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Wrapped Two-Parameter Lindley Distribution for Modelling Circular Data

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Abstract

In this paper, a new circular probability model with two parameters namely, the wrapped two-parameter Lindley distribution is proposed. Behavior of the density function with variation in the values of the parameters is examined and expressions for its characteristic function, trigonometric moments and other related descriptive measures are derived. Operation of wrapping and convoluting linear distribution around a unit circle is investigated. The method of maximum likelihood is used to estimate the unknown parameters of this distribution and a simulation study is conducted to check the consistency of these estimates. Finally, the proposed model is fitted to two real-life circular data sets and the goodness-of-fit of this distribution is assessed in comparison to wrapped exponential and wrapped Lindley distributions to elucidate the modelling potential of the proposed distribution.

Keywords: Two-parameter Lindley distribution, wrapped distribution, trigonometric moments, simulation, convolution.

1. Introduction

Circular data also known as two-dimensional directional data, corresponds to those occurrences where the observations are recorded in terms of radians or degrees. Circular data, being directions, bear no magnitude, and therefore, are conveniently represented as points on a circle of unit radius, centered at the origin or as a unit vector in the plane, connecting the origin to the corresponding point (Rao and SenGupta 2001). Common instances of record, are evident in various natural and physical sciences such as geology (orientations of cross-beds in rivers, measured in degrees), meteorology (wind direction), biology (vanishing angles of birds soon after their release) (Schmidt-Koenig 1963), etc. Analyzing the behavior of these situations requires a special class of distributions, typically known as circular probability distributions.

The wrapping of linear distributions around a unit circle, generates a rich and useful class of circular models known as wrapped distributions. In this approach, a linear random variable (r.v.) is transformed into a circular random variable (r.v.) by reducing its modulo (Mardia and Jupp 2000). Lévy (1939), first introduced this idea and obtained wrapped variables from the corresponding symmetric as well as non-symmetric distributions on the real line. Many authors, since then, have carried out comprehensive work on wrapped distributions.

A targeted area of interest for the researchers, however, remains modelling occurrences, where the data is rightly-skewed. In this endeavor, Jammalamadaka and Kozubowski (2004) developed the wrapped exponential distribution obtained by wrapping the classical exponential distributions on the real line around the unit circle. Later, Roy and Adnan (2012) incorporated the concept of weights, and established a new class of circular distributions namely wrapped weighted exponential distribution. Recently, Joshi and Jose (2018) projected the wrapped Lindley density by wrapping a Lindley density and exhibited its superiority over the wrapped exponential distribution while modelling skewed situations. Empirical observations have justified the extensive use of Lindley density as a probability model against various other life-time distributions. The two-parameter Lindley density (Shanker et al. 2013), being a generalization of the Lindley and exponential density also offers similar advantage over the other existing distributions. Therefore, in order to investigate its usefulness as a circular model, the wrapped version of the two-parameter Lindley distribution is introduced through the classical wrapping approach and is named as wrapped two-parameter Lindley distribution.

The contents of this research manuscript are as follows: Section 1 presents a review of existing literature in analyzing circular data. Section 2 introduces the density function of the proposed distribution. Section 3 discusses the various distributional properties of the proposed circular model and establishes the commutativity of the operations of wrapping and convoluting linear distributions on the unit circle. Section 4 considers the maximum likelihood estimation of the parameters and Section 5 conducts a simulation study to show the consistency of these estimates. Section 6 displays the modelling potential of the proposed model to two real-life data sets and finally, Section 7 summarizes the findings of the study.

2. Wrapped Two-Parameter Lindley Distribution

In this section, a new circular probability model is proposed, which is obtained by wrapping a Two-parameter Lindley distribution defined on $\Re^{+}$, around the circumference of a circle with unit radius. Alternatively, the distribution is also obtainable by a two-component mixing of the wrapped exponential distribution and the wrapped gamma distribution. A general definition of this distribution is provided which subsequently introduces its density function.

Definition 1 *Let Y be a random variable on the real line with density function $f(y)$. The wrapped circular variable ζ corresponding to Y can be obtained by defining (Rao and SenGupta 2001)*

$$\zeta = Y[\text{mod } 2\pi]. \tag{1}$$

Proposition 1 *Let ζ be a circular random variable following the wrapped two-parameter Lindley probability law with parameters θ and α, denoted by $\zeta \sim WTPLD(\theta,\alpha)$. The probability density function of ζ is then given by*

$$f_w(\zeta) = \left(\frac{\theta^2}{\theta+\alpha}e^{-\theta\zeta}\right)\left\{\frac{1+\alpha\zeta}{1-e^{-2\pi\theta}} + \frac{2\pi\alpha e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right\}, \quad \zeta \in [0,2\pi), \theta > 0, \alpha > -\theta. \tag{2}$$

Proof: Let Y follow a two-parameter Lindley distribution (TPLD) (Shanker et al. 2013). The probability density function (p.d.f.) of Y is

$$f(y)=\frac{\theta^2}{\theta+\alpha}(1+\alpha y)\,e^{-\theta y},\quad y>0,\theta>0,\alpha>-\theta,$$

where θ and α are the shape and scale parameters of the distribution, respectively.

