta^v$$
$$\times \frac{\left(\left(\frac{i-b}{\alpha} + u + r \right)(1-\theta) + (1+v)(1+\theta) \right)}{\left(\frac{i-b}{\alpha} + u + v + r + 1 \right)} B\left(\frac{i-b}{\alpha} + u + r, v+1 \right), \tag{6}$$

where $B(a,b)$ *is a beta function.*

Proof:

Using (1), (3) and (4), we have

$$E[X_{r:n}^{i-b}] = \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} \int_0^\beta x^{i-b-1} \left(\frac{x}{\beta} \right)^{\alpha(u+r)} \left[1 + \theta - \theta \left(\frac{x}{\beta} \right)^\alpha \right]^{u+r-1}$$
$$\times \left[1 + \theta - 2\theta \left(\frac{x}{\beta} \right)^\alpha \right] dx. \tag{7}$$

Now, expanding binomially in (7), we get

$$E[X_{r:n}^{i-b}] = \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} \sum_{v=0}^{u+r-1} (-1)^u \binom{n-r}{u} \binom{u+r-1}{v} \theta^v$$
$$\times \int_0^\beta x^{i-b-1} \left(\frac{x}{\beta} \right)^{\alpha(u+r)} \left[1 - \left(\frac{x}{\beta} \right)^\alpha \right]^v \left[1 + \theta - 2\theta \left(\frac{x}{\beta} \right)^\alpha \right] dx.$$

$$E[X_{r:n}^{i-b}] = \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} \sum_{v=0}^{u+r-1} (-1)^u \binom{n-r}{u} \binom{u+r-1}{v} \theta^v$$
$$\times \left\{ (1+\theta) \int_0^\beta x^{i-b-1} \left(\frac{x}{\beta} \right)^{\alpha(u+r)} \left(1 - \left(\frac{x}{\beta} \right)^\alpha \right)^v dx - 2\theta \int_0^\beta x^{i-b-1} \left(\frac{x}{\beta} \right)^{\alpha(u+r+1)} \left(1 - \left(\frac{x}{\beta} \right)^\alpha \right)^v dx \right\}. \tag{8}$$

Substituting $\left(\frac{x}{\beta}\right)^{\alpha}=z$ in the (8), we get

$$E[X_{r:n}^{i-b}]=\frac{n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}\sum_{v=0}^{u+r-1}(-1)^{u}\binom{n-r}{u}\binom{u+r-1}{v}\beta^{i-b}\theta^{v}$$
$$\times\left\{(1+\theta)\int_0^1 z^{\frac{(i-b)}{\alpha}+u+r-1}(1-z)^{v}dz-2\theta\int_0^1 z^{\frac{(i-b)}{\alpha}+u+r}(1-z)^{v}dz\right\}. \tag{9}$$

Simplifying (9) by using the beta function, we obtain (6).

Special Cases:

(i) For the special case, $r=1$ in (6), we obtain an exact expression for single moments of the first order statistics which is also denoted as the sample minimum

$$E[X_{1:n}^{i-b}]=n\beta^{i-b}\sum_{u=0}^{n-1}\sum_{v=0}^{u}(-1)^{u}\binom{n-1}{u}\binom{u}{v}\theta^{v}$$
$$\times\frac{\left(\frac{i-b}{\alpha}+u+1\right)(1-\theta)+(1+v)(1+\theta)}{\left(\frac{i-b}{\alpha}+u+v+2\right)}B\left(\frac{i-b}{\alpha}+u+1,v+1\right).$$

(ii) In the other case for sample maximum, putting $r=n$ in (6), we obtain the exact expression for single moments of the largest order statistic as

$$E[X_{n:n}^{i-b}]=n\beta^{i-b}\sum_{v=0}^{n-1}\binom{n-1}{v}\theta^{v}\times\frac{\left(\frac{i-b}{\alpha}+n\right)(1-\theta)+(1+v)(1+\theta)}{\left(\frac{i-b}{\alpha}+v+n+1\right)}B\left(\frac{i-b}{\alpha}+n,v+1\right).$$

(iii) More specifically, the result obtained in (6) reduces to the inverse moments for TPF distribution for $i<b$, while for $i>b$, we get the simple moments for TPF distribution.

