Thailand Statistician
October 2020; 18(4): 491-506
http//:statassoc.or.th
Contributed paper

An Extension of Generalized Extreme Value Distribution: Uniform-GEV Distribution and Its Application to Earthquake Data

Cigdem Topcu Guloksuz*[a] and Nuri Celik [b]

[a] Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, British Columbia, Canada.

[b] Department of Mathematics, Gebze Technical University, Gebze, Kocaeli, Turkey.

*Corresponding author; e-mail: topcucigdem@gmail.com

Received: 23 May 2019
Revised: 30 August 2019
Accepted: 2 October 2019

Abstract

This paper presents an extension of generalized extreme value distribution (GEV) which is generated based on the method which is proposed by Alzaatreh et al. (2013). The new extension is called Uniform-GEV distribution and its properties, maximum likelihood estimation are discussed. A data set is used to demonstrate the potentiality of the proposed distribution. The studied data set refers to the magnitudes of Turkey's earthquakes from 1970 (January) to 2018 (October) and Uniform-GEV distribution is considered as a model for the data. The results are compared to the considered distributions. Uniform-GEV distribution gives the best fit to the studied data over the considered distributions.

Keywords: Skewness, maximum likelihood estimation, extension, generated distributions.

1. Introduction

Statistical distributions are used to model various types of data which are collected from different research fields. They are important tools for drawing conclusions from data. Therefore, developing more flexible distributions provide modelling a broad range of data types. In recent years, there has been a great interest to generalize the distributions to get more efficient models. We give some examples of these kinds of works.

Eugene et al. (2002) proposed a class of new distribution using beta distribution as a generator. These new type of distributions are so-called beta-generated (BG) distributions. This generalization proceeds as follows.

Let $F(x)$ be the cumulative distribution function (cdf) of a random variable X, then the cdf of $G(x)$ is defined as

$$G(x) = \int_0^{F(x)} b(t)dt, \tag{1}$$

where $b(t)$ is the probability density function (pdf) of beta random variable. The pdf of (2) can be obtained by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)\left[F(x)\right]^{\alpha-1}\left[1-F(x)\right]^{\beta-1}. \tag{2}$$

In literature, there are various types of studies including beta-generated distributions, for example Kotz and Nadarajah (2004, 2005), Famoye (2005), Akinsete et al. (2008), Barreto-Souza et al. (2011) and Alshawarbeh et al. (2013). Following Eugene's work, Jones (2009) introduced Kumaraswamy generalized distribution (KWG) by using Kumaraswamy distribution instead of beta distribution in (1). Many researchers proposed some kind of KWG distributions, see Corderio et al. (2010, 2012) and De Pascoa et al. (2011).

In this paper, we consider the method for generating families of continuous distributions which is proposed by Alzaatreh et al. (2013). This method is based on using a random variable X to transform another random variable T through the function $W[F(x)]$ where $F(x)$ is a cdf. The generated families by this method are called T-X family of distributions. As different from the Alzaatreh et al. (2013)'s work, we study with the different $W[F(x);\theta]$ function which is proposed and used to generate a new life time distribution in Celik and Guloksuz (2017). In Section 2 and Subsection 2.1 we give a brief summary for the method which is proposed by Alzaatreh et al. (2013) and for the studied $W[F(x);\theta]$ function, respectively. Based on the $W[F(x);\theta]$, we obtain an extension of the generalized extreme value (GEV) distribution. We call it Uniform-GEV distribution because it is transformed from a uniform random variable through the $W[F(x);\theta]$, where $F(x)$ is cdf of GEV. Subsection 2.2 gives a brief introduction of the GEV distribution. In Section 3, the Uniform-GEV distribution is introduced, the probability density function (pdf), cumulative distribution function (cdf) and hazard function of the Uniform-GEV are presented. Additionally, the moment generating function (mgf) and the quantile function of Uniform-GEV distribution are also studied. The some simulation studies are conducted to present how θ effects skewness of the Uniform-GEV distribution. In Subsection 3.1, the maximum likelihood estimation for Uniform-GEV distribution is discussed. Since GEV is a well-known model for extreme events, the Uniform-GEV distribution can be considered as a model as well. In Section 4, the data set refers the magnitudes of earthquakes which occurred in Turkey from 1970 (January) to 2018 (October) and the frame of the application study are introduced. In Section 5, the Uniform-GEV distribution is fitted to the data along with two existing distributions. The results are presented and discussed. Section 6 concludes the paper.

2. Methodology

In this paper, we consider the approach of generating distribution function which is proposed in Alzaatreh et al. (2013). The main idea of the technique can be summarized as follows.

Definition 1 Let T be continuous random variable with pdf $h(t)$ defined on $[a,b]$. $F(x)$ is a cdf and $f(x)$ is a pdf of any random variable X. A function of $F(x)$ which is represented by

$W[F(x)]$ satisfies the following conditions:

1. $W\left[F(x)\right] \in \left[a,b\right]$
2. $W\left[F(x)\right]$ is differentiable and monotonically non-decreasing on $\left[a,b\right]$
3. $W\left[F(x)\right] \to a$ as $x \to -\infty$ and $W\left[F(x)\right] \to b$ as $x \to \infty$.

Considering $W\left[F(x)\right]$, the cdf and pdf of the new family of distribution are defined, respectively.

$$G(x) = \int_{a}^{W[F(x)]} h(y)dy, \tag{3}$$

$$g(x) = \left\{\frac{d}{dx}W[F(x)]\right\} h\left\{W[F(x)]\right\}. \tag{4}$$

So, the new generated distributions $G(x)$ is called T-X family distributions. The random variable T is called "transformed" into a new cdf $G(x)$ through the function $W[F(x)]$ which is called "transformer". They also notice that different $W[F(x)]$ gives a new family of distributions.

