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Moments of Dual Generalized Order Statistics from Generalized Inverted Kumaraswamy Distribution and Characterization

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Abstract

In this paper, some simple recurrence relations for single and product moments of dual generalized order statistics from generalized inverted Kumaraswamy distribution have been derived. These relations are deduced for moments of reversed order statistic and lower record values. Further, this distribution has been characterized through the recurrence relations for a single moment, conditional expectations and truncation moment. In addition, some statistical calculations are also carried out.

Keywords: Reversed order statistics, lower record value, single moments, product moments, recurrence relations, conditional expectations, truncation moment.

1. Introduction

Often it happens that the sample is arranged in descending order for example the life length of an electric bulb arranged from highest to lowest. In such situations the distributional properties of variables cannot be studied by using the models of ordered random variables. The study of distributional properties of such random variables is studied by using the inverse image of generalized order statistics (gos) and is popularly known as dual generalized order statistics. The dual generalized order statistics (dgos) was introduced by Burkschat et al. (2003) as a unified model for descendingly ordered random variables like reverse order statistics, lower record values and lower Pfeifer record values.

The dual generalized order statistics (dgos) sometimes called lower generalized order statistics (lgos) is a combined mechanism of studying random variables arranged in descending order. The technique was introduced by Burkschat et al. (2003) and is defined in the following.

Let $F(x)$ be an absolutely continuous distribution function (df) with the probability density function pdf $f(x)$. Further let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called (dgos) if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n), \tag{1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$. Here two cases may be considered:

1.1. Case I: $m_i = m_j = m,\ i, j = 1, 2, \ldots, n-1$

The probability density function of the r^{th} – dgos is given by,

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}[F(x)], \quad -\infty < x < y < \infty. \tag{2}$$

The joint probability density function of the r^{th} and s^{th} – dgos is given by,

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)\, g_m^{r-1} F(x)$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \ -\infty < x < \infty. \tag{3}$$

The conditional pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, the r^{th} and s^{th} $m-$ dgos, $1 \leq r < s \leq n$, is

$$f_{s|r}(y \mid x) = \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}[F(y)]^{\gamma_s - 1}}{(m+1)^{s-r-1}[F(x)]^{\gamma_{r+1}}} f(y), \quad y < x, \tag{4}$$

where $C_{r-1} = \prod_{i=1}^{r}\gamma_i$, $\gamma_i = k + (n-i)(m+1)$, $h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1} & , m \neq -1 \\ -\log x & , m = -1 \end{cases}$

and $g_m(x) = h_m(x) - h_m(1)$, $x \in [0, 1)$.

1.2. Case II: $\gamma_i \neq \gamma_j,\ i \neq j,\ i, j = 1, 2, \ldots, n-1$

The probability density function of the r^{th} – dgos is given by

$$f_{X'(r,n,\tilde{m},k)}(x) = C_{r-1} f(x)\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1}, \ -\infty < x < \infty. \tag{5}$$

The joint probability density function of the r^{th} and s^{th} – dgos is given by,

$$f_{X'(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x, y) = C_{s-1}\sum_{i=1}^{s} a_i^{(r)}(s)\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}, \tag{6}$$

where

$$a_i(r) = \prod_{j(\neq i)=1}^{r}\frac{1}{(\gamma_i - \gamma_j)}, \ \gamma_i \neq \gamma_j, \ 1 \leq i \leq r \leq n. \tag{7}$$

and

$$a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s}\frac{1}{(\gamma_i - \gamma_j)}, \ \gamma_i \neq \gamma_j, \ r+1 \leq i \leq s \leq n. \tag{8}$$

The conditional pdf of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k) = x$, $1 \leq r < s \leq n$, is

$$f_{s|r}(x) = \frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(S)\left[\frac{F(y)}{F(x)}\right]^{\gamma_{i-1}}\frac{f(y)}{F(x)}, \ x > y. \tag{9}$$

If $m=0$, $k=1$, then $X'(r,n,m,k)$ reduces to $(n-r+1)^{\text{th}}$ order statistic $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and when $m \to -1$ then $X'(r,n,m,k)$ reduces to r^{th} lower k record values.

Characterization of probability distributions play an important role in probability and statistics. A probability distribution can be characterized by several methods. In recent years there has been a great interest in the characterization of probability distributions through recurrence relations, conditional expectations and truncation moment.

