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Properties and Applications of a New Member of the T-X Family of Distributions

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Abstract

In this paper, we introduce a new family of distributions by the use of the so-called T-X transformation defined with the Weibull distribution as generator and a sophisticated transformation involving logarithmic and power functions. We motivate the interest of this family in the fields of probability and statistics by complete theoretical and practical studies. Our theoretical investigations show how the family can be handle analytically, with formula for moments, moment generating function and order statistics. Then, the applied side is considered with a special focus on the member based on the so-called Lomax distribution. The motivation behind this member is to define a new five parameter lifetime distribution having a very flexible probability density function. We apply the related model by the means of two well-known real-life data sets, showing that the new distribution fits better than seven recent competitors.

Keywords: Weibull-X family, Akaike information criterion, Anderson-Darling test, Cramer von-Mises test.

1. Introduction

The recent literature has suggested several ways of extending well-known distributions. A common way consists in defining new classes of univariate continuous distributions by introducing additional shapes parameter(s) to a baseline distribution. The role of this additional parameter(s) is useful in exploring tail properties and also for improving the goodness-of-fit of the generator family. The well-known families are: the beta-G family (B-G) by Eugene et al. (2002) and Jones (2004), Kumaraswamy-G family (Kw-G) by Cordeiro and de Castro (2011), McDonald-G family (Mc-G) by Alexander et al. (2012), gamma-X family by Alzaatreh et al. (2014), gamma-G family (type I) by Zografos and Balakrishanan (2009), gamma-G family (type II) by Ristic and Balakrishanan (2012), gamma-G family (type III) by Torabi and Montazari (2012), log-gamma-G family by Amini et al. (2014), logistic-G family by Torabi and Montazari (2014), transformed-transformer family (T-X) by Alzaatreh et al. (2013), Exponentiated T-X family by Alzaghal et al. (2013), Logistic-X family by

Tahir et al. (2016), new Weibull-X family by Ahmad et al. (2018) and some new member of T-X family by Jamal and Nasir (2019) among others.

Let $\nu(t)$ be the probability density function (pdf) of a random variable, say T, where $T \in [m,n]$ for $-\infty \le m < n < \infty$. Let $\mathrm{W}[\mathrm{F}(x;\xi)]$ be a function of cumulative distribution function (cdf) $\mathrm{F}(x;\xi)$ of a random variable, say X, depending on the vector parameter ξ so that $\mathrm{W}[\mathrm{F}(x;\xi)]$ satisfying the conditions given below:

$$\begin{cases} W[F(x;\xi)] \in [m,n], \\ W[F(x;\xi)] \text{ is differentiable and monotonically increasing,} \\ W[F(x;\xi)] \to m \text{ as } x \to -\infty \text{ and } W[F(x;\xi)] \to n \text{ as } x \to \infty. \end{cases}$$

Alzaatreh et al. (2013), defined the cdf of the T-X family of distributions by

$$G(x;\xi) = \int_{m}^{W[F(x;\xi)]} \nu(t)\,dt,\ x \in R, \tag{1}$$

where $W[F(x;\xi)]$ satisfies the conditions stated above. The pdf corresponding to (1) is given by

$$g(x;\xi) = \left\{ \frac{\partial}{\partial x} W[F(x;\xi)] \right\} \nu\{W[F(x;\xi)]\},\ x \in R.$$

Using the T-X family idea, several new classes of distributions have been introduced in the literature. Table 1 and Table 2 provide some $W[F(x;\xi)]$ and $W[1-F(x;\xi)]$ functions for some members of the T-X family.