2.1. The $W[F(x);\theta]$ Transformer

Considering the method that is proposed in Alzaatreh et al. (2013), a new function $W[F(x);\theta]$ is proposed in Celik and Guloksuz (2017). The properties of this function are summarized as

$$W[F(x);\theta] = \frac{e^{-\theta F(x)} - 1}{e^{-\theta} - 1}, \tag{5}$$

where $\theta \in R$. The limits of $W\left[F(x);\theta\right]$ are $W\left[F(x);\theta\right] \to 0$ as $x \to -\infty$ and $W\left[F(x);\theta\right] \to 1$ as $x \to \infty$. Additionally, Figure 1 illustrates that $W\left[F(x);\theta\right]$ satisfies the all conditions to be a transformer which is defined in Alzaatreh et al. (2013).

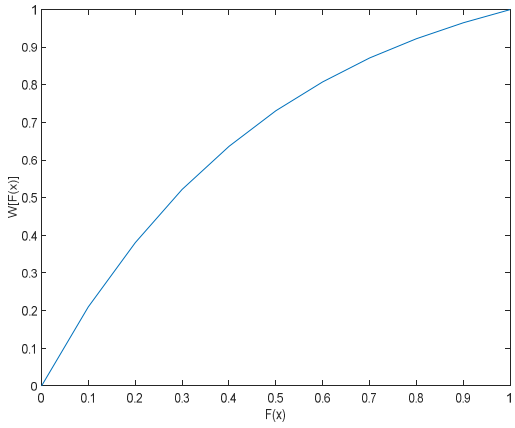


Figure 1 Graph of $W\left[F(x)\right]$

Recalling the method Definition 1, it is seen that $W\left[F(x);\theta\right]\in\left[0,1\right]$, so the transformed random variable with a pdf which is defined on $\left[0,1\right]$ must be chosen. For the transformed variable, we consider a uniform random variable whose pdf is $h(y)=1,\ 0<y<1$. The detailed definition is given below.

Definition 2 Let X be a random variable whose pdf is $f(x)$ and cdf is $F(x)$. Let Y be a uniform random variable which is defined on $\left[0,1\right]$. Then,

$$G(x)=\int_0^{\frac{e^{-\theta F(x)}-1}{e^{-\theta}-1}} dy=\frac{e^{-\theta F(x)}-1}{e^{-\theta}-1} \tag{6}$$

is a cdf of a new family which is called Uniform-X distribution. The corresponding pdf of Uniform-X family is also defined as

$$g(x)=\frac{\theta f(x)e^{-\theta F(x)}}{1-e^{-\theta}}. \tag{7}$$

It should be also noted that for all bounded or unbounded intervals, this new extension of distributions satisfy the properties of being the pdf and the cdf. It can be easily shown by using simple integration techniques.

Additionally, if θ tends to 0, the pdf in (7) coincide with the original distributions $f(x)$. It is proved by using the L'Hospital rule.

$$\lim_{x\to\theta} g(x)=\lim_{x\to\theta}\frac{\theta f(x)e^{-\theta F(x)}}{1-e^{-\theta}}\to f(x). \tag{8}$$

In this study, we consider the GEV distribution as $f(x)$ and obtain a new extension of the GEV distribution. Rest of the paper, we call Uniform-GEV distribution to this new extension.

2.2. The GEV Distribution

In most fields in science such as engineering, finance, medicine, there are extreme and rare events to observe. Some kinds of events such as extreme floods, rainfalls, temperatures, earthquake records, market crashes and high level ozone concentration may possess undesired consequences. Therefore, estimating the possible damages or risks that are caused by these extreme events have become a main concern. The GEV distribution which based on Fisher-Tippet theorem is commonly used to model extreme events. The GEV distribution covers three well known probability distributions which are Gumbel, Frechet and reversed Weibull (or negative Weibull) distributions. These distributions are also known as extreme value type I, extreme value type II and extreme value type III, respectively. The class of GEV distributions is very flexible, because it can be represented by single shape parameter which controls the tail behavior. It means that if $k=0$, the GEV distribution has a thin tail behavior, if $k>0$, the GEV distribution has a fat tail behavior and if $k<0$, the GEV distribution has a short tail behavior. Finally, the sub families with $k=0$, $k>0$ and $k<0$ are linked to Gumbel, Frechet and reversed Weibull, respectively.

Shukla et al. (2010), Castillo et al. (2004), Beirlant et al. (1996) are some studies which present applications of GEV in engineering. Also, there are some other excellent works dealing

with the theory and the applications for extreme values, see Nadarajah and Kotz (2002), Eljabri, (2013), Nadarajah and Pogany (2013), Coles (2001).

The GEV distribution has the following probability density function (pdf) and cumulative distribution function (cdf), respectively

$$f(x;\mu,\sigma,k)=\frac{1}{\sigma}\left[t(x)\right]^{k+1}e^{-t(x)}, \tag{9}$$

$$F(x;\mu,\sigma,k)=e^{-t(x)}, \tag{10}$$

$$t(x)=\begin{cases}\left[1+k\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{k}}, & k\neq 0,\\ e^{-\left(\frac{x-\mu}{\sigma}\right)}, & k=0.\end{cases} \tag{11}$$

Here $x\in\left[\mu-\dfrac{\sigma}{k},\infty\right]$ when $k>0$, $x\in[-\infty,\infty]$ when $k=0$ and $x\in\left[-\infty,\mu-\dfrac{\sigma}{k}\right]$ when $k<0$. Additionally, $\mu\in\Re$ is a location parameter, $\sigma>0$ is a scale parameter and $k\in\Re$ is a shape parameter. The expected value and variance are given as follows:

$$E(X)=\mu-\frac{\sigma}{k}+\left(\frac{\sigma}{k}\right)g_1, \tag{12}$$

$$V(X)=\frac{\sigma^2}{k^2}\left(g_2-g_1^2\right). \tag{13}$$

Here, $g_i=\Gamma(1-ik)$, $i=1,2,...$ and $\Gamma(t)$ is a gamma function.