Several authors have utilized the concept of dual generalized order statistics in characterization of distributions including Ahsanullah (2004, 2005), Mbah and Ahasanullah (2007), Khan et al. (2010a, 2010b), Faizan and Khan (2011), Tavanagar (2011), Khan and Faizan (2012), Khan and Khan (2015), Khan and Khan (2017), Gupta and Anwar (2019) and Khan (2019) among others.

The recurrence relations based on dual generalized order statistics have received considerable attention in recent years. Recurrence relations are interesting in their own right. They are useful in reducing the number of operations necessary to obtain the general form for the function under consideration. Furthermore, they are used in characterizing distributions which in important area permitting the identification of population distribution from the properties of the sample. Many authors derived the recurrence relations for dual generalized order statistics for different distributions. See (Pawlas and Szynal 2001, Khan et al. 2008, Khan and Kumar 2010, 2011a, 2011b) among others.

The results are given in this paper can be used to compute the moments of decreasingly ordered random variable, if parent distribution follows the generalized inverted Kumaraswamy (GIKum) distribution. Since recurrence relations reduce the amount of direct computation and hence reduce the time and labor.

The rest of the paper is organized as follows. Section 2 discusses generalized inverted Kumaraswamy distribution, its some sub-models and statistical properties. Some simple recurrence relations for single and product moments of dual generalized order statistics from generalized inverted Kumaraswamy distribution have been derived in Sections 3 and 4. Characterization results from different techniques are given in Section 5. Conclusions are summarized in Section 6.

2. Generalized Inverted Kumaraswamy Distribution

A number of researchers studied the inverted distributions and its applications for example, Prakash (2012) studied the inverted exponential model and Aljuaid (2013) presented exponentiated inverted Weibull distribution. The inverted distributions are important in problems related to econometrics, engineering sciences, life testing, financial studies and environmental sciences.

Kumaraswamy (1980) obtained a distribution which is derived from beta distribution after fixing some properties in beta distribution. But it has a closed-form cumulative distribution function which is invertible and for which moments do exist. The distribution is appropriate to the natural phenomena whose outcome are bound from both sides, such as the individual's heights, test score, temperatures and hydrological daily data of rain fall (for more details, see Kumaraswamy 1980, Jones 2009 and Sharaf El-Deen et al. 2014).

Abd Al-Fattah et al. (2017) derived the inverted Kumaraswamy (IKum) distribution from Kumaraswamy (Kum) distribution using the transformation $T = x^{-1} - 1$. When $X \sim Kum(\alpha, \beta)$, where α and β are shape parameters, then T has a IKum distribution with probability density function,

$$F(x,\alpha,\beta) = [1-(1+x)^{-\alpha}]^{\beta}, \quad x>0,\ \alpha,\beta>0. \tag{10}$$

Iqbal et al. (2017) generalized the continuous distribution by using power transformation. Here, we use the same technique to find the cdf of generalized inverted Kumaraswamy (GIKum) distribution and is derived by using transformation $T = x^{\gamma}$ which has closed form and is as under

$$F(x) = [1-(1+x^{\gamma})^{-\alpha}]^{\beta}, \quad x > 0,\ \alpha, \beta,\ \gamma > 0. \tag{11}$$

Assuming X is a random variable with shape parameters $\alpha, \beta,\ \gamma$ the probability density function of (GIKum) is as

$$f(x) = \alpha\beta\gamma\, x^{\gamma-1}(1+x^{\gamma})^{-(\alpha+1)}[1-(1+x^{\gamma})^{-\alpha}]^{\beta-1}, \quad x > 0,\ \alpha, \beta,\ \gamma > 0. \tag{12}$$

This model is flexible enough to accommodate both monotone as well as non-monotonic failure rates. Further the probabilistic properties of this distribution and its applications are given, for example, in (Iqbal et al. 2017).

From Equations (11) and (12), we note that the characterizing differential equation for GIKum distribution is given by

$$\alpha\beta\gamma\, F(x,\alpha,\beta,\gamma) = \left[\beta\, x + \sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu} x^{\gamma(u-1)+1}\right] f(x), \tag{13}$$

which will be utilized for deriving the recurrence relation for single and product moments from GIKum distribution. The recurrence relations for moments of dual generalized order statistics from GIKum distribution have not been considered in the earlier literature.