When $b=0$, we obtain

$$E[X_{r:n}^{i}]=\frac{\beta^{i}n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}\sum_{v=0}^{u+r-1}(-1)^{u}\binom{n-r}{u}\binom{u+r-1}{v}\theta^{v}$$
$$\times\frac{\left(\left(\frac{i}{\alpha}+u+r\right)(1-\theta)+(1+v)(1+\theta)\right)}{\left(\frac{i}{\alpha}+u+v+r+1\right)}B\left(\frac{i}{\alpha}+u+r,v+1\right).$$

This expression will further be used to compute the first four moments and variances of the TPF distribution based on order statistics.

Theorem 2.2 *For the TPF distribution with pdf and cdf as in (3) and (4), we have for* $r, n, i\in N$, *and* $r, n>2$

$$E[X_{r:n}^{i-b}] = \frac{r\alpha}{(i-b+r\alpha)} E[X_{r+1:n}^{i-b}] + \frac{(i-b)n\alpha\theta}{\beta^{2\alpha}(i-b+2\alpha)(i-b+r\alpha)} \left[E[X_{r-1:n-1}^{i-b+2\alpha}] - E[X_{r:n-1}^{i-b+2\alpha}] \right]. \tag{10}$$

Proof:
By using (1) and (5), we obtain

$$\begin{aligned} E[X_{r:n}^{i-b}] &= \frac{\alpha n!}{(r-1)!(n-r)!} \int_0^\beta x^{i-b-1} [F(x)]^r [1-F(x)]^{n-r} dx \\ &\quad - \frac{\alpha\theta n!}{\beta^{2\alpha}(r-1)!(n-r)!} \int_0^\beta x^{i-b+2\alpha-1} [F(x)]^{r-1} [1-F(x)]^{n-r} dx \\ &= \frac{\alpha n!}{(r-1)!(n-r)!} I_1 - \frac{\alpha\theta n!}{\beta^{\alpha}(r-1)!(n-r)!} I_2, \end{aligned} \tag{11}$$

where $I_1 = \int_0^\beta x^{i-b-1}[F(x)]^r [1-F(x)]^{n-r} dx$ and $I_2 = \int_0^\beta x^{i-b+2\alpha-1}[F(x)]^{r-1}[1-F(x)]^{n-r} dx.$

After simplifying I_1 and I_2, we get

$$I_1 = -\frac{r}{(i-b)} \int_0^\beta x^{i-b} [F(x)]^{r-1} f(x) [1-F(x)]^{n-r} dx + \frac{n-r}{(i-b)} \int_0^\beta x^{i-b} [F(x)]^{r} f(x) [1-F(x)]^{n-r-1} dx,$$

$$I_2 = -\frac{r-1}{(i-b+2\alpha)} \int_0^\beta x^{i-b+2\alpha} [F(x)]^{r-2} f(x) [1-F(x)]^{n-r} dx + \frac{n-r}{(i-b+2\alpha)} \int_0^\beta x^{i-b+2\alpha} [F(x)]^{r-1} f(x) [1-F(x)]^{n-r-1} dx.$$

Substituting I_1 and I_2 in (11), we obtain relation (10). More specifically, for $i < b$ the result obtained in (10) reduces to recurrence relation for inverse moments from the TPF distribution while for $i > b$ we get the recurrence relation for simple moments.

We can test the validity of the calculated results for the moments of order statistics by using the fact $E\left(\sum_{i=1}^{n} X_{i:n}^{j} \right) = nE(X)^j$ given by David and Nagaraja (2003). It can be seen from Table 1 that as the sample size increases the variance decreases.