Table 1 Different $W[F(x;\xi)]$ functions for special members of the T-X family

S.No.	$W[F(x;\xi)]$	Range of X	Members of T-X family
1	$F(x;\xi)$	$[0,1]$	Beta-G,Eugene et al. (2002), Mc-G, Alexander et al. (2012)
2	$-\log[1-F(x;\xi)]$	$[0,\infty]$	Gamma-G Type-1, Zografos and Balakrishnan (2009)
3	$\frac{F(x;\xi)}{1-F(x;\xi)}$	$[0,\infty]$	Gamma-G Type-3, Torabi and Montazeri (2012)
4	$-\log[1-F^{a}(x;\xi)]$	$[0,\infty]$	Exponentiated T-X, Alzaghal et al. (2013)
5	$\log\left[\frac{F(x;\xi)}{1-F(x;\xi)}\right]$	$[-\infty,\infty]$	Logistic-G, Torabi and Montazeri (2014)
6	$\log\left[-\log\left[1-F^{a}(x;\xi)\right]\right]$	$[-\infty,\infty]$	The Logistic-X Family, Tahir et al. (2016)
7	$\frac{F^{a}(x;\xi)}{1-F^{a}(x;\xi)}$	$[0,\infty]$	The generalized odd log-logistic family, Cordeiro et al. (2017)
8	$\frac{-\log[1-F(x;\xi)]}{1-F(x;\xi)}$	$[0,\infty]$	New Weibull-X Family, Ahmad et al. (2018)
9	$\frac{-\log[1-F^{a}(x;\xi)]}{1-F^{a}(x;\xi)}$	$[0,\infty]$	Jamal Weibull-X Family, Jamal and Nasir (2019)

Table 1 (Continued)

S.No.	$W[F(x;\xi)]$	Range of X	Members of T-X family
10	$\dfrac{\log[-\log\{1-F^{a}(\mathrm{x};\xi)\}]}{1-F^{a}(\mathrm{x};\xi)}$	$[-\infty,\infty]$	Nasir Logistic-X Family, Jamal and Nasir (2019)
11	$\log\left[\dfrac{F^{a}(\mathrm{x};\xi)}{1-F^{a}(\mathrm{x};\xi)}\right]$	$[-\infty,\infty]$	Jamal Logistic-X Family, Jamal and Nasir(2019)

Table 2 Different $W[1-F(x;\xi)]$ functions for special members of survival family

S.No.	$W[F(x;\xi)]$	Range of X	Members of T-X family
1	$-\log\left[F(x;\xi)\right]$	$[0,\infty]$	Gamma-G Type-2, Ristic and Balakrishnan (2012)
2	$\dfrac{-\log\left[F(x;\xi)\right]}{F(x;\xi)}$	$[0,\infty]$	Nasir Weibull-Generalized Family, Jamal and Nasir (2019)

2. A Generalized Families Based on a New Member of the T-X Family

2.1. Jamal Weibull-X (JW-X) family

If X is a Weibull random variable with parameter $\alpha,\beta>0,$ then from Table 1, S.No. 9 (called new exponentiated X family, in short NEX) and form (1), (By taking $W[F(x;\xi)]=\dfrac{-\log[1-F^{a}(x;\xi)]}{1-F^{a}(x;\xi)}$) the new proposed cdf and pdf of the Jamal Weibull-X (JW-X(α,β,a)) family is given by

$$G(x;\alpha,\beta,a,\xi) \;=\; \int_{0}^{\frac{-\log[1-F^{a}(x;\xi)]}{1-F^{a}(x;\xi)}} v(t)\,dt = 1-\exp\left[-\alpha\left\{\frac{-\log[1-F^{a}(x;\xi)]}{1-F^{a}(x;\xi)}\right\}^{\beta}\right] \tag{2}$$

and

$$g\,(x\,;\alpha,\beta,a,\xi=\alpha\beta\,a\;\frac{f(x;\xi)\,F^{a-1}(x;\xi)\,[\log\{1-F^{a}(x;\xi)\}]^{\beta}\,[1-[\log\{1-F^{a}(x;\xi)\}]^{-1}]}{[1\text{-}F^{a}(x;\xi)]^{\beta+1}}$$
$$\times\exp\left[-\alpha\left\{\frac{-\log[1-F^{a}(x;\xi)]}{1-F^{a}(x;\xi)}\right\}^{\beta}\right]. \tag{3}$$