3. The Uniform-GEV Distribution

Considering the Definition 2, $F(x)$ is considered as the GEV distribution in (10), then the Uniform-GEV distribution can be obtained. The general form of cdf of the Uniform-GEV is given below

$$G(x)=\frac{\exp(-\theta e^{-t(x)})-1}{e^{-\theta}-1}. \tag{14}$$

Using (11) for $t(x)$ and rewriting (14), we obtain the explicit form of cdf and pdf of the Uniform-GEV, respectively

$$G(x)=\begin{cases}\dfrac{\exp\left[-\theta e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}}\right]-1}{e^{-\theta}-1}, & k\neq 0\begin{cases}k<0, & x\in[-\infty,\mu-\sigma/k],\\ k>0, & x\in[\mu-\sigma/k,\infty],\end{cases}\\ \dfrac{\exp\left[-\theta e^{-\left[\left(\frac{x-\mu}{\sigma}\right)\right]}\right]-1}{e^{-\theta}-1}, & k=0.\end{cases} \tag{15}$$

Considering (15), the explicit form of the pdf of Uniform-GEV distribution can be written as

$$g(x)=\begin{cases}\dfrac{\theta\left[1+k\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-\left(\frac{k+1}{k}\right)} e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}} \exp\left[-\theta e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}}\right]}{\sigma\left(1-e^{-\theta}\right)}, & k\neq 0,\\ \dfrac{\theta e^{-\left(\frac{x-\mu}{\sigma}\right)} \exp\left[-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right]\exp\left[-\theta\exp\left[-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right]\right]-1}{\sigma\left(1-e^{-\theta}\right)}, & k=0.\end{cases} \tag{16}$$

Figures 2 to 4 illustrate that the pdfs and cdfs of the Uniform-GEV distribution with $k=0$, $k>0$ and $k<0$, respectively.

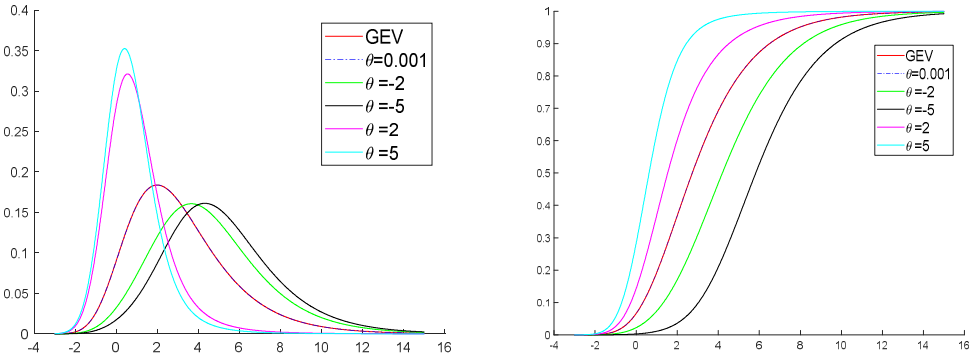


Figure 2 The pdfs and cdfs of the Uniform-GEV and GEV distributions with $\mu=2$, $\sigma=2$ and $k=0$ for $\theta=0.001$, 2, 5, −2 and −5

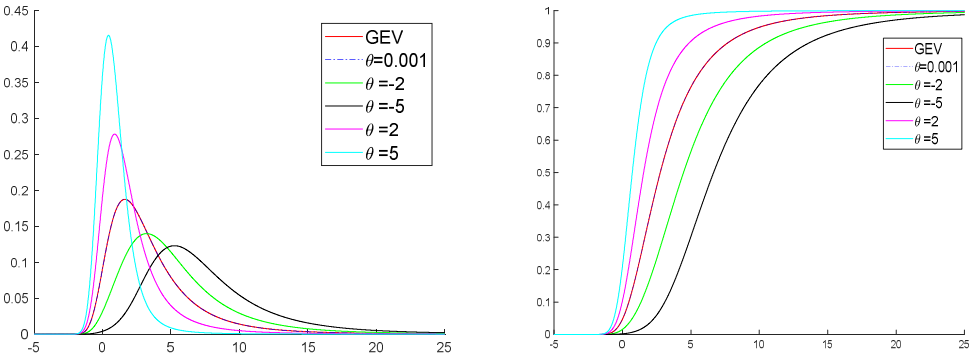


Figure 3 The pdfs and cdfs of the Uniform-GEV and GEV distributions with $\mu=2$, $\sigma=2$ and $k=0.2$ for $\theta=0.001$, 2, 5, −2 and −5

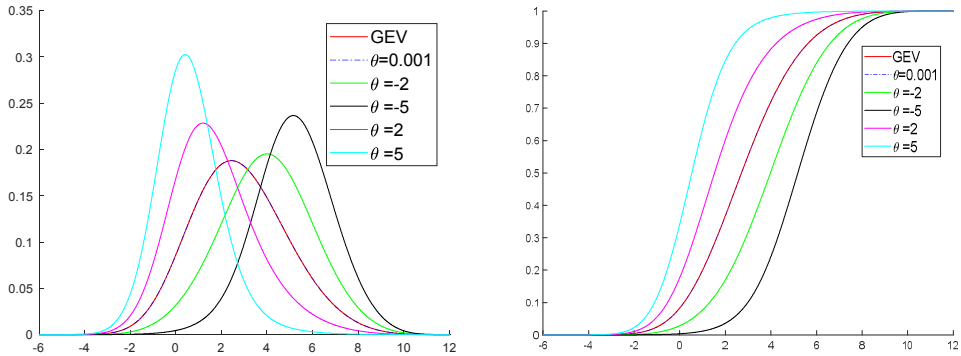


Figure 4 The pdfs and cdfs of the Uniform-GEV and GEV distributions with $\mu = 2$, $\sigma = 2$ and $k = -0.2$ for $\theta = 0.001, 2, 5, -2$ and -5