3. L-Moments

The term L-moments was first introduced by Hosking (1990), as a linear combination of order statistics and can also be used in summarizing the shape of a distribution analogous to the conventional moments. L-moments always exist whenever the mean of the distribution exists,

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\left(X_{r-k:r} \right), \quad r = 1, 2, ..., n. \tag{11}$$

Table 1 The first four moments and variances for the TPF distribution based on order statistics have computed for arbitrary values of α, β and θ for various sample sizes $n = 1, 2, ..., 5$

n	r	$\alpha = 1, \beta = 0.5, \theta = 0.5$				
		$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	Variance
1	1	0.208333	0.062500	0.021875	0.008333	0.019110
2	1	0.129167	0.027083	0.007143	0.002158	0.010399
	2	0.287500	0.097917	0.036607	0.014509	0.015261
3	1	0.093304	0.014955	0.003144	0.000780	0.006250
	2	0.200893	0.051339	0.015141	0.004911	0.010981
	3	0.330804	0.121205	0.047340	0.019308	0.011774
4	1	0.072966	0.009449	0.001647	0.000347	0.004125
	2	0.154315	0.031473	0.007633	0.002085	0.007660
	3	0.247470	0.071205	0.022649	0.007736	0.009964
	4	0.358581	0.137872	0.055570	0.023165	0.009291
5	1	0.059890	0.006502	0.000967	0.000176	0.002915
	2	0.125271	0.021239	0.004367	0.001028	0.005546
	3	0.197883	0.046824	0.012532	0.003670	0.007667
	4	0.280528	0.087459	0.029394	0.010440	0.008763
	5	0.378095	0.150475	0.062114	0.026345	0.007520
		$\alpha = 1.5, \beta = 1, \theta = 0.5$				
		$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	Variance
1	1	0.525000	0.342857	0.250000	0.194805	0.067232
2	1	0.375487	0.186676	0.108333	0.069442	0.045685
	2	0.674513	0.499038	0.391667	0.320168	0.044071
3	1	0.300115	0.122630	0.059821	0.0328333	0.032561
	2	0.526232	0.314768	0.205357	0.142660	0.037848
	3	0.748654	0.591174	0.484821	0.408922	0.030691
4	1	0.253723	0.089029	0.037798	0.018264	0.024654
	2	0.439289	0.223430	0.125893	0.076542	0.030455
	3	0.613175	0.406106	0.284821	0.208779	0.030123
	4	0.793813	0.652863	0.551488	0.475636	0.022723
5	1	0.221859	0.068754	0.026008	0.011285	0.019533
	2	0.381180	0.170131	0.084957	0.046179	0.024832
	3	0.526452	0.303379	0.187297	0.122085	0.026227
	4	0.670991	0.474592	0.349838	0.266575	0.024363
	5	0.824519	0.697430	0.601901	0.527901	0.017599

In this section, we discuss the closed-form expressions of the first four L-moments of TPF distribution. For $r = 1, 2, 3, 4,$ the L-moments are given as

$$\lambda_1 = E[X_{1:1}], \tag{12}$$

$$\lambda_2 = \frac{1}{2}\left[E(X_{2:2}) - E(X_{1:2})\right], \tag{13}$$

$$\lambda_3 = \frac{1}{3}\left[\left(E(X_{3:3}) - 2E(X_{2:3}) + E(X_{1:3})\right)\right], \tag{14}$$

$$\lambda_4 = \frac{1}{4}\left[E(X_{4:4}) - 3E(X_{3:4}) + 3E(X_{2:4}) - E(X_{1:4})\right]. \tag{15}$$

The complete expressions for $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are given in Appendix. Note that, the L-moment ratios defined by Hosking (1990) are taken as the L-coefficient of variation (L-CV) (a dimensionless measure of variability) is denoted by λ_2 / λ_1. Also, the L-skewness (L-SK) (a dimensionless measure of asymmetry) and the L-kurtosis (L-KU) (a dimensionless measure of kurtosis) are defined as $\tau_3 = \lambda_3 / \lambda_2$ and $\tau_4 = \lambda_4 / \lambda_2$, respectively.