2.2. Special model

By taking Lomax distribution as a baseline distribution with pdf and cdf respectively, $f(x;\theta,\sigma)=\dfrac{\sigma}{\theta}\left(1+\dfrac{x}{\theta}\right)^{-\sigma-1}$ and $F(x;\theta,\sigma)=1-\left(1+\dfrac{x}{\theta}\right)^{-\sigma}$, where $\theta,\sigma\geq 0.$ From (2) and (3), the new cdf and pdf of Jamal Weibull-Lomax (JW-Lx) distribution are as follow

$$G(x;\alpha,\beta,a,\theta,\sigma)=1-\exp\left[-\left\{\frac{-\log\left[1-\{1-(1+\frac{x}{\theta})^{-\sigma}\}^{a}\right]}{1-\left\{1-(1+\frac{x}{\theta})^{-\sigma}\right\}^{a}}\right\}^{\beta}\right],$$

and

$$g(x;\alpha,\beta,a,\theta,\sigma)=\frac{\sigma\alpha\beta a}{\theta(1+\frac{x}{\theta})^{\sigma+1}}\frac{\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a-1}\left[\log\left[1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a}\right]\right]^{\beta}\left[1-\left[\log\left[1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a}\right]\right]\right]^{-1}}{\left[1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a}\right]^{\beta+1}}$$

$$\times\exp\left[-\alpha\left\{\frac{-\log\left[1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a}\right]}{1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\sigma}\right\}^{a}}\right\}^{\beta}\right].$$

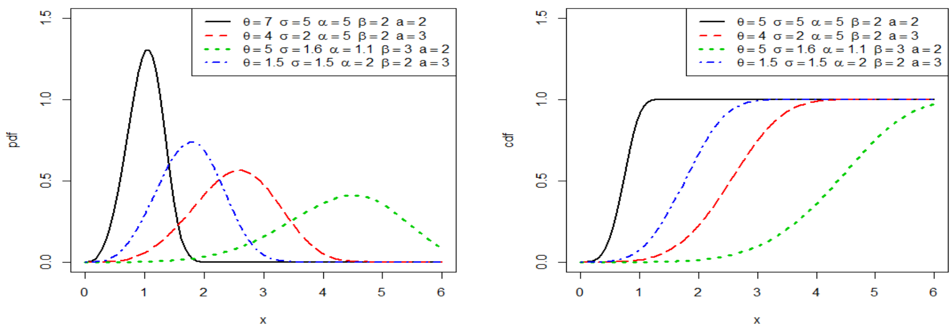


Figure 1 Plots of pdf and cdf for JW-Lx distribution for different values of the parameters

3. Mathematical and Statistical Properties of JW-X Family

Here we express the cdf and pdf of the JW-X(α,β,a) as infinite linear mixture of the corresponding functions of NEX(α) distribution.

3.1. Expansion of the cdf and pdf of JW-X family

By using Taylor series expansion cdf of (2), we can write

$$G^{JWX}(x;\alpha,\beta,a)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\alpha^{k}}{k!}[F^{NEX}(x;a)]^{\beta k} \tag{4}$$

$$=\sum_{j=0}^{\infty}t_j[\bar{F}^{NEX}(x;a)]^{j}. \tag{5}$$

By differentiating (5), we obtain

$$g^{JWX}(x;\alpha,\beta,a)=f^{NEX}(x;a)\sum_{j=0}^{\infty}t'_j[\bar{F}^{NEX}(x;a)]^{j-1} \tag{6}$$

$$=\sum_{j=0}^{\infty}t_j\frac{d}{dx}[\bar{F}^{NEX}(x;a)]^j, \tag{7}$$

where $t_j=\sum_{k=0}^{\infty}\frac{(-1)^{j+k+1}\alpha^k}{k!}\binom{\beta k}{j}$ and $t'_j=-jt_j$.

3.2. Probability weighted moments

The probability weighted moments (PWMs), first proposed by Greenwood et al. (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{\text{th}}$ PWM of X is having cdf $F(x)$, defined by

$$\Gamma_{p,q,r}=\int_{-\infty}^{\infty}x^p[F(x)]^q[1-F(x)]^r f(x)dx.$$

From (6), the s^{th} moment of X can be expressed as

$$\begin{aligned}E(X^s)&=\int_{-\infty}^{+\infty}x^s f^{NEX}(x;a)\sum_{j=0}^{\infty}t'_j[\bar{F}^{NEX}(x;a)]^{j-1}dx\\&=\sum_{j=0}^{\infty}t'_j\int_{-\infty}^{+\infty}x^s[\bar{F}^{NEX}(x;a)]^{j-1}f^{NEX}(x;a)\,dx\\&=\sum_{j=0}^{\infty}t'_j\Gamma^{NEX}_{s,0,j-1},\end{aligned}$$

where $\Gamma^{NEX}_{p,q,r}=\int_{-\infty}^{\infty}x^p\{F^{NEX}(x;\alpha)\}^q\ \{\bar{F}^{NEX}(x;\alpha)\}^r[f^{NEX}(x;\alpha)]dx$ is the PWM of NEX(α) distribution.