In the wrapping approach, the circular (wrapped) two-parameter Lindley variate ζ is generated using the relation (1) and accordingly the pdf of $\mathrm{WTPLD}(\theta,\alpha)$ is obtained by wrapping $f(y)$ around the circumference of a circle of unit radius as follows (Rao and SenGupta 2001):

$$\begin{aligned}
f_w(\zeta) &= \sum_{k=0}^{\infty} f(\zeta+2k\pi)\\
&= \sum_k \frac{\theta^2}{\theta+\alpha}\left[\{1+\alpha(\zeta+2k\pi)\}e^{-\theta(\zeta+2k\pi)}\right]\\
&= \frac{\theta^2}{\theta+\alpha}e^{-\theta\zeta}\left\{(1+\alpha\zeta)\sum_k e^{-2\pi\theta k}+2\pi\alpha\sum_k k\,e^{-2\pi\theta k}\right\}\\
&= \frac{\theta^2}{\theta+\alpha}e^{-\theta\zeta}\left\{\frac{(1+\alpha\zeta)}{1-e^{-2\pi\theta}}+\frac{2\pi\alpha e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right\},\quad \zeta\in[0,2\pi),\theta>0,\alpha>-\theta.
\end{aligned}$$

The operation of wrapping and mixing commute (Jammalamadaka and Kozubowski 2017). The two-parameter Lindley distribution defined on the real line arises as a mixture of the exponential and the gamma distribution (Shanker et al. 2013). Consequently, the $\mathrm{WTPLD}(\theta,\alpha)$ arises as a mixture of wrapped exponential (θ) and wrapped gamma $(2,\theta)$ with mixing proportion $\frac{\theta}{\theta+\alpha}$ as shown below

$$\begin{aligned}
f_w(\zeta) &= \frac{\theta}{\theta+\alpha}\left(\frac{\theta e^{-\theta\zeta}}{1-e^{-2\pi\theta}}\right)+\frac{1}{\theta+\alpha}\left[\sum_{k=0}^{\infty}\theta^2 e^{-\theta(\zeta+2\pi k)}(\zeta+2\pi k)\right]\\
&= \frac{\theta^2}{\theta+\alpha}e^{-\theta\zeta}\left[\frac{(1+\alpha\zeta)}{1-e^{-2\pi\theta}}+\frac{2\pi\alpha e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right]\quad,\zeta\in\left[0,2\pi\right),\theta>0,\alpha>-\theta.
\end{aligned}$$

Thus, the $\mathrm{WTPLD}(\theta,\alpha)$ random variable Θ admits the representation

$$\Theta \equiv I\,\Theta_1+\left(1-I\right)\Theta_2,$$

where Θ_1 and Θ_2 are independent wrapped exponential (θ) and wrapped gamma $(2,\theta)$ random variables, respectively and I is an indicator random variable which takes on values 1 and 0 with probabilities $\frac{\theta}{\theta+\alpha}$ and $\frac{1}{\theta+\alpha}$, respectively, independently of Θ_1 and Θ_2. Here, $\equiv$ denotes distributional equivalence.

Remarks:

I. For $\alpha=1$, (2) reduces to the density function of the wrapped Lindley distribution with parameter θ (Joshi and Jose 2018).

II. For $\alpha=0$, (2) gives the expression of the density function of the wrapped exponential distribution with parameter θ (Jammalamadaka and Kozubowski 2004).

If $Y\sim\mathrm{TPLD}(\theta,\alpha)$, the cumulative distribution function (c.d.f.) of Y denoted by $F(y)$, is given by

$$F(y)=1-\frac{\theta+\alpha+\alpha\theta y}{\theta+\alpha}e^{-\theta y},y>0,\theta>0,\alpha>-\theta. \tag{3}$$

The c.d.f. of the corresponding wrapped variable is obtained by the relation

$$F_w(\zeta)=\sum_{k=0}^{\infty}\left\{F(\zeta+2\pi k)-F(2\pi k)\right\}. \tag{4}$$