Table 2 The computed values of L-moments and corresponding characteristics for different values of parameters are reported as

	$\beta = 0.5, \theta = 0.5$			$\beta = 1, \theta = 0.5$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
λ_1	0.208330	0.333333	0.348214	0.416667	0.600000	0.696429
λ_2	0.791667	0.068254	0.056456	0.158333	0.136508	0.112912
λ_3	0.007441	−0.00456	−0.00742	0.014881	0.009124	0.014829
λ_4	0.001538	0.003136	0.003927	0.003075	0.006272	0.007853
L-CV	3.800010	0.204762	0.162130	0.379122	0.227513	0.162130
L-SK	0.009398	−0.06684	−0.13134	0.093985	0.066840	0.131339
L-KU	0.001942	0.045948	0.069553	0.019424	0.045947	0.069553

It is obvious from the Table 2 that the L-CV decreases as α increases. Also, the L-SK decreases as α increases while the L-SK increases as α increases for fixed values of β and θ.

4. Product Moments

Theorem 4.1 *For the TPF distribution with pdf as given in (3) with* $n \in N, 1 \le r < s \le n, i, j \in N$,

$$E[X_{r:n}^{i-b} X_{s:n}^{j}] = \frac{\beta^{i+j-b} n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{u=0}^{s-r-1} \sum_{v=0}^{n-s} \sum_{w=0}^{u+r-1} \sum_{p=0}^{w} \sum_{q=0}^{s-r-1-u+v} (-1)^{u+v+p}$$
$$\times \binom{s-r-1}{u}\binom{n-s}{v}\binom{u+r-1}{w}\binom{w}{p}\binom{s-r-1-u+v}{q}\theta^{w+q}$$
$$\times\left\{(1+\theta)\frac{((i+j-b)/\alpha+s+v+p)(1-\theta)+(1+q)(1+\theta)}{((i+j)/\alpha+s+v+p+q+1)((i-b)/\alpha+r+u+p)}\right.$$
$$\left.-2\theta\frac{(((i+j-b)/\alpha+s+v+p+1)(1-\theta)+(1+q)(1+\theta))((i+j)/\alpha+s+v+p)}{((i+j-b)/\alpha+s+v+p+q+1)((i+j-b)/\alpha+s+v+p+q+2)((i-b)/\alpha+r+u+p+1)}\right\}$$
$$\times B((i+j-b)/\alpha+s+p+v, 1+q). \tag{16}$$

Proof:

By using (2) and expanding binomially, we have

$$E[X_{r:n}^{i-b}X_{s:n}^{j}]=\sum_{u=0}^{s-r-1}\sum_{v=0}^{n-s}(-1)^{u+v}\binom{s-r-1}{u}\binom{n-s}{v}\int_0^{\beta}y^j[F(y)]^{s-r-1-u+v}f(y)I(y)dy, \quad (17)$$

where

$$I(y)=\int_0^y x^{i-b}[F(x)]^{u+r-1}f(x)dx.$$

Now using (3) and (4), in the above expression, we obtain

$$I(y)=\int_0^y x^{i-b}\left(\frac{x}{\beta}\right)^{\alpha(u+r-1)}\left\{1+\theta-\theta\left(\frac{x}{\beta}\right)^{\alpha}\right\}^{u+r-1}\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\left\{1+\theta-2\theta\left(\frac{x}{\beta}\right)^{\alpha}\right\}dx.$$

After simplification, we get

$$I(y)=\alpha\sum_{w=0}^{u+r-1}\theta^{w}\binom{u+r-1}{w}\left\{(1+\theta)\int_0^y x^{i-b-1}\left(\frac{x}{\beta}\right)^{\alpha(u+r)}\left(1-\left(\frac{x}{\beta}\right)^{\alpha}\right)^{w}dx\right.$$

$$\left.-2\theta\int_0^y x^{i-b-1}\left(\frac{x}{\beta}\right)^{\alpha(u+r+1)}\left(1-\left(\frac{x}{\beta}\right)^{\alpha}\right)^{w}dx\right\}.$$