For the sake of the simplicity, we use the following substitutions rest of the paper,

$$A = 1 + \frac{k(x-\mu)}{\sigma} \text{ and } B = e^{-\frac{(x-\mu)}{\sigma}}.$$

The hazard function of the Uniform-GEV is defined

$$h(x) = \frac{g(x)}{1 - G(x)}, \tag{17}$$

where $G(x)$ is the cdf and $g(x)$ is the pdf of the Uniform-GEV and are defined in (15) and (16), respectively. The explicit form of the hazard in (17) is

$$h(x) = \begin{cases} \dfrac{\theta A^{-(k+1/k)} e^{-A^{-1/k}} \exp\left[-\theta e^{-A^{-1/k}}\right]}{\sigma\left(e^{-\theta e^{-A^{-1/k}}} - e^{-\theta}\right)}, & k \neq 0, \\ \dfrac{\theta B e^{-B} \exp\left[-\theta e^{-B}\right]}{\sigma\left(e^{-\theta B} - e^{-\theta}\right)}, & k = 0. \end{cases} \tag{18}$$

Figure 5 illustrates the hazard functions of Uniform-GEV distribution for different θ values and hazard function of GEV distribution. When θ tends to 0, Uniform-GEV converges to GEV Figure 5 states as well.

The moment generator function of Uniform-GEV distribution can be obtained as follows

$$M_X(t) = \int_{\mu - \frac{\sigma}{k}}^{\infty} e^{tx} g(x) dx, \tag{19}$$

where $g(x)$ is defined in (16). Let $e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}} = u$,

$$M_X(t) = \int_{\mu - \frac{\sigma}{k}}^{\infty} e^{tx} \frac{\theta \frac{1}{\sigma}\left[1 + k\left(\frac{x-\mu}{\sigma}\right)^{-1/k}\right]^{k+1} e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}} e^{-\theta e^{-\left[1+k\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/k}}}}{1 - e^{-\theta}} dx. \tag{20}$$

Using the Taylor expansion of $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$, we get

$$M_X(t) = \frac{\theta}{1-e^{-\theta}} e^{-\frac{\sigma t}{k}} e^{\frac{\mu t}{\sigma}} \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!} \int_0^1 e^{\frac{\sigma t}{k}(-\ln u)^{-k}} u^i du. \tag{21}$$

By substituting $-\ln u = w$ and using the Taylor expansion again,

$$M_X(t) = \frac{\theta}{1-e^{-\theta}} e^{-\frac{\sigma t}{k}} e^{\frac{\mu t}{\sigma}} \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!} \sum_{n=0}^{\infty} \frac{(\sigma t/k)^n}{n!} \int_0^{\infty} w^{-kn} e^{-w(i+1)} dw. \tag{22}$$

Finally, the mgf can be obtained as,

$$M_X(t) = \frac{\theta}{1-e^{-\theta}} e^{-\frac{\sigma t}{k}} e^{\frac{\mu t}{\sigma}} \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!} \sum_{n=0}^{\infty} \frac{(\sigma t/k)^n}{n!} \frac{\Gamma(1-kn)}{(i+1)^{(1-kn)}}. \tag{23}$$

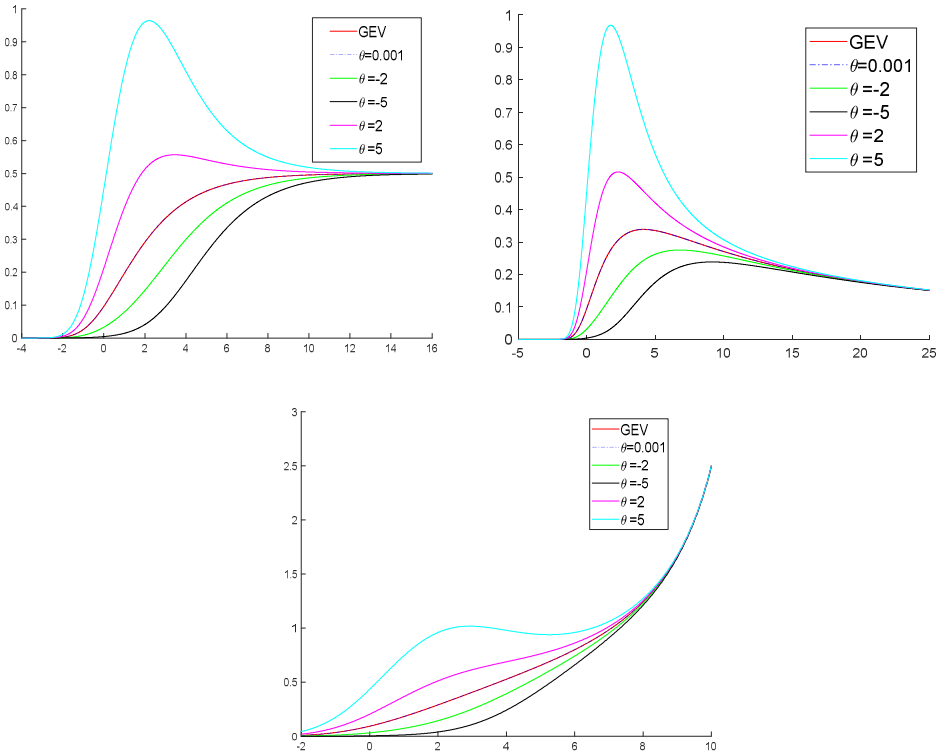


Figure 5 The hazard functions of the Uniform-GEV and GEV distributions with $\mu = 2$, $\sigma = 2$ and $k = 0$, 0.2, −0.2 for $\theta = 0.001$, 2, 5, −2 and −5