2.1. Some sub-models of generalized inverted Kumaraswamy distribution

The generalized inverted Kumaraswamy distribution is very flexible as this distribution includes several well-known distributions as sub-models based on special values of parameters (α, β, γ). The sub-models are,

(i) Setting $\beta = 1,\ \gamma = 1,$ we obtain Lomax (Pareto-type II) distribution with following pdf

$$f(x) = \frac{\alpha}{(1+x)^{\alpha-1}}, \quad \alpha, x \geq 0.$$

(ii) Setting $\alpha = 1,\ \gamma = 1,$ we obtain inverted beta type II $(\beta, 1)$ distribution with following pdf

$$f(x,\beta) = \frac{1}{B(\beta,1)}\frac{x^{\beta-1}}{(1+x)^{-(\beta+1)}}, \quad \alpha, x \geq 0.$$

(iii) Setting $\alpha = \beta = \gamma = 1,$ we obtain log logistic distribution with following pdf

$$f(x) = \frac{1}{(1+x)^{2}}, \quad x \geq 0.$$

2.2. Statistical computations

The mean and variance of GIKum distribution when $\gamma = 1$ are given by

$$\mu = \beta B\left(1-\frac{1}{\alpha}, \beta\right) - 1,\ \alpha > 1.$$

$$\mu_2 = \beta B\left(1-\frac{2}{\alpha}, \beta\right) - \left[\beta\, B\left(1-\frac{1}{\alpha}, \beta\right)\right]^2,\ \alpha > 2.$$

The mode and quantile function of GIKum distribution when $\gamma = 1$ are given by

$$Mode=\left(\frac{\alpha+1}{\alpha\beta+1}\right)^{-\frac{1}{\alpha}}-1,\ \ \beta\geq 2,$$

$$t_q=\left(1-q^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1,\ \ 0<q<1.$$

We can also use t_q to define well known quantile measures as (skewness and kurtosis). The statistical measures of skewness and kurtosis play important role in describing shape characteristic of the probability distribution. The Bowely's skewness measure based on quartiles (Kenney and Keeping 1962) is given by

$$B=\frac{t_{3/4}+t_{1/4}-2t_{1/2}}{t_{3/4}-t_{1/4}},$$

and Moors' kurtosis measures based on octiles (Moors 1988) is given by

$$M=\frac{t_{7/8}-t_{5/8}+t_{3/8}-t_{1/8}}{t_{6/8}-t_{2/8}}.$$

Table 1 Values of mean for different values of α and β when $\gamma=1$

α	β				
	1	2	3	4	5
2	1.00	1.66	2.18	2.64	3.05
3	0.49	0.78	0.98	1.16	1.30
4	0.33	0.52	0.65	0.76	0.85
5	0.25	0.38	0.47	0.56	0.60

Table 2 Values of variance for different values of α and β when $\gamma=1$

α	β				
	1	2	3	4	5
3	0.81	1.38	1.93	2.37	2.81
4	0.44	0.35	0.46	0.55	0.63
5	0.42	0.18	0.24	0.25	0.34

Table 3 Values of mode for different values of α and β when $\gamma=1$ (Iqbal et al. 2017)

α	β			
	2	3	4	5
1	0.50	1.00	1.50	2.00
2	0.29	0.52	0.73	0.91
3	0.20	0.35	0.48	0.58
4	0.15	0.26	0.35	0.43
5	0.12	0.21	0.28	0.34

Table 4 Values of median for different values of α and β when $\gamma = 1$

α	β				
	1	2	3	4	5
1	1.00	2.33	3.76	5.25	6.69
2	0.41	0.81	1.17	1.50	1.78
3	0.26	0.49	0.69	0.85	1.00
4	0.19	0.35	0.47	0.58	0.67
5	0.14	0.26	0.36	0.44	0.51

Table 5 Values of skewness for different values of α and β when $\gamma = 1$

α	β				
	1	2	3	4	5
1	0.49	0.53	0.51	0.48	0.56
2	0.37	0.42	0.36	0.35	0.41
3	0.31	0.34	0.26	0.27	0.33
4	0.29	0.33	0.28	0.29	0.29
5	0.38	0.35	0.25	0.21	0.25

Table 6 Values of kurtosis for different values of α and β when $\gamma = 1$

α	β				
	1	2	3	4	5
1	2.17	2.31	2.36	2.50	1.57
2	−1.26	1.71	1.69	1.78	1.29

From the above tables, it is shown that when increasing the value of parameter α corresponding value of parameter β is decreasing.