Using (3) in (4), the c.d.f. of ζ, a wrapped two-parameter Lindley variate with parameters θ and α, is obtained as

$$F_w(\zeta)=\frac{1}{1-e^{-2\pi\theta}}\left(1-e^{-\theta\zeta}-\frac{\alpha\theta\zeta}{\theta+\alpha}\right)+\frac{2\pi\alpha\theta}{\theta+\alpha}(1-e^{-\theta\zeta})\left\{\frac{e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right\}, \tag{5}$$

$$\zeta\in[0,2\pi),\theta>0,\alpha>-\theta.$$

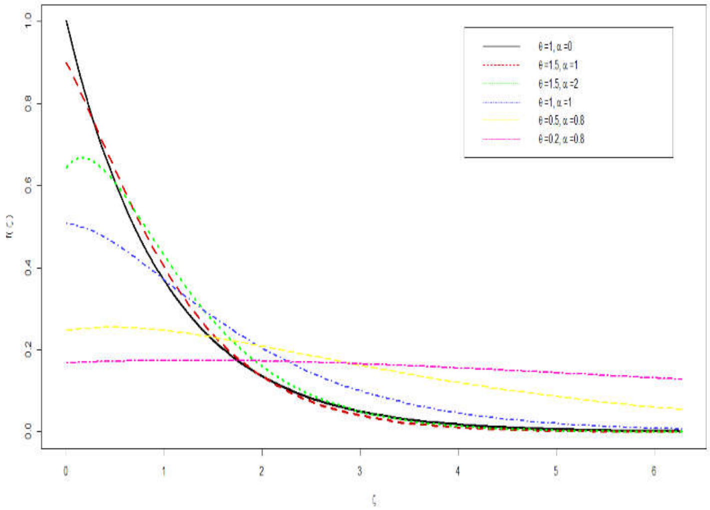

Figure 1 Density plots of the $\text{WTPLD}(\theta,\alpha)$ for different values of θ and α

Figure 1 illustrates the behavior of the density function of wrapped two-parameter Lindley distribution for some selected values of the parameters θ and α. It is found out that the new distribution has a tendency to accommodate right tail and for higher values of θ, the tail approaches to zero at a faster rate. The figure also indicates that a decrease in the value of the parameter α leads to an increase in the steepness of the density curve. Figure 1 further shows that for $\alpha=0$ and $\alpha=1$, the density model coincides with that of wrapped exponential and wrapped Lindley densities, respectively.

It is evident from the density plots of the $\text{WTPLD}(\theta,\alpha)$ that the distribution is most likely to appropriately model those data sets where the observations of lower magnitude appear more frequently than that of the higher magnitude values. Such data sets are quite prevalent in the field of entomology and meteorology.

3. Properties of Wrapped Two-Parameter Lindley Distribution

In this section, the expressions for the characteristic function, trigonometric moments, coefficient of skewness and kurtosis and the median of the $\text{WTPLD}(\theta,\alpha)$ are derived. The operation of wrapping and convoluting linear distribution around a unit circle is investigated and the density of convolution of two wrapped two-parameter Lindley variates is obtained

Proposition 2 *The characteristic function of a wrapped two-parameter Lindley variate* ζ *with parameters* θ *and* α *is given by*

$$\varphi_p = \frac{\theta^2 \left\{(\theta+\alpha)^2 + p^2\right\}^{\frac{1}{2}}}{(\theta+\alpha)(\theta^2+p^2)} \exp\left(2i\arctan\left(\frac{p}{\theta}\right) - i\arctan\left(\frac{p}{\theta+\alpha}\right)\right), \quad p = \pm 1, \pm 2, \ldots \tag{6}$$

Proof: The characteristic function (c.f.) of a wrapped circular variable, say φ_p at an integer value p can be obtained from the c.f. of the corresponding linear random variable, say $\phi_y(t)$ via the following relation (Jammalamadaka and SenGupta 2001)

$$\varphi_p = \phi_y(p).$$

The c.f. of the two-parameter Lindley (θ,α) distribution is given by

$$\phi_y(t) = \frac{\theta^2(\theta+\alpha-it)}{(\theta+\alpha)(\theta-it)^2},\ i = \sqrt{-1}.$$

Therefore, the c.f. of $\text{WTPLD}(\theta,\alpha)$ is obtained as

$$\varphi_p = \frac{\theta^2\left(\theta+\alpha-ip\right)}{\left(\theta+\alpha\right)\left(\theta-ip\right)^2},\ i = \sqrt{-1}\ ;\ p = \pm 1, \pm 2, \ldots \tag{7}$$

Using the result of Roy and Adnan (2012) which gives $\forall a,b,r \in R^+$,

$$(a-ib)^{-r} = (a^2+b^2)^{-\frac{r}{2}} \exp\left(ir\arctan\left(\frac{b}{a}\right)\right).$$

The following expressions are obtained

$$(\theta-ip)^{-2} = (\theta^2+p^2)^{-1} \exp\left(2i\arctan\left(\frac{p}{\theta}\right)\right),$$

$$\theta+\alpha-ip)^{-1} = \left\{(\theta+\alpha)^2+p^2\right\}^{-1/2} \exp\left(i\arctan\left(\frac{p}{\theta+\alpha}\right)\right).$$

In reference to the above expressions, (7) may finally be written as

$$\varphi_p = \frac{\theta^2 \left\{(\theta+\alpha)^2 + p^2\right\}^{1/2}}{(\theta+\alpha)(\theta^2+p^2)} \exp\left(2i\arctan\left(\frac{p}{\theta}\right) - i\arctan\left(\frac{p}{\theta+\alpha}\right)\right)$$

which completes the proof.