When θ tends to 0, the expected value of Uniform-GEV distribution is $\mu + \sigma \frac{\Gamma(1-k)-1}{k}$, which is the expected value of original GEV distribution. The quantile function of the Uniform-GEV distribution for $k = 0$ and $k \neq 0$ are given below, respectively

$$Q(p) = (-\sigma)\log(-\log((-1/\theta)\log(1-p+pe^{-\theta}))) + \mu, \tag{24}$$

$$Q(p) = (-\sigma/k)(-\log((-1/\theta)\log(1-p+pe^{-\theta})))^{-k} - (\sigma/k) + \mu. \tag{25}$$

Some simulations are performed to present how the skewness of the Uniform-GEV is changed for different θ values. Random samples with size 1,000 are generated 1,000 times by using (24) and (25), The results for sample means $(\bar{X})$ standard deviations (SD), skewness values (S.V.) and kurtosis values (K.V.) when $\mu = 2,\ \sigma = 2$ and $k = 0, 0.2, -0.2,$ respectively are listed in Tables 1 to 3.

Table 1 The simulation results for the Uniform-GEV distributions with $\mu = 2, \sigma = 2$ and $k = 0$

θ	Mean	SD	S.V.	K.V.
GEV	3.1544	2.5651	1.1399	5.4000
0.001	3.1545	2.5661	1.1401	5.3670
−5.000	6.1624	2.7538	0.8640	4.7487
−4.000	5.7073	2.7846	0.8436	4.6843
−3.000	5.1650	2.8039	0.8374	4.5891
−2.000	4.5392	2.7851	0.8879	4.6590
2.000	1.9101	2.0778	1.4673	7.1708
3.000	1.4410	1.8109	1.5551	8.0857
4.000	1.0780	1.5665	1.5113	8.5059
5.000	0.8015	1.3762	1.4062	8.6523

Table 2 The simulation results for the Uniform-GEV distributions with $\mu = 2, \sigma = 2$ and $k = 0.2$

θ	Mean	SD	S.V.	K.V.
GEV	3.6448	3.5526	2.1499	10.6477
0.001	3.6403	3.5585	2.1735	10.8541
−5.000	7.8074	5.0775	1.9963	9.9074
−4.000	7.1233	4.9442	2.0313	10.2513
−3.000	6.3320	4.7112	1.9987	9.9530
−2.000	5.4440	4.3577	1.9985	10.0473
2.000	2.1580	2.5022	2.3529	12.2089
3.000	1.6242	2.0458	2.3687	12.8299
4.000	1.2431	1.6623	2.2066	12.1812
5.000	1.0131	1.5918	2.1968	12.9116

As it is mentioned in (8), Figures 2 to 4 and Tables 1 to 3 indicate that when θ tends to 0 the Uniform-GEV distribution converges to the GEV distribution. It shows that the Uniform-GEV distribution can be treated as new distribution and also for very small θ values converges to the original GEV distribution. It provides a flexibility to Uniform-GEV distribution. When we compare the Uniform-GEV distribution to the GEV distribution, the sign of the parameter θ effects the skewness and kurtosis of the Uniform-GEV distribution. Positive values of the θ make the Uniform-GEV distribution right-skewed and more peaked and

negative values of θ make Uniform-GEV left-skewed and less peaked comparing to GEV distribution, (see Figures 2 and 4, Tables 1 and 3). However, Figure 4 and Table 2 indicate that while the Uniform-GEV is right-skewed for positive θ values and left-skewed for negative θ values, Uniform-GEV is more peaked for all values of θ.

Table 3 The simulation results for the Uniform-GEV distributions with $\mu = 2,\ \sigma = 2$ and $k = -0.2$

θ	Mean	SD	S.V.	K.V.
GEV	2.8180	2.1026	0.2263	2.4007
0.001	2.8187	2.1010	0.2368	2.8461
−5.000	5.1694	1.7286	-0.1122	3.1137
−4.000	4.8375	1.8370	-0.1295	3.0955
−3.000	4.4335	1.9320	-0.1098	2.9818
−2.000	4.9599	2.0458	-0.0411	2.8392
2.000	1.7063	1.8890	0.5175	3.3547
3.000	1.2746	1.7402	0.5873	3.7141
4.000	0.9048	1.5797	0.5879	3.9627
5.000	0.6230	1.4515	0.5317	3.9625

3.1. Maximum Likelihood Estimation for Uniform-GEV Distribution

Suppose $X_1,...,X_n$ are random sample from a Uniform-GEV distribution defined in (17). The likelihood functions for two cases are given below

Case 1: $k = 0$,

$$L = \frac{\theta^n}{\sigma^n (1-e^{-\theta})^n} e^{\sum_{i=1}^{n}\left(\frac{\mu - x_i}{\sigma}\right)} \exp\left[-\sum_{i=1}^{n} e^{\left(\frac{\mu - x_i}{\sigma}\right)}\right] \exp\left[\theta \sum_{i=1}^{n} e^{\left(\frac{\mu - x_i}{\sigma}\right)}\right]. \tag{26}$$

The log-likelihood function is

$$\ln L = n\ln\theta - n\ln\left(1-e^{-\theta}\right) + \sum_{i=1}^{n}\left(\frac{\mu - x_i}{\sigma}\right) - \sum_{i=1}^{n} e^{\left(\frac{\mu - x_i}{\sigma}\right)} - \theta \sum_{i=1}^{n} \exp\left[-e^{\left(\frac{\mu - x_i}{\sigma}\right)}\right]. \tag{27}$$