Assuming $z=\left(\frac{x}{\beta}\right)^{\alpha}$ in the above expression and simplifying, we obtain the resulting expression as

$$I(y)=\sum_{w=0}^{u+r-1}\sum_{p=0}^{w}(-1)^{p}\theta^{w}\beta^{i-b}\binom{u+r-1}{w}\binom{w}{p}\left\{\frac{(1+\theta)(y/\beta)^{((i-b)/\alpha+r+u+p)}}{((i-b)/\alpha+r+u+p)}-2\theta\frac{(y/\beta)^{((i-b)/\alpha+r+u+p+1)}}{((i-b)/\alpha+r+u+p+1)}\right\}.$$

Now substituting $I(y)$ in (17), we get

$$E[X_{r:n}^{i-b}X_{s:n}^{j}]=\sum_{u=0}^{s-r-1}\sum_{v=0}^{n-s}\sum_{w=0}^{u+r-1}\sum_{p=0}^{w}(-1)^{u+v+p}\binom{s-r-1}{u}\binom{n-s}{v}\binom{u+r-1}{w}\binom{w}{p}\theta^{w}, \quad (18)$$

where

$$I_1'=\sum_{q=0}^{s-r-1-u+v}\binom{s-r-1-u+v}{q}\theta^{q}\int_0^{\beta}y^{j-1}\left(\frac{y}{\beta}\right)^{\alpha((i-b)/\alpha+s+v+p)}\left(1-\left(\frac{y}{\beta}\right)^{\alpha}\right)^{q}\left(1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right)dy,$$

$$I_2'=\sum_{q=0}^{s-r-1-u+v}\binom{s-r-1-u+v}{q}\theta^{q}\int_0^{\beta}y^{j-1}\left(\frac{y}{\beta}\right)^{\alpha((i-b)/\alpha+s+v+p+1)}\left(1-\left(\frac{y}{\beta}\right)^{\alpha}\right)^{q}\left(1+\theta-2\theta\left(\frac{y}{\beta}\right)^{\alpha}\right)dy,$$

and $A=\sum_{u=0}^{s-r-1}\sum_{v=0}^{n-s}\sum_{w=0}^{u+r-1}\sum_{p=0}^{w}(-1)^{u+v+p}\binom{s-r-1}{u}\binom{n-s}{v}\binom{u+r-1}{w}\binom{w}{p}\theta^{w}$

$$\times\int_0^{\beta}y^{j}[F(y)]^{s-r-1-u+v}f(y)\times\left\{\frac{\beta^{i-b}(1+\theta)(y/\beta)^{\alpha((i-b)/\alpha+r+u+p)}}{\left((i-b)/\alpha+r+u+p\right)}-2\theta\beta^{i-b}\frac{(y/\beta)^{\alpha((i-b)/\alpha+r+u+p+1)}}{\left((i-b)/\alpha+r+u+p+1\right)}\right\}dy$$

$$=(1+\theta)A\frac{\alpha\beta^{i-b}}{(i-b)/\alpha+(r+u+p)}I_1'-2\theta A\frac{\alpha\beta^{i-b}}{(i-b)/\alpha+(r+u+p+1)}I_2'. \quad (18)$$

After substitution $z = \left(\dfrac{y}{\beta}\right)^{\alpha}$ in the above expressions and simplifying, we get

$$I_1' = \frac{\beta^j}{\alpha} \sum_{q=0}^{s-r-1-u+v} \binom{s-r-1-u+v}{q} \theta^q \frac{[((i+j-b)/\alpha+s+p)(1-\theta)+(1+q)(1+\theta)]}{((i+j-b)/\alpha+s+v+p+q+1))}$$
$$\times B((i+j-b)/\alpha+s+v+p, 1+q), \tag{19}$$

$$I_2' = \frac{\beta^j}{\alpha} \sum_{q=0}^{s-r-1-u+v} \binom{s-r-1-u+v}{q} \theta^q \frac{[((i+j-b)/\alpha+s+p+1)(1-\theta)+(1+q)(1+\theta)]}{((i+j-b)/\alpha+s+v+p+q+2))}. \tag{20}$$