Proposition 3 *The* p^{th} *non-central trigonometric moment of* $\zeta \sim WTPLD(\theta,\alpha)$ *are*

$$\alpha_p = \frac{\theta^2 \left\{(\theta+\alpha)^2 + p^2\right\}^{1/2}}{(\theta+\alpha)(\theta^2+p^2)} \cos\left(2\arctan\left(\frac{p}{\theta}\right) - \arctan\left(\frac{p}{\theta+\alpha}\right)\right),$$

$$\beta_p = \frac{\theta^2 \left\{(\theta+\alpha)^2 + p^2\right\}^{1/2}}{(\theta+\alpha)(\theta^2+p^2)} \sin\left(2\arctan\left(\frac{p}{\theta}\right) - \arctan\left(\frac{p}{\theta+\alpha}\right)\right).$$

Proof: An alternative expression for φ_p defined in (6) is as follows (Rao and SenGupta 2001)

$$\varphi_p = \rho_p e^{i\mu_p}, \tag{8}$$

where ρ_p and μ_p denote the concentration measure and mean direction of the p^{th} trigonometric moment, respectively.

Comparing (8) to (6), we may write,

$$\rho_p = \frac{\theta^2\left\{(\theta+\alpha)^2+p^2\right\}^{1/2}}{(\theta+\alpha)(\theta^2+p^2)} \tag{9}$$

and

$$\mu_p = 2\arctan\left(\frac{p}{\theta}\right) - \arctan\left(\frac{p}{\theta+\alpha}\right). \tag{10}$$

Again, from (8), the p^{th} non-central trigonometric moment of ζ can be written as

$$\varphi_p = \alpha_p + i\beta_p, \tag{11}$$

where $\alpha_p = \rho_p\cos(\mu_p)$ and $\beta_p = \rho_p\sin(\mu_p)$. Substituting (9) and (10) in (11), result becomes obvious.

Corollary *The p^{th} central trigonometric moments of $\zeta \sim \text{WTPLD}(\theta,\alpha)$ are obtained using the relations stated below*

$$\begin{aligned}\bar{\alpha}_p &= \rho_p\cos(\mu_p - p\mu) \\ \bar{\beta}_p &= \rho_p\sin(\mu_p - p\mu).\end{aligned} \tag{12}$$

Proposition 4 *The mean resultant length of $\zeta \sim \text{WTPLD}(\theta,\alpha)$ is $\rho = \dfrac{\theta^2\left\{(\theta+\alpha)^2+1\right\}^{1/2}}{(\theta+\alpha)(\theta^2+1)}$ and the mean direction is $\mu = 2\arctan\left(\dfrac{1}{\theta}\right) - \arctan\left(\dfrac{1}{\theta+\alpha}\right)$.*

Proof: Results are obvious by setting $p=1$ in (9) and (10).

Remark: ρ indicates the extent of concentration of ζ towards the mean direction μ and it lies between 0 and 1. The closer it is to 1, the higher is the concentration towards μ.

Proposition 5 T*he circular variance of $\zeta \sim \text{WTPLD}(\theta,\alpha)$ is given by* $1-\dfrac{\theta^2\left\{\theta+\alpha)^2+1\right\}^{1/2}}{(\theta+\alpha)(\theta^2+1)}$.

Proof: The circular variance of $\zeta \sim \text{WTPLD}(\theta,\alpha)$ denoted by V is obtained by the relation

$$V = 1-\rho.$$

Using the results achieved in Proposition 4, the circular variance is given by

$$V = 1-\frac{\theta^2\left\{(\theta+\alpha)^2+1\right\}^{1/2}}{(\theta+\alpha)(\theta^2+1)}. \tag{13}$$

Remark: The interpretation of V is contrary to that of ρ.

3.1. Measure of skewness and kurtosis of wrapped two-parameter Lindley distribution

The measures of skewness and kurtosis, denoted by ξ_1^0 and ξ_2^0, respectively is defined as

$$\xi_1^0 = \frac{\bar{\beta}_2}{V^{3/2}} \quad \text{and} \quad \xi_2^0 = \frac{\bar{\alpha}_2 - \rho^4}{V^2}.$$

The expressions for the above measures are large but however, can be obtained by using the results derived in (9), (12) and (13).

ξ_1^0 is nearly zero for unimodal symmetric data sets and ξ_2^0 is close to zero for the data sets which are single peaked and for which the wrapped normal distribution provides a good fit (Mardia and Jupp, 2000).