3. Relations for Single Moments

In this section, we derive some simple recurrence relations for single moments of dual generalized order statistics from the GIKum distribution.

Theorem 1 *Let X be a non-negative continuous random variable follows the GIKum distribution given* (11). *Suppose that $j > 0$ and $1 \le r \le n$, then*

$$\left(1+\frac{j}{\alpha\gamma\gamma_r}\right)E[X_d^j(r,n,m,k)] = E[X_d^j(r-1,n,m,k)] - \frac{j}{\alpha\gamma\gamma_r}\left\{\left[\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}\right]E[X_d^{j+\gamma(u-1)}(r,n,m,k)]\right\} \tag{14}$$

Proof: We have from (2),

$$E[X_d^j(r,n,m,k)] = \frac{c_{r-1}}{(r-1)!}\int_0^\infty x^j [F(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x)dx. \tag{15}$$

Integrating by parts $[F(x)]^{\gamma_r - 1}$ as part to be integrated, we get

$$E[X_d^j(r,n,m,k)] = -\frac{jc_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)]f(x)dx$$
$$+\frac{jc_{r-2}}{(r-1)!\gamma_r}\int_0^\infty x^{j}[F(x)]^{\gamma_{r-1}-1} g_m^{r-2}[F(x)]f(x)dx,$$

which implies that,

$$E[X_d^j(r,n,m,k)] - E[X_d^j(r-1,n,m,k)] = -\frac{jc_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx\ .$$

Now in view of (13), we have,

$$E[X_d^j(r,n,m,k)] - E[X_d^j(r-1,n,m,k)] =$$
$$-\frac{jc_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)]\left\{\frac{1}{\alpha\beta\gamma}\left(\beta x+\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}x^{\gamma(u-1)+1}f(x)\right)\right\}dx$$

after simplification,

$$\left(1+\frac{j}{\alpha\gamma\gamma_r}\right)E[X_d^j(r,n,m,k)] = E[X_d^j(r-1,n,m,k)] - \frac{j}{\alpha\beta\gamma\gamma_r}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^{j+\gamma(u-1)}(r,n,m,k)]\right\}.$$

This completes the proof of Theorem 1.

Corollary 1 *For* $2\le r\le n,\ n\ge 2$ *and* $k=1,2,...,$

$$\left(1+\frac{j}{\alpha\gamma\gamma_1}\right)E[X_d^j(r,n,m,k)] = E[X_d^j(r-1,n-1,m,k)] - \frac{j}{\alpha\beta\gamma\gamma_1}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^{j+\gamma(u-1)}(r,n,m,k)]\right\}$$

if and only if (11) holds.

Proof: From Khan et al. (2008),

$$E[X_d^j(r,n,m,k)] - E[X_d^j(r-1,n-1,m,k)] = -\frac{jc_{r-1}}{(r-1)!\gamma_1}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx.$$

Now in view of (13), corollary 1 is proved.

Remark 1 Setting $\gamma=1$ in (14) result reduced for inverted Kumaraswamy distribution as follows,

$$\left(1+\frac{j}{\alpha\gamma_r}\right)E[X_d^j(r,n,m,k)] = E[X_d^j(r-1,n,m,k)] - \frac{j}{\alpha\beta\gamma_r}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^{j+\gamma(u-1)}(r,n,m,k)]\right\}.$$

Remark 2 Setting $m=0,\ \ k=1$ in (14) result reduced for reversed order statistic for GIKum distribution as follows

$$\left(1+\frac{j}{\alpha\gamma(n-r+1)}\right)E[X'^{j}_{r:n}] = E[X'^{j}_{r-1:n}] - \frac{j}{\alpha\beta(n-r+1)}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_{r:n}^{j+(u-1)}]\right\}.$$

Remark 3 Setting $m=-1,\ \ k\ge 1$ in (14) result reduced for k^{th} lower record values for GIKum distribution as follows

$$\left(1+\frac{j}{\alpha\gamma\, k}\right)E[X'^{j}_{L(r)}] = E[X'^{j}_{L(r-1)}] - \frac{j}{\alpha\beta\gamma\, k}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_{L(r)}^{j+(u-1)}]\right\}.$$

4. Relations for Product Moments

In this section, we derive some simple recurrence relations for product moments of dual generalized order statistics from the GIKum distribution.