Substituting (19) and (20) in (18) and simplifying we get the required result in (16). For $i < b$, the result obtained in (16) reduces to the ratio moments for two order statistics for the TPF distribution while $i > b$ we get the expression for product moments.The validity of (16) can be checked by using the following identity,

$$\sum_{r=1}^{n} \sum_{s=r+1}^{n} \mu_{r,s:n} = \binom{n}{2} [E(X)]^2,$$

(See Balakrishnan and Cohen1991, p.24), where $\mu_{r,s:n} = E[X_{r,s:n}]$ and $E(X) =$ Mean of the distribution.

Theorem 4.2 *For the TPF distribution with the conditions of Theorem 4.1, we have*

$$E[X_{r:n}^{i-b} X_{s:n}^{j}] = \frac{r\alpha}{(i-b+r\alpha)} E[X_{r+1:n}^{i-b} X_{s:n}^{j}] + \frac{n\alpha\theta(i-b)}{\beta^{2\alpha}(i-b+r\alpha)(i-b+2\alpha)}$$
$$\times \left\{ E\left(X_{r-1:n-1}^{i-b+2\alpha} X_{s-1:n-1}^{j}\right) - E\left(X_{r:n-1}^{i-b+2\alpha} X_{s-1:n-1}^{j}\right) \right\}. \tag{21}$$

Proof:

From (2) and (5), we have

$$E[X_{r:n}^{i-b} X_{s:n}^{j}] = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{\beta} \int_0^{y} x^{i-b} y^j [F(x)]^{r-1} f(x)$$
$$\times [F(y)-F(x)]^{s-r-1} f(y) [1-F(y)]^{n-s} dydx. \tag{22}$$

Using (3) in (22),

$$E[X_{r:n}^{i-b} X_{s:n}^{j}] = \frac{\alpha n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{\beta} \int_0^{y} x^{i-b-1} y^j [F(x)]^{r} [F(y)-F(x)]^{s-r-1}$$
$$\times [1-F(y)]^{n-s} f(y) dxdy - \frac{\alpha\theta n!}{\beta^{2\alpha}(r-1)!(s-r-1)!(n-s)!}$$
$$\times \int_0^{\beta} \int_0^{y} x^{i-b+2\alpha-1} y^j [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1} f(y) [1-F(y)]^{n-s} dxdy, \tag{23}$$

$$E[X_{r:n}^{i-b} X_{s:n}^{j}] = \frac{\alpha n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{\beta} y^j [1-F(y)]^{n-s} f(y) I_1(y) dy$$
$$- \frac{\alpha\theta n!}{\beta^{2\alpha}(r-1)!(s-r-1)!(n-s)!} \int_0^{\beta} y^j [1-F(y)]^{n-s} f(y) I_2(y) dy,$$

where

$$I_1(y) = \int_0^{y} x^{i-b-1} [F(y)-F(x)]^{s-r-1} [F(x)]^{r} dx,$$

and

$$I_2(y)=\int_0^y x^{i-b+2\alpha-1}[F(y)-F(x)]^{s-r-1}[F(x)]^{r-1}\,dx.$$

To simplifying integration given above, integrating by parts treating x^{i-b-1} and $x^{i-b+2\alpha-1}$ for integration and the remaining integrand for differentiation, we get

$$I_1(y)=\frac{(s-r-1)}{(i-b)}\int_0^y x^{i-b}[F(y)-F(x)]^{s-r-2} f(x)[F(x)]^{r}\,dx$$
$$-\frac{r}{(i-b)}\int_0^y x^{i-b}[F(y)-F(x)]^{s-r-1} f(x)[F(x)]^{r-1}\,dx,$$
$$I_2(y)=\frac{(s-r-1)}{(i-b+2\alpha)}\int_0^y x^{i-b+2\alpha}[F(y)-F(x)]^{s-r-2} f(x)[F(x)]^{r-1}\,dx$$
$$-\frac{(r-1)}{(i-b+2\alpha)}\int_0^y x^{i-b+2\alpha}[F(y)-F(x)]^{s-r-1} f(x)[F(x)]^{r-2}\,dx,$$

on substituting $I_1(y)$ and $I_2(y)$ in (23), we obtain (21).