3.2. Median direction of wrapped two-parameter Lindley distribution

The median direction of a circular distribution having density $f_w(\cdot)$, denoted by η_0 is the solution of the following equation in the interval $[0, 2\pi)$ (Jammalamadaka and Kozubowski 2004)

$$\int_{\eta_0}^{\eta_0+\pi} f_w(\zeta)\, d\zeta = \frac{1}{2}$$

where f_w is such that, $f_w(\eta_0) > f_w(\eta_0 + \pi)$. Thus, we have

$$\int_{\eta_0}^{\eta_0+\pi} \left(\frac{\theta^2}{\theta+\alpha} e^{-\theta\zeta} \right) \left\{ \frac{1+\alpha\zeta}{1-e^{-2\pi\theta}} + \frac{2\pi e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2} \right\} d\zeta = \frac{1}{2}$$

$$\Rightarrow \frac{\theta}{(\theta+\alpha)(1-e^{-2\pi\theta})} e^{-\theta\eta_0}(1-e^{-\theta\pi})$$
$$+ \frac{\alpha}{(\theta+\alpha)(1-e^{-2\pi\theta})} e^{-\theta\eta_0} \left\{ (1+\theta\eta_0)(1-e^{-\theta\pi}) - e^{-\theta\pi}\theta\pi \right\}$$
$$+ \frac{2\pi\alpha\theta e^{-2\pi\theta}}{(\theta+\alpha)(1-e^{-2\pi\theta})^2} e^{-\theta\eta_0}(1-e^{-\theta\pi}) = \frac{1}{2}$$

η_0 is obtained by solving (14). The values of the various descriptive measures for some particular values of θ and α are listed in Table 1.

Table 1 Values of different characteristic measures of $WTPLD(\theta, \alpha)$

Measure	$\theta = 1.5, \alpha = 1$	$\theta = 1.5, \alpha = 2$	$\theta = 1, \alpha = 1$	$\theta = 1, \alpha = 2$	$\theta = 0.5, \alpha = 0.8$
μ	0.795	0.898	1.107	1.249	1.559
ρ	0.746	0.720	0.559	0.527	0.252
V	0.254	0.280	0.441	0.473	0.748
ξ_1^0	−1.436	−1.242	−0.683	−0.566	−0.166
ξ_2^0	1.754	1.311	0.526	0.347	0.014
η_0	0.675	0.792	1.018	1.182	1.549

Remarks:

I. Mean and median directions are large when α is large and θ is small.

II. Extent of concentration towards the mean direction increases (i.e. becomes closer to 1) as θ increases.

III. Skewness and kurtosis measures, tend to zero with a decrease in the value of θ and an increase in the value of α.

3.3. Wrapping and convolution of linear distributions around a unit circle

Proposition 6 *The circular distribution obtained by wrapping the convolution of two linear distributions around a unit circle coincides with the convolution of their corresponding wrapped distributions.*

Proof: Suppose X and Y be two independently distributed random variables with the density functions $f_X(\cdot)$ and $f_Y(\cdot)$, respectively. Further, let $X \in S \subseteq \Re$. Then, the density function of $Z = X + Y$ is obtained as

$$f_Z(z) = \int_S f_X(t) f_Y(z-t)\,dt.$$

The wrapped distribution corresponding to $f_Z(\cdot)$ is given by

$$\begin{aligned}
f_Z^w(z) &= \sum_{k=-\infty}^{\infty} f_Z(z + 2\pi k) \\
&= \sum_k \int_S f_X(t) f_Y(z + 2\pi k - t)\,dt \\
&= \int_S \sum_k f_Y(z + 2\pi k - t) f_X(t)dt \\
&= \int_S f_Y^w(z-t) f_X(t)dt \\
&= \sum_{j=-\infty}^{\infty} \left[\int_0^{2\pi} f_Y^w\{z - (t + 2j\pi)\} f_X(t + 2j\pi)dt \right] \\
&= \sum_j \int_0^{2\pi} f_Y^w(z-t) f_X(t + 2j\pi)dt \quad (\text{since } f_Y^w(t) = f_Y^w(t + 2j\pi)) \\
&= \int_0^{2\pi} \sum_j f_X(t + 2j\pi) f_Y^w(z-t)dt \\
&= \int_0^{2\pi} f_X^w(t) f_Y^w(z-t)dt,
\end{aligned}$$

which coincides with the convolution of the wrapped densities corresponding to X and Y. By virtue of the Fubini's theorem, since the integral is bounded and the integrands are non-negative, the interchanging of the order of integration and summation is valid. Hence, the proof.

4. Maximum Likelihood Estimates of the Unknown Parameters

This section discusses the estimation of the parameters of WTPLD(θ, α) through the maximum likelihood method.