By differentiating the log-likelihood function with respect to the unknown parameters, the following equations are obtained

$$\frac{\partial}{\partial\theta}\ln L = \frac{n}{\theta} - \frac{ne^{-\theta}}{e^{-\theta}-1} - \sum_{i=1}^{n} \exp\left[-e^{\left(\frac{\mu - x_i}{\sigma}\right)}\right], \tag{28}$$

$$\frac{\partial}{\partial\sigma}\ln L = -\frac{n}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(\mu - x_i) + \frac{1}{\sigma^2}\sum_{i=1}^{n}(\mu - x_i)e^{\left(\frac{\mu - x_i}{\sigma}\right)} - \frac{\theta}{\sigma^2}\sum_{i=1}^{n}(\mu - x_i)e^{\left(\frac{\mu - x_i}{\sigma}\right)} \exp\left[-e^{\left(\frac{\mu - x_i}{\sigma}\right)}\right], \tag{29}$$

$$\frac{\partial}{\partial\mu}\ln L = \frac{n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n} e^{\left(\frac{\mu - x_i}{\sigma}\right)} + \frac{\theta}{\sigma}\sum_{i=1}^{n} e^{\left(\frac{\mu - x_i}{\sigma}\right)} \exp\left[-e^{\frac{\mu - x_i}{\sigma}}\right]. \tag{30}$$

Maximum likelihood estimations can be obtained by equating these equations to zero.

Case 2: $k \neq 0$,

The likelihood function is given below

$$L = \frac{\theta^n}{\sigma^n \left(1-e^{-\theta}\right)^n}\left[\prod_{i=1}^{n}\left(1+k\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-\left(\frac{k+1}{k}\right)}\right] e^{-\sum_{i=1}^{n}\left[1+k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-1/k}} \exp\left[-\theta\sum_{i=1}^{n} e^{-\left[1+k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-1/k}}\right]. \tag{31}$$

The log-likelihood function is obtained as follows

$$\begin{aligned}\ln L = {} & n\ln(\theta) - n\ln\left(\sigma\right) - n\ln\left(1-e^{-\theta}\right) - \left(\frac{k+1}{k}\right)\sum_{i=1}^{n}\ln\left(1+k\left(\frac{x_i-\mu}{\sigma}\right)\right) \\ & -\sum_{i=1}^{n}\left(1+k\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-\frac{1}{k}} - \theta\sum_{i=1}^{n} e^{-\left(1+k\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-\frac{1}{k}}}.\end{aligned} \tag{32}$$

By differentiating the log-likelihood function with respect to the unknown parameters, the following equations are obtained

$$\frac{\partial}{\partial\theta}\ln L = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n} e^{-\left(1+k\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-\frac{1}{k}}}, \tag{33}$$

$$\begin{aligned}\frac{\partial}{\partial\sigma}\ln L = {} & -\frac{n}{\sigma} - \left(\frac{k+1}{k}\right)\sum_{i=1}^{n}\frac{k\left(\mu-x_i\right)}{\sigma^2 A} - \sum_{i=1}^{n}\frac{A^{-1-\frac{1}{k}}\left(x_i-\mu\right)}{\sigma^2} \\ & + \theta\sum_{i=1}^{n}\frac{e^{-A^{-1/k}}\left(x_i-\mu\right)A^{-1/k}}{\sigma^2},\end{aligned} \tag{34}$$

$$\frac{\partial}{\partial\mu}\ln L = \frac{k+1}{\sigma}\sum_{i=1}^{n}A^{-1} - \sum_{i=1}^{n}\sigma^{-1}A^{-\frac{1}{k}-1} + \left(\frac{\theta}{\sigma}\right)\sum_{i=1}^{n}\left[A^{-\frac{1}{k}-1}e^{-A^{-1/k}}\right], \tag{35}$$

$$\begin{aligned}\frac{\partial}{\partial k}\ln L = {} & \frac{1}{k^2}\sum_{i=1}^{n}\log(A) - \frac{k+1}{k}\sum_{i=1}^{n}\frac{(x_i-\mu)}{\left(\sigma+k(x_i-\mu)\right)} - \sum_{i=1}^{n}\left[\frac{A^{-\frac{1}{k}-1}(\mu-x_i)}{k\sigma} + \frac{(A)^{-1/k}\ln(A)}{k^2}\right] \\ & + \sum_{i=1}^{n}\left\{\left[\frac{A^{-\frac{1}{k}-1}\left(\mu-x_i\right)}{k\sigma} + \frac{(A)^{-1/k}\ln(A)}{k^2}\right]e^{-(A)^{-1/k}}\right\}.\end{aligned} \tag{36}$$

Maximum likelihood estimations can be obtained by equating these equations to zero. It should be also noted that only numerical solutions are available for solving these equations. So, in this study iterative algorithm method (IRA) is used to solve likelihood equations.

4. A Real Data Example

The seismic map of Turkey, see Figure 6, most lands of Turkey have located on a first degree earthquake zone. Because of Turkey has an active tectonic system, determining the risk of the occurrence of earthquakes may become an important interest. In this study, our aim is to present the potentiality of the Uniform-GEV distribution to model the magnitudes

of earthquake and to estimate the extreme quantiles of the annual maximum. It should be said that the detailed seismic analysis is out of the scope of this paper.