The relation established in (21) reduces to the recurrence relation for ratio moments for two order statistics when we take $i<b$ from the TPF distribution while $i>b$ we get the recurrence relation for product moments.

Corollary 1 *For the TPF distribution with pdf (1) for* $1\le r\le n,$ *the following relation*

$$E[X_{r:n}^{i-b}X_{r+1:n}^{j}]=\frac{\alpha r}{(i-b+\alpha r)}E[X_{r+1:n}^{i+j-b}]$$
$$+\frac{n\alpha\theta(i-b)}{\beta^{2\alpha}(i-b+\alpha r)(i-b+2\alpha)}\left[E[X_{r-1:n-1}^{i-b+2\alpha}X_{r:n-1}^{j}]-E[X_{r:n-1}^{i+j-b+2\alpha}]\right] \tag{24}$$

Proof:

On taking $s=r+1$ in (21), we obtain the result given in (24).

5. Conclusions

In this paper, we derived a closed-form expressions and some recurrence relations for the single and product moments of order statistics for the TPF distribution. The aim for establishing a relationship between moments of order statistics is to get a general form for calculating higher-order moments in terms of lower-order moments. The recurrence relations are very useful in saving the time of computation by giving an iterative form. Numerical computation is also given to calculate the first four moments of order statistics and variance to show the behaviour of the distribution. Further, we derived the first four L-moments for this distribution and also a numerical computation is given for L-moments and L-moment ratios.

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Appendix

The L-moments of TPF distribution is obtained by using (12)-(15), which can be given as

$$\lambda_1 = \frac{\alpha\beta[2\alpha+(1-\theta)]}{(1+2\alpha)(1+\alpha)}.$$

The first L-moment is the mean of the distribution and here we can see that λ_1 is the mean of the TPF distribution. Equations (13)-(15) gives the expression for second, third and fourth L-moments, respectively.

$$\lambda_2 = \frac{(1+1/\alpha)\left\{\left(2+\frac{1}{\alpha}\right)\left(4+\theta+\frac{1}{\alpha}\right)-\frac{1}{\alpha}(1-\theta)-2(2-\theta)\right\}(1-\theta) + \left\{\left(4+2\theta+\frac{1}{\alpha}\right)-2\left(\frac{1}{\alpha}(1-\theta)+(4-\theta)\right)\right\}(1+\theta)}{\left(1+\frac{1}{\alpha}\right)\left(2+\frac{1}{\alpha}\right)\left(3+\frac{1}{\alpha}\right)\left(4+\frac{1}{\alpha}\right)},$$

$$\lambda_3 = \sum_{k=0}^{2}\sum_{u=0}^{k}\sum_{v=0}^{u+2-k}(-1)^{u+k}\binom{2}{k}\binom{k}{u}\binom{u+2-k}{v}\frac{2\beta^j}{(2-k)!k!}\theta^v \times \frac{\left(\left(\frac{j}{\alpha}+u+3-k\right)(1-\theta)+(1+v)(1+\theta)\right)}{\left(\frac{j}{\alpha}+u+v+3-k+1\right)} B\left(\frac{j}{\alpha}+u-k+3, v+1\right), \tag{A1}$$