Let $\zeta_1, , ..., \zeta_n$ be a random sample of size n from the proposed WTPLD(θ, α). Then the log-likelihood function for the vector of parameters $\vartheta = (\theta, \alpha)^T$ of this sample is given by

$$\log L = 2n\log\theta - \theta\sum_{i=1}^{n}\zeta_i - n\log(\theta+\alpha) + \sum_{i=1}^{n}\log\left[\left(\frac{1+\alpha\zeta_i}{1-e^{-2\pi\theta}}\right) + \left\{\frac{2\pi\alpha e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right\}\right]. \quad (15)$$

The maximum likelihood estimates (MLE) of the parameters are computed by solving the maximum likelihood equations

$$\frac{\delta}{\delta\theta}\log L = 0, \quad \frac{\delta}{\delta\alpha}\log L = 0. \quad (16)$$

Since the maximum likelihood equations are non-linear in nature and difficult to deal analytically, they are to be solved by using some suitable numerical technique.

A detailed simulation algorithm for generating samples from (2) and obtaining the MLE for ϑ is enclosed in Section 5.

5. Simulation Study

In this section, a simulation study is conducted to generate data from the proposed distribution and to check for the consistency behavior of its maximum likelihood estimates.

At first, random samples of size 25, 50, 100, 250, 500 and 800 are generated from $\text{WTPLD}(\theta,\alpha)$ for different values of θ and α. Next, MLE of the parameters are obtained by solving the set of equations (16) for each of the respective samples. Finally, the average values of bias and mean squared error (MSE) of these estimates are calculated by the Monte Carlo approximation technique, taking $N = 1{,}000$ replicates. Delineated below, is the algorithm for the desired study:

Step I: Generating a random sample from $\mathbf{WTPLD(\theta,\alpha)}$

Step 1: A random variable is generated from the $U(0,1)$ distribution, say u.

Step 2: The expression of c.d.f. given in (5) is equated with u and is solved for ζ which is a random variable from the $\text{WTPLD}(\theta,\alpha)$.

Steps 1 and 2 are repeated as many numbers of times as the desirable sample size is, to obtain our desired sample from $\text{WTPLD}(\theta,\alpha)$.

The above method is called the inverse transform method for generating ζ.

Step II: Obtaining maximum likelihood estimates of the parameters

The MLE of θ and α is obtained by substituting the values of ζ generated in Step I in (15) and maximizing this with respect to θ and α, respectively.

Step III: Calculating the average Bias and MSE of the maximum likelihood estimates.

Let the true value of the parameter θ be θ^* and the MLE be $\hat{\theta}$. Then the average bias and MSE of $\hat{\theta}$ in estimating θ is given by

$$Bias(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\theta_i - \theta^*), \quad MSE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\theta_i - \theta^*)^2,$$

where N is the number of replications and $\hat{\theta}_i$ is the MLE of θ obtained in the i^{th} replicate.

Similarly, the bias and MSE of the MLE of α are calculated. The MLE is said to be consistent if the bias and MSE decreases (approaches to zero) with an increase in the sample size.

Table 2 shows the average values of the bias and MSE of the MLE of θ and α for different sample sizes and for different set of values of θ and α.

From Table 2 it is seen that the bias and MSE of the MLE of both θ and α approaches towards zero with an increase in the sample size. This shows that the estimates of the parameters are accurate, precise and hence, consistent.

Calculations pertaining to this study is carried out using the R software, version 3.5.0, through the user-contributed packages viz. *CircStats* (Lund and Agostinelli 2018) and *circular* (Lund and Agostinelli 2017) with the help of self-programmed codes. The *maxLik* package (Henningsen and Toomet 2011) is used to obtain the MLE of the parameters and *rootSolve* package (Soetaert 2016) is used to generate random variables from WTPLD(θ, α) by solving the non-linear system of equations.

Table 2 Average values of bias and MSE for θ^* and α^*

$\theta = 1.2$, $\alpha = 0.6$				
n	$Bias(\theta)$	$MSE(\theta)$	$Bias(\alpha)$	$MSE(\alpha)$
25	0.0512	0.4667	0.0782	0.0852
50	0.0418	0.0387	−0.0692	0.0801
100	−0.0325	0.0209	0.0589	0.0752
250	0.0197	0.0127	0.0358	0.0455
500	−0.0109	0.0070	−0.0179	0.0250
800	0.0080	0.0052	0.0146	0.0185
$\theta = 0.5$, $\alpha = 0.7$				
n	$Bias(\theta)$	$MSE(\theta)$	$Bias(\alpha)$	$MSE(\alpha)$
25	0.0399	0.0238	0.0118	0.0103
50	−0.0313	0.0166	0.0097	0.0089
100	0.0288	0.0124	0.0049	0.0067
250	−0.0174	0.0075	0.0029	0.0053
500	−0.0096	0.0041	0.0016	0.0041
800	−0.0071	0.0030	−0.0012	0.0029

6. Data analysis and Results

This section comprises of applying the wrapped two-parameter Lindley distribution to two real-life directional data set having an extended right tail. The fit of this distribution is compared to that of the wrapped Lindley distribution (Joshi and Jose 2018) and wrapped exponential distribution (Jammalamadaka and Kozubowski 2004) with the help of the statistics- log likelihood, AIC (Akaike information criterion) and BIC (Bayesian information criterion). A comparatively smaller value of the AIC and BIC test statistic would construe of a better fit to the data set.