Theorem 2 *Let X be a non-negative continuous random variable follows the GIKum distribution given* (11). *Suppose that $i, j > 0$ and $1 \le r < s \le n$, then,*

$$\left(1+\frac{j}{\alpha\gamma\gamma_s}\right)E[X_d^i(r,n,m,k), X_d^j(s,n,m,k)] = E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)$$
$$-\frac{j}{\alpha\beta\gamma\gamma_s}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^i(r,n,m,k), X_d^{j+\gamma(u-1)}(s,n,m,k)]\right\}. \tag{16}$$

Proof: From (3), we have

$$E[X_d^i(r,n,m,k), X_d^j(s,n,m,k)] = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i [F(x)]^m g_m^{r-1}[F(x)] f(x) I(x)dx, \tag{17}$$

where $I(x) = \int_0^\infty y^j [F(y)]^{\gamma_s - 1}[h_m F(y) - h_m F(x)]^{s-r-1} f(y)dy$. Solving the integral $I(x)$ by parts and substituting the resulting expression in (17), we get

$$E[X_d^i(r,n,m,k), X_d^j(s,n,m,k)] - E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k) =$$
$$-\frac{jc_{s-1}}{(r-1)!(s-r-1)!\gamma_s}\int_0^\infty\int_0^x x^i y^{j-1}[F(x)]^m g_m^{r-1}[F(x)][h_m F(y) - h_m F(x)]^{s-r-1}[F(x)]^{\gamma_s}$$
$$\times\left[\frac{1}{\alpha\beta\gamma}\left\{\beta y + \sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}y^{\gamma(u-1)+1}\right\}\right] f(x)f(y)dx,$$

after simplification (16) yields. i.e.,

$$\left(1+\frac{j}{\alpha\gamma\gamma_s}\right)E[X_d^i(r,n,m,k), X_d^j(s,n,m,k)] = E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)$$
$$-\frac{j}{\alpha\beta\gamma\gamma_s}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^i(r,n,m,k), X_d^{j+\gamma(u-1)}(s,n,m,k)]\right\}.$$

This completes the proof of Theorem 2.

Remark 4 Setting $\gamma = 1$ in (16) result reduced for inverted Kumaraswamy distribution as follows,

$$\left(1+\frac{j}{\alpha\gamma_s}\right)E[X_d^i(r,n,m,k), X_d^j(s,n,m,k)] = E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)$$
$$-\frac{j}{\alpha\beta\gamma_s}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_d^i(r,n,m,k), X_d^{j+\gamma(u-1)}(s,n,m,k)]\right\}.$$

Remark 5 Setting $m = 0$, $k = 1$ in (16) result reduced for reversed order statistic for GIKum distribution as follows,

$$\left(1+\frac{j}{\alpha\gamma(n-s+1)}\right)E[X'^{i,j}_{r,s:n}] = E[X'^{i,j}_{r,s-1:n}] - \frac{j}{\alpha\beta\gamma(n-s+1)}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X'^{i,j+\gamma(u-1)}_{r,s:n}]\right\}.$$

Remark 6 Setting $m=-1,\ \ k\geq 1$ in (16) result reduced for k^{th} lower record values for GIKum distribution as follows

$$\left(1+\frac{j}{\alpha\gamma\, k}\right)E[X_{L(r,s)}^{i,j}]=E[X_{L(r,s-1)}^{i,j}]-\frac{j}{\alpha\beta\,\gamma\, k}\left\{\sum_{\mu=2}^{\lambda+1}\binom{\lambda+1}{\mu}E[X_{L(r,s)}^{i,j+(u-1)}]\right\}.$$

5. Characterizations

This section discusses the characterization results of GIKum distribution. Characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various method. Various characterizations of distributions have been established in many different directions. In recent years, there has been a great interest in the characterizations of probability distributions through recurrence relations, conditional expectations and truncation moment.

The following theorem contains, the characterization of GIKum distribution by a recurrence relation for the single moments of dual generalized order statistics.