Expanding (A1), we obtain

$$\lambda_3 = \frac{\beta^j\left(1+\left(1+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(1+\frac{j}{\alpha},1\right)}{2+\frac{j}{\alpha}} - \frac{6\beta^j\left(1+\left(2+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(2+\frac{j}{\alpha},1\right)}{3+\frac{j}{\alpha}}$$
$$-\frac{6\beta^j\theta\left(\left(2+\frac{j}{\alpha}\right)(1-\theta)+2(1+\theta)\right)B\left(2+\frac{j}{\alpha},2\right)}{\left(4+\frac{j}{\alpha}\right)} + \frac{6\beta^j\left(1+\left(3+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(3+\frac{j}{\alpha},1\right)}{\left(4+\frac{j}{\alpha}\right)}$$
$$+\frac{12\beta^j\theta\left(\left(3+\frac{j}{\alpha}\right)(1-\theta)+2(1-\theta)\right)B\left(3+\frac{j}{\alpha},2\right)}{\left(5+\frac{j}{\alpha}\right)} + \frac{6\beta^j\theta^2\left(\left(3+\frac{j}{\alpha}\right)(1-\theta)+3(1+\theta)\right)B\left(3+\frac{j}{\alpha},3\right)}{\left(6+\frac{j}{\alpha}\right)} \tag{A2}$$

It is a difficult task to carry out (A2) in a simple form, so we left it here and proceeded to derive the fourth L-moment.

$$\lambda_4 = \sum_{k=0}^{3}\sum_{u=0}^{k}\sum_{v=0}^{u-k+3}(-1)^u(-1)^k\binom{3}{k}\binom{k}{u}\binom{u-k+3}{v}\frac{4\beta^j}{(4-k)!k!}\theta^v \tag{A3}$$

$$\times \frac{\left\{\left(\frac{j}{\alpha}+u-k+4\right)(1-\theta)+(1+v)(1+\theta)\right\}}{\left(\frac{j}{\alpha}+u+v-k+5\right)} B\left(\frac{j}{\alpha}+u-k+4, v+1\right),$$

on expanding (A3), we obtain

$$\lambda_4 = -\frac{2\beta^j\left(1+\left(1+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(1+\frac{j}{\alpha},1\right)}{3\left(2+\frac{j}{\alpha}\right)} + \frac{5\beta^j\left(1+\left(2+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(2+\frac{j}{\alpha},1\right)}{\left(3+\frac{j}{\alpha}\right)}$$

$$+\frac{5\beta^j\theta\left(\left(2+\frac{j}{\alpha}\right)(1-\theta)+2(1+\theta)\right)B\left(2+\frac{j}{\alpha},2\right)}{\left(4+\frac{j}{\alpha}\right)} - \frac{10\beta^j\left(1+\left(3+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(3+\frac{j}{\alpha},1\right)}{\left(4+\frac{j}{\alpha}\right)}$$

$$-\frac{20\beta^j\theta\left(\left(3+\frac{j}{\alpha}\right)(1-\theta)+2(1+\theta)\right)B\left(3+\frac{j}{\alpha},2\right)}{\left(5+\frac{j}{\alpha}\right)} - \frac{10\beta^j\theta^2\left(\left(3+\frac{j}{\alpha}\right)(1-\theta)+3(1+\theta)\right)B\left(3+\frac{j}{\alpha},3\right)}{\left(6+\frac{j}{\alpha}\right)}$$

$$+\frac{35\beta^j\left(1+\left(4+\frac{j}{\alpha}\right)(1-\theta)+\theta\right)B\left(4+\frac{j}{\alpha},1\right)}{6\left(5+\frac{j}{\alpha}\right)} + \frac{35\beta^j\theta\left(\left(4+\frac{j}{\alpha}\right)(1-\theta)+2(1+\theta)\right)B\left(4+\frac{j}{\alpha},2\right)}{2\left(6+\frac{j}{\alpha}\right)}$$

$$+\frac{35\beta^j\theta^2\left(\left(4+\frac{j}{\alpha}\right)(1-\theta)+3(1+\theta)\right)B\left(4+\frac{j}{\alpha},3\right)}{2\left(7+\frac{j}{\alpha}\right)}$$

$$+\frac{35\beta^j\theta^3\left(\left(4+\frac{j}{\alpha}\right)(1-\theta)+4(1+\theta)\right)B\left(4+\frac{j}{\alpha},4\right)}{6\left(8+\frac{j}{\alpha}\right)}. \tag{A4}$$

Equation (A4) gives the expression for fourth L-moments, which is too complicated for further simplification.