The first data set considered here, is the measurements of long-axis orientation of feldspar laths in basalt, which is procured from Smith (1988, set 24-6-5 co.prn) and published in Fisher (1993), Appendix B5. By drawing a histogram, it is observed that the data depicts a right skew behavior, which may be aptly modelled by the proposed distribution.

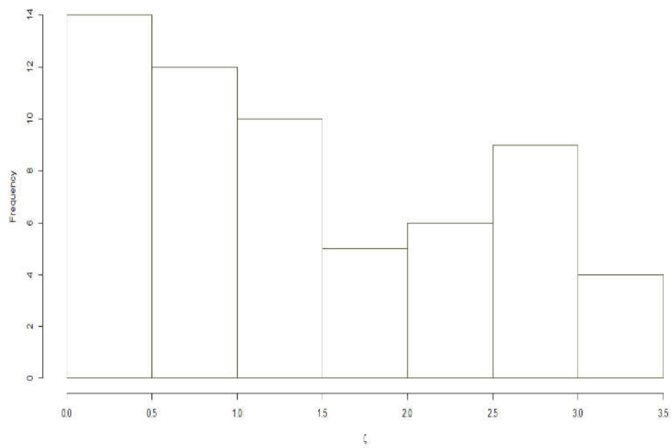

Figure 2 Histogram of the measurements of long-axis orientation of Feldspar laths in basalt

Table 3 summarizes the findings of the fitting of each of the three distributions to the data set. Smallest values of the AIC and BIC statistic for $\text{WTPLD}(\theta,\alpha)$ clearly demonstrate that the proposed distribution provides the best fit to the data under consideration.

Table 3 Values of the statistics and other measures for the $\text{WTPLD}(\theta,\alpha)$, $\text{WLD}(\theta)$ and $\text{WED}(\theta)$ fitted to the data on the measurements of long-axis orientation of feldspar laths in basalt

Distribution	MLE	Log-likelihood	AIC	BIC
$\text{WTPLD}(\theta,\alpha)$	$\theta = 1.1006$, $\alpha = 1.5443$	−78.31766	160.6353	164.8240
$\text{WLD}(\theta)$	$\theta = 1.0309$	−78.82681	160.9836	165.7423
$\text{WED}(\theta)$	$\theta = 0.6640$	−79.71503	161.4301	167.6188

The goodness-of-fit of $\text{WTPLD}(\theta,\alpha)$ is checked using Watson's U^2 one sample test for circular data (Bhattacharjee and Das 2017). The observed value of the test is found to be 0.0917, whereas the critical value at 5% level of significance is given by 0.187. Since, the observed value is less than the tabulated value, it can be concluded that the wrapped two-parameter Lindley distribution is a good fit to the given data set. Figure 3 exhibits the distribution function plot of WTPLD(1.1006,1.5443) fitted to the data.

The second data set is that on wind directions in degrees at Gorleston, England recorded between 11 a.m. and 12 noon on Sundays in 1968, measured during the Summer season (Mardia and Jupp, 2000, Table 7.2, pp.137). The histogram of the data reveals its right skewed behavior, and so, the proposed distribution may be thought to be appropriate in modeling the data.

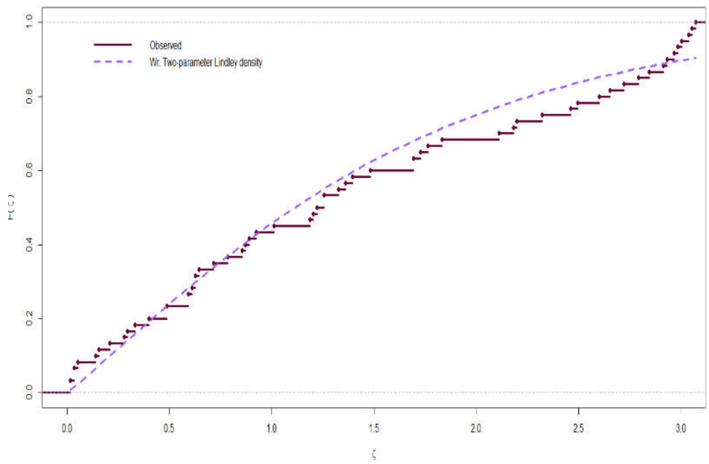

Figure 3 Distribution function plot of WTPLD(1.1006, 1.5443) fitted to the data on measurements of long-axis orientation of feldspar laths in basalt