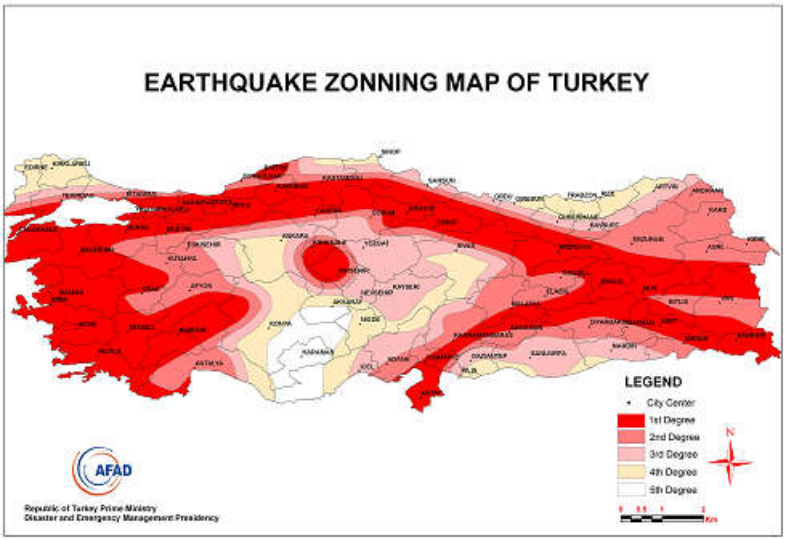


Figure 6 The earthquake risk map of Turkey

The data which is used in this study is obtained from the web page of Bogazici University Kandilli Observatory and Earthquake Research Institute (http://www.koeri.boun.edu.tr). The magnitudes of the earthquakes that occurred in the area located 36^{o}-42^{o}N and 26^{o}-45^{o}E (Turkey) between the years 1970 (January)-2018 (October) are considered. Among all the considered records, we focus on the records of the earthquakes have 5 and bigger than 5 magnitude values. The maximum magnitude value of each year are considered by using block maxima method and an extreme series is obtained. The extreme is used for obtaining all estimates.

The calculation of the return level of the extreme events is an important application of GEV distribution. The return is defined as a value that is expected to be exceeded on average once every period of time with probability $p = 1/T$. From the distribution which gives best possible fit to the data, how often the extreme quantiles occur with a certain return level, T, can be estimated. For this purpose, we fitted four possible distributions to the extreme series data. We denote Uniform-GEV distribution with parameter k by Uniform-GEV$_{k\neq 0}$ and Uniform-GEV without parameter k by Uniform-GEV$_{k=0}$. The GEV distribution and Gumbel distribution which is a special case of the GEV are the appropriate statistical distributions to model for block maxima. We fitted these four competing distributions which are GEV, Gumbel, Uniform-GEV$_{k=0}$ and Uniform-GEV$_{k\neq 0}$ to model the considered data.

5. Results and Discussion

The maximum likelihood estimates of the parameters with standard errors of the all competing distributions are listed in Table 4. We use the iterative methods to obtain the

parameter estimates of the Uniform-GEV distributions. To specify the distribution that gives the best fit among the all candidates, three selection criteria are used. One of them is the distance measure (DM) between theoretical and empirical distribution, the other one is the root mean squared error (RMSE) and the last one is the log likelihood (Loglik) values and their calculated values are listed in Table 5.

Table 4 Parameter estimates

	θ	μ	σ	k
GEV	-	5.5144	0.4463	0.2022
		(0.0714)	(0.0618)	(0.0714)
Gumbel	-	5.5622	0.4932	-
		(0.0748)	(0.0586)	
Uniform-GEV$_{k=0}$	−0.1210	5.5501	0.4887	-
	(0.0812)	(0.0695)	(0.0548)	
Uniform-GEV$_{k\neq 0}$	−0.8103	5.4000	0.3795	0.2510
	(0.0781)	(0.0548)	(0.0514)	(0.0745)

Minimum of the DM and RMSE and maximum of the Loglik indicate the best fit. Additionally, the histogram of the data with the fitted distributions illustrates that the Uniform-GEV distribution with k parameter fits better than other candidates (see Figure 7). Figure 8 demonstrates the comparison of the best two candidates distributions which are GEV and Uniform-GEV$_{k\neq 0}$.

Table 5 The values of DM, Log likelihood, RMSE

	DM	Loglik	RMSE
GEV	0.1120	−42.6593	0.0483
Gumbel	0.1655	−43.3952	0.0587
Uniform-GEV$_{k=0}$	0.1783	−43.4383	0.0609
Uniform-GEV$_{k\neq 0}$	0.1036	−42.6204	0.0464

Estimates of the extreme quantiles of the annual maximum are obtained by

$$y_p = (-\sigma/k)(-\log((-1/\theta)\log(1-p+pe^{-\theta})))^{-k} - (\sigma/k) + \mu,$$

where $G(y_p) = 1-p$. Here, y_p is called return level with the return period $1/p$. It means that y_p is exceeded by the annual maximum in any particular year with probability p (Coles 2001). T is called return period then $p = 1/T$. Return levels for Uniform-GEV$_{k\neq 0}$ distribution are listed in Table 6.