Theorem 3 *The necessary and sufficient condition for a random variable* X *to be distributed with pdf given by* (12) *is that*

$$E[X_d^j(r,n,m,k)]-E[X_d^j(r-1,n,m,k)]$$
$$=-\frac{j}{\alpha\gamma\gamma_r}\left\{E[X_d^j(r,n,m,k)]+\frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}E[X_d^{j+(u-1)}(r,n,m,k)]\right\},\tag{18}$$

if and only if

$$F(x)=[1-(1+x^{\gamma})^{-\alpha}]^{\beta},\ \ x>0,\ \alpha,\beta,\gamma>0.\tag{19}$$

Proof: The necessary part follows immediately from (14). On the other hand, if the recurrence relation (18) is satisfied, then on rearranging the terms in (18),

$$\frac{c_{r-1}}{(r-1)!}\int_0^{\infty}x^j[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx-\frac{c_{r-1}(r-1)}{(r-1)!\gamma_r}\int_0^{\infty}x^j[F(x)]^{\gamma_r+m}g_m^{r-2}[F(x)]f(x)dx$$
$$=-\frac{j}{\gamma\gamma_r}\left\{\int_0^{\infty}x^j[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx\right.$$
$$\left.+\frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}\int_0^{\infty}x^{j+\gamma(u-1)}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx\right\}$$
$$\frac{c_{r-1}}{(r-1)!}\int_0^{\infty}x^j[F(x)]^{\gamma_r-1}g_m^{r-2}[F(x)]f(x)\left[\frac{g_m[F(x)]}{[F(x)]}-\frac{(r-1)[F(x)]^m}{\gamma_r}\right]dx$$
$$=-\frac{j}{\gamma\gamma_r}\frac{c_{r-1}}{(r-1)!}\left\{\int_0^{\infty}x^j[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx\right.$$
$$\left.+\frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}\int_0^{\infty}x^{j+\gamma(u-1)}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx\right\}.\tag{20}$$

Let

$$h(x)=\frac{[F(x)]^{\gamma_r}\,{g^{r-1}}_m[F(x)]}{\gamma_r}.\tag{21}$$

Differentiating both sides of (21), we get

$$h'(x) = -[F(x)]^{\gamma_r} g_m^{r-2}[F(x)] f(x)\left[\frac{g_m[F(x)]}{[F(x)]} - \frac{(r-1)[F(x)]^m}{\gamma_r}\right].$$

Thus,

$$\frac{c_{r-1}}{(r-1)!}\int_0^\infty x^j h'(x)dx = \frac{j}{\gamma\gamma_r}\frac{c_{r-1}}{(r-1)!}\left\{\int_0^\infty x^j [F(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx\right.$$
$$\left.+\frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}\int_0^\infty x^{j+\gamma(u-1)}[F(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx\right\}. \tag{22}$$

Integrating left hand side in (22) by parts and using the value of $h(x)$ from (21),

$$\frac{c_{r-1}}{(r-1)!}\int_0^\infty jx^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx - \frac{c_{r-1}(r-1)}{(r-1)!\gamma_r}\int_0^\infty x^j [F(x)]^{\gamma_r+m} g_m^{r-2}[F(x)] f(x)dx$$
$$-\left\{\frac{j}{\gamma\gamma_r}\frac{c_{r-1}}{(r-1)!}\left\{\int_0^\infty x^j [F(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx\right.\right.$$
$$\left.\left.+\frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}\int_0^\infty x^{j+\gamma(u-1)}[F(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx\right\}\right\},$$

which reduces to,

$$\frac{c_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}[F(x)] f(x)\left[\frac{F(x)}{f(x)} - \frac{1}{\alpha\gamma}\left(x + \frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}x^{\gamma(u-1)+1}\right)\right]dx = 0. \tag{23}$$

Now applying the generalization of the Müntz-Szász theorem (see for example Hwang and Lin 1984) to (23), we get

$$F(x,\alpha,\beta,\gamma) = \frac{1}{\alpha\beta\gamma}\left(\beta x + \frac{1}{\beta}\sum_{u=2}^{\lambda+1}\binom{\lambda+1}{u}x^{\gamma(u-1)+1}\right)f(x),$$

which proves that $f(x)$ has the form as in (12) i.e.,

$$F(x) = [1-(1+x^\gamma)^{-\alpha}]^\beta,\ \ x>0,\ \alpha,\beta,\gamma>0.$$

Following theorem deals with the characterization of GIKum distribution through conditional expectations.