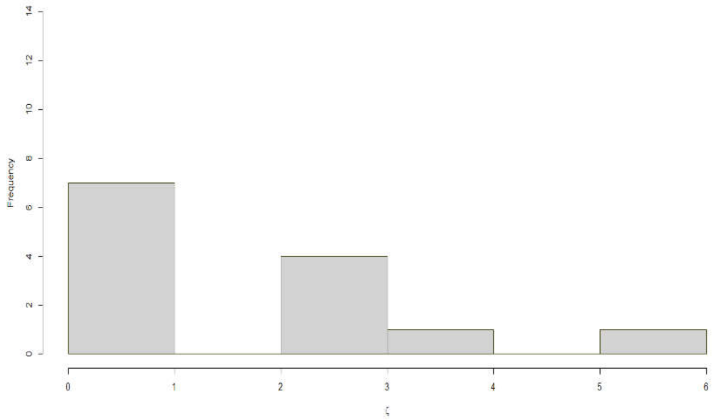

Figure 4 Histogram on wind directions in degrees at Gorleston, England recorded between 11 a.m. and 12 noon on Sundays in 1968.

Table 4 summarizes the findings of the fitting of WTPLD and WLD to the second data set. Smaller values of the AIC and BIC statistic for $\text{WTPLD}(\theta,\alpha)$ clearly demonstrate that the proposed distribution provides the better fit to the data under consideration.

Table 4. Values of the statistics and other measures for the $\text{WTPLD}(\theta,\alpha)$ and $\text{WLD}(\theta)$ fitted to the data on wind directions

Distribution	MLE	Log-likelihood	AIC	BIC
$\text{WTPLD}(\theta,\alpha)$	$\theta = 0.5116$ $\alpha = 2.1630\times 10^{-8}$	-19.43036	42.8607	43.9906
$\text{WLD}(\theta)$	$\theta = 0.8476$	-19.8638	43.1275	44.8574

The goodness-of-fit of $\text{WTPLD}(\theta,\alpha)$ to the data is checked using Watson's U^2 one sample test for circular data. The observed value of the test is found to be 0.1465, whereas the critical value at 5% level of significance is given by 0.267. Since, the observed value is less than the tabulated value, it can be concluded that the wrapped two-parameter Lindley distribution is a good fit to the given data set.

Figure 5 exhibits the distribution function plot of $\text{WTPLD}(0.5116,\ 2.1630\times10^{-8})$ fitted to the data.

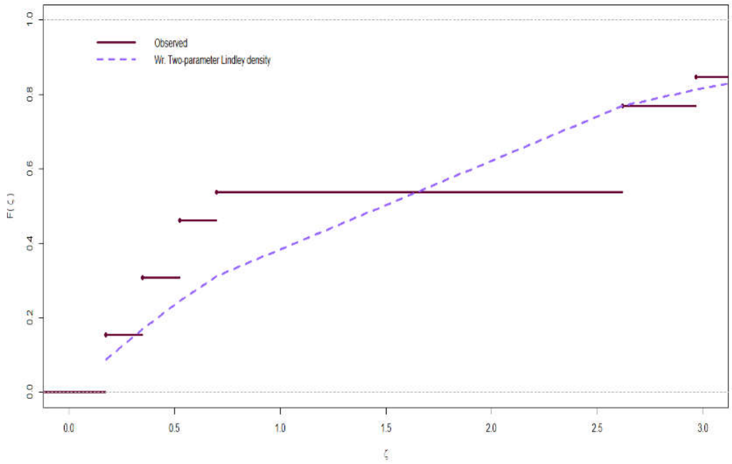

Figure 5 Distribution function plot of $\text{WTPLD}(0.5116,\ 2.1630\times10^{-8})$ fitted to the data on wind directions in degrees at Gorleston, England recorded between 11 a.m. and 12 noon on Sundays in 1968

7. Conclusion

Through this manuscript, a new circular probability model viz. wrapped two-parameter Lindley distribution (WTPLD) is proposed, which is obtained by wrapping a two-parameter Lindley distribution on around a unit circle. Alternatively, the proposed distribution is also shown to be obtainable through a finite mixture of wrapped exponential and wrapped gamma distribution. A few special cases of the WTPLD are showcased and the pictorial behavior of the density function for varying values of the parameters is illustrated. Expressions for the characteristic function, trigonometric moments and other related descriptive measures of this distribution are derived. It is found out that the operations of wrapping and convoluting linear distributions around a unit circle are commutative in nature. Parameter estimation is carried out by the method of maximum likelihood and a simulation study is conducted to check the consistency of these estimates. From the results obtained, MLE are found to be consistent and precise in estimating the true value of the parameters. Finally, the proposed distribution is applied to two real-life directional data sets and the goodness-of-fit of the distribution is assessed and compared to that of the wrapped exponential and wrapped Lindley distribution with the help of the log-likelihood, AIC and BIC measures. Results indicate that WTPLD is a better fit and more flexible as a model as against the others, for modelling the situations where the directions having lower magnitude have higher likelihood of occurrence.

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