Table 6 Return levels for Uniform-GEV$_{k\neq 0}$ distribution

2 years level	10 years level	20 years level
5.6	6.8	7.4

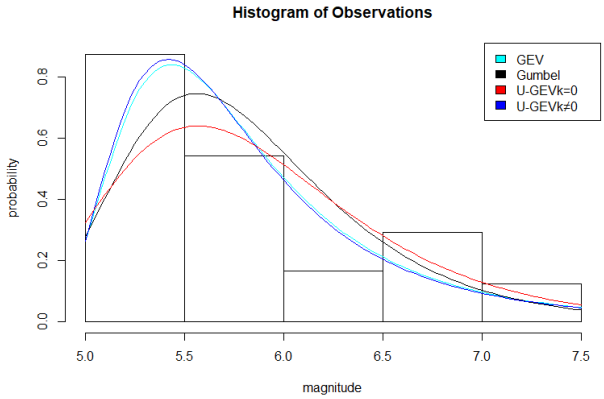


Figure 7 Histogram with fitted pdfs

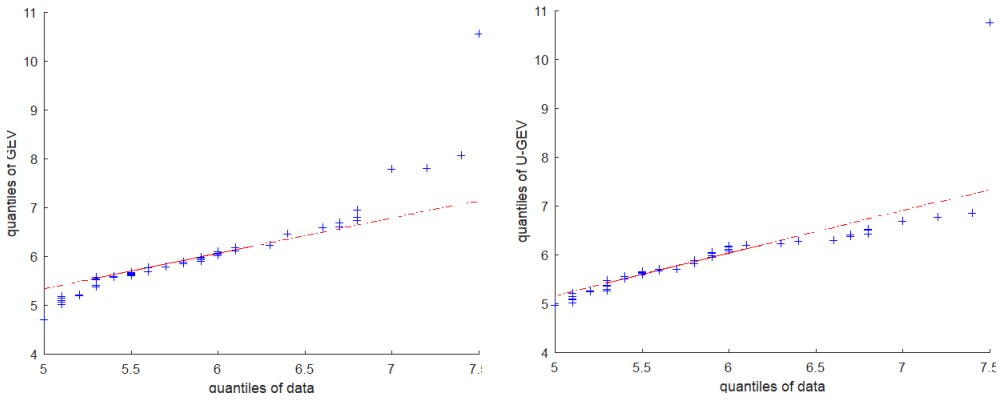


Figure 8 Q-Q plots

The probability of occurring an earthquake that exceeds the magnitude 5.6 is 0.5 for next two years; the probability of occurring an earthquake that exceeds the magnitude 6.8 is 0.05 for next ten years and the probability of occurring an earthquake that exceed the magnitude 7.4, is 0.01 for next twenty years (see Table 6).

6. Conclusions

In this paper, following the method of the generating T-X family of distributions we introduced a new extension of the generalized extreme value distribution which is called the Uniform-GEV distribution. The Uniform-GEV distribution has two cases and both cases were studied in this paper. We considered the Uniform-GEV distribution as a model for extreme events data and presented an example of the usefulness in modelling of such kind of data. According to the selected selection criteria, the Uniform-GEV distribution gave the best possible fit to the studied data over the other candidate models. Having four parameters and complex structure are the main disadvantages of the Uniform-GEV distribution. Although the Uniform-GEV distribution has an extra parameter it also provides a flexibility to the Uniform-GEV distribution

in modelling data. We don't claim that a general statement about the superiority of the Uniform-GEV distribution to the existing distributions. The true model of the data is never known and the researchers choose the appropriate ones according to the considered selection criteria. In this study, the Uniform-GEV distribution was the appropriate model for the studied data. We have only presented an extended distribution with the hope it will be a useful model for skewed data. The improvement in computational ability in modern computers and new numerical techniques may help researchers to overcome the estimation problem in complex functions.

We consider to extend our study to the different data sets and with the other existing distributions.

Acknowledgements

The authors wish to place on record their appreciation to the editor and the anonymous reviewers for critically examining the paper, providing constructive comments and suggestions. The research was supported by the award of Scientific and Technological Research Council of Turkey (TUBITAK) Grant No. 2219 to the first author. First author also gratefully acknowledges the hospitality extended by the University of Northern British Columbia.

References

Akinsete A, Famoye F, Lee C. The beta-Pareto distribution. Statistics. 2008; 42(6): 547-563.

Alshawarbeh E, Famoye F, Lee C. The beta-Cauchy distribution: some properties and applications. J Stat Theory Appl. 2013; 12(4): 378-391.

Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. Metron. 2013; 71: 63-79.

Barreto-Souza W, Morais ALD, Cordeiro GM. The Weibull-geometric distribution. J Stat Comput Simul. 2011; 81(5): 645-657.

Beirlant J, Teugels JL, Vynckier P. Practical analysis of extreme values. Belgium: Leuven University Press; 1996.

Castillo E, Hadi AS, Balakrishnan N, Sarabia JM. Extreme value and related models with applications in engineering and science. New York: John Wiley; 2004.

Celik N, Guloksuz CT. A new lifetime distribution. Eksploat Niezawodn. 2017; 19(4): 634-639.

Coles S. An introduction to statistical modelling of extreme values. London: Springer-Verlag; 2001.

Cordeiro GM, Ortega EMM, Nadarajah S. The Kumaraswamy Weibull distribution with application to failure data. J Franklin Inst. 2010; 347(8): 1399-1429.

Cordeiro GM, Pescim RR, Ortega EMM. The Kumaraswamy generalized half-normal distribution for skewed positive data. J Data Sci. 2012; 10: 195-224.

De Pascoa M, Ortega EMM, Cordeiro GM. The Kumaraswamy generalized gamma distribution with application in survival analysis. Stat Methodol. 2011; 8(5): 411-433.

Eljabri SS. New statistical models for extreme values. PhD [dissertation]. Manchester: The University of Manchester; 2013.

Eugene E, Lee C, Famoy F. Beta-normal distribution and its application. Commun Stat - Theory Methods. 2002; 31(4): 497-512.

Famoye F, Lee C, Olumolade O. The beta-Weibull distribution. J Stat Theory Appl. 2005; 4(2): 121-136.

Jones MC. Kumaraswamy distribution: A beta type distribution with tractability advantages. Stat

Methodol. 2009; 6(1): 70-81.
Kotz S, Nadarajah S. Extreme value distributions theory and applications. London: Imperial College Press; 2000.
Nadarajah S, Kotz S. The beta exponential distribution. Reliab Eng Syst Safe 2006; 91(6): 689-697.
Nadarajah S, Pogany TK. On the characteristic functions for extreme value distributions. Extremes. 2013; 16(1): 27-38.
Shukla RK, Trivedi M, Kumar M. On the proficient use of GEV distribution: a case study of subtropical monsoon region in India. An Ser Inform. 2010; 7(1): 81-92.