Theorem 4 *Let* X *be a non-negative random variable having an absolutely continuous df* $F(x)$ *with* $F(0)=0$ *and* $0 \le F(x) \le 1$ *for all* $x>0$, *then,*

$$E[\xi\{X'(s,n,m,k)\} \mid X'(l,n,m,k)=x] = [1-(1+x^\gamma)^{-\alpha}]^\beta \prod_{j=1}^{s-l}\left(\frac{\gamma_{l+j}}{\gamma_{l+j}+1}\right),\ \ l=r,\ \ r+1,\ \ m\ne -1, \tag{24}$$

if and only if

$$F(x) = [1-(1+x^\gamma)^{-\alpha}]^\beta,\ x>0,\ \alpha,\beta,\gamma>0, \tag{25}$$

where $\xi(y) = [1-(1+y^\gamma)^{-\alpha}]^\beta$.

Proof: From (4) for $s > r+1$, we have

$$E[\xi\{X'(s,n,m,k)\}\,|\,X'(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_0^x [1-(1+y^{\gamma})^{-\alpha}]^{\beta}\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \quad (26)$$

Setting $u=\dfrac{F(y)}{F(x)}=\dfrac{[1-(1+y^{\gamma})^{-\alpha}]^{\beta}}{[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}$ in (26), we have

$$E[\xi\{X'(s,n,m,k)\}\,|\,X'(r,n,m,k)=x]=\frac{C_{s-1}[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\times\int_0^1 u^{\gamma_s}(1-u^{m+1})^{s-r-1}du \quad (27)$$

Again by setting $t=u^{m+1}$ in (27), we get

$$E[\xi\{X(s,n,m,k)\}\,|\,X(r,n,m,k)=x]=\frac{C_{s-1}[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\times\int_0^1 t^{\frac{\gamma_s+1}{m+1}-1}(1-t)^{s-r-1}dt$$

$$=\frac{C_{s-1}[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}{C_{r-1}(s-r-1)!(m+1)^{s-r}}\frac{\Gamma\left(\frac{k+1}{m+1}+n-s\right)}{\Gamma\left(\frac{k+1}{m+1}+n-r\right)}$$

$$=\frac{C_{s-1}[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}{C_{r-1}}\frac{1}{\prod_{j=1}^{s-r}(\gamma_{r+j}+1)},$$

where $\dfrac{C_{s-1}}{C_{r-1}}=\prod_{j=1}^{s-r}\gamma_{r+j}$, and hence the necessary part is proved. To prove the sufficiency part, we have from (4) and (24)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^x [1-(1+y^{\gamma})^{-\alpha}]^{\beta}[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy$$

$$=g_{s|r}(x)[F(x)]^{\gamma_{r+1}}, \quad (28)$$

where $g_{s|r}(x)=[1-(1+x^{\gamma})^{-\alpha}]^{\beta}\prod_{j=1}^{s-r}\left(\dfrac{\gamma_{l+j}}{\gamma_{l+j}+1}\right)$. Differentiating (28) both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_0^x [1-(1+y^{\gamma})^{-\alpha}]^{\beta}[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-2}[F(y)]^{\gamma_s-1}f(y)dy$$

$$=g'_{s|r}[F(x)]^{\gamma_{r+1}}+\gamma_{r+1}g_{s|r}(x)[F(x)]^{\gamma_{r+1}-1}f(x),$$

where

$$g'_{s|r}=\alpha\beta\gamma x^{\gamma-1}(1+x^{\gamma})^{-(\alpha+1)}[1-(1+x^{\gamma})^{-\alpha}]^{\beta-1}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right)$$

$$g_{s|r+1}-g_{s|r}=[1-(1+x^{\gamma})^{-\alpha}]^{\beta}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right).$$

Therefore,

$$\frac{f(x)}{F(x)}=\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x)-g_{s|r}(x)]}=\frac{\alpha\beta\gamma x^{\gamma-1}(1+x^{\gamma})^{-(\alpha+1)}}{[1-[(1+x^{\gamma})]^{-\alpha}]}, \text{ (Khan et al. 2010a).} \quad (29)$$

Integrating (29) on both the sides with respect to $x,$ the sufficiency part is proved. Following theorem presents the characterization result for of GIKum distribution based on truncated moment.

Theorem 5 *Suppose an absolutely continuous (with respect to Lebesgue measure) random variable* X *has the df* $F(x)$ *and pdf* $f(x)$ *for* $0<x<\infty,$ *such that* $f'(x)$ *and* $E(X\mid X\leq x)$ *exist for all* $x,$ $0<x<\infty,$ *then*

$$E(X\mid X\leq x)=g(x)\eta(x), \tag{30}$$

where

$$g(x)=\left\{\frac{(1+x)^{(\alpha+1)}[1-(1+x^{\alpha})]^{-\alpha}}{\lambda\beta\gamma x^{\gamma-1}}\left[-x+\frac{\int_0^x[1-(1+u^{\gamma})^{-\alpha}]^{\beta}du}{[1-(1+x^{\gamma})^{-\alpha}]^{\beta}}\right]\right\} \text{ and } \eta(x)=\frac{f(x)}{F(x)},$$

if and only if (12) *holds.*

Proof: We have,

$$E(X\mid X\leq x)=\frac{1}{F(x)}\int_0^x u\,f(u)\,du$$

$$=\frac{\alpha\beta\gamma}{F(x)}\int_0^x u\,u^{\gamma-1}(1+u^{\gamma})^{-(\alpha-1)}[1-(1+u^{\gamma})^{-\alpha}]^{\beta-1}du. \tag{31}$$

Integrating (31) by parts treating $u^{\gamma-1}(1+u^{\gamma})^{-(\alpha-1)}[1-(1+u^{\gamma})^{-\alpha}]^{\beta-1}$ for integration and rest for the integrand for differentiation, we get

$$E(X\mid X\leq x)=\frac{1}{F(x)}\{-x[1-(1+x^{\gamma})^{-\alpha}]^{\beta}+\int_0^x[1-(1+u^{\gamma})^{-\alpha}]^{\beta}du\}. \tag{32}$$

After multiplying and dividing by $f(x)$ in (32), we have the result given in (30). To prove sufficient part, we have from (30)

$$\frac{1}{F(x)}\int_0^x u\,f(u)\,du=\frac{g(x)f(x)}{F(x)}$$

or

$$\int_0^x u\,f(u)\,du=g(x)f(x). \tag{33}$$

Differentiating (33) on both the sides with respect to $x,$ we find that

$$x\,f(x)=g'(x)f(x)+g(x)f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)} \text{ (Ahsanullah et al. 2016).}$$

$$g'(x)=x+g(x)\left[\frac{f'(x)}{f(x)}\right]. \tag{34}$$

Integrating both sides in (34), with respect to $x,$ we get

$$f(x)=cx^{\gamma-1}(1+x^{\gamma})^{-(\alpha+1)}[1-(1+x^{\gamma})^{-\alpha}]^{\beta-1}, \tag{35}$$

where c is determined such that,

$$\int_{-\infty}^{\infty}f(x)\,dx=1$$

$$\int_0^\infty cx^{\gamma-1}(1+x^\gamma)^{-(\alpha+1)}[1-(1+x^\gamma)^{-\alpha}]^{\beta-1}dx=1$$

$$\frac{1}{c}=\int_0^\infty x^{\gamma-1}(1+x^\gamma)^{-(\alpha+1)}[1-(1+x^\gamma)^{-\alpha}]^{\beta-1}dx$$

$$\frac{1}{c}=\frac{1}{\alpha\beta\gamma}.$$

This proves that,

$$f(x)=\alpha\beta\gamma\, x^{\gamma-1}(1+x^\gamma)^{-(\alpha+1)}[1-(1+x^\gamma)^{-\alpha}]^{\beta-1},\ \ x>0,\ \alpha,\beta,\ \gamma>0.$$

Remark 7 Putting $\gamma=1$ in Theorem (5), we get the characterization result for inverted Kumaraswamy distribution based on truncation moment.

6. Conclusions

Characterization of probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various method. These characterization results are useful in the field of ordered random variables. Findings of this paper will be useful for researchers in the fields of econometrics, engineering sciences, life testing, financial studies and environmental sciences.

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