3.3. Distribution of order statistics

Suppose $X_1,X_2,...,X_n$ is a random sample from any distribution belonging JW-X(α,β,a) family. Let $X_{r:n}$ denote the r^{th} order statistics. The pdf of $X_{r:n}$ can be expressed as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}f^{JWX}(x)F^{JWX}(x)^{i+r-1}.$$

We get the pdf of the r^{th} order statistics of the JW-X(α,β,a) by using the general expansion of the JW-X(α,β,a) distribution as given in Section 3.1,

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}f^{NEX}(x;a)\sum_{j=0}^{\infty}t'_j[\bar{F}^{NEX}(x;a)]^{j-1}\sum_{p=0}^{\infty}t_p[\bar{F}^{NEX}(x;a)]^{p(i+r-1)},$$

where t_p and t'_j are defined earlier and

$$\begin{aligned}f_{r:n}(x)&=\frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}f^{NEX}(x;a)\sum_{j,p=0}^{\infty}t_pt'_j[\bar{F}^{NEX}(x;a)]^{j+p(i+r-1)-1}\\&=f^{NEX}(x;a)\sum_{j,p=0}^{\infty}M_{j,p}[\bar{F}^{NEX}(x;a)]^{j+p(i+r-1)-1}\\&=-\sum_{j,p=0}^{\infty}M'_{j,p}\frac{d}{dx}[\bar{F}^{NEX}(x;a)]^{j+p(i+r-1)}\end{aligned}$$

$$= \sum_{j,p=0}^{\infty} M'_{j,p} f^{NEX}(x; a(j + p(i + r - 1)))],$$

where $M_{j,p} = t_p \rightleftarrows t'_j \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i}$ and $M'_{j,p} = M_{j,p} / (j + p(i + r - 1))$.

3.4. Moment generating function (mgf)

The mgf of JW-X(α, β, a) family can be easily expressed in terms of those of the exponentiated NEX(α) distribution, using the results of Section 3.1. For example, using (7), it can be seen that

$$M_X(s) = E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} f^{JWX}(x) dx$$

$$= \int_{-\infty}^{\infty} e^{st} \sum_{j=0}^{\infty} t_j \frac{d}{dx} [\bar{F}^{NEX}(x; a)]^j dx$$

$$= \sum_{j=0}^{\infty} t_j \int_{-\infty}^{\infty} e^{sx} \frac{d}{dx} [\bar{F}^{\mathrm{NEX}}(x; a)]^j dx$$

$$= \sum_{j=0}^{\infty} t'_j M_X(s),$$

where t'_j defined in Section 3.1, X has exponentiated NEX(α) distribution.

4. Maximum Likelihood Estimation

Let $x = (x_1, x_2, ..., x_n)^X$ be a random sample of size n from JW-X(α, β, a) with parameter vector $\boldsymbol{\theta} = (\alpha, \beta, a, \boldsymbol{\eta}^T)^T$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, ..., \eta_q)^T$ corresponds to the parameter vector of the baseline distribution G. Then the log-likelihood function for ℓ is given by

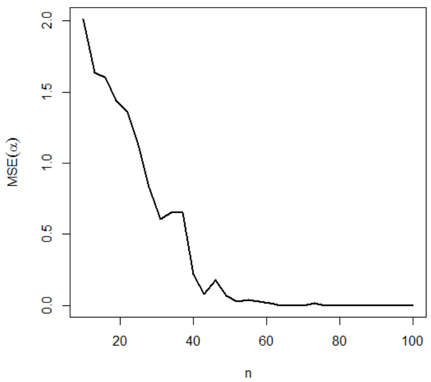
$$\ell = \ell(\boldsymbol{\theta}) = n \log(\alpha \beta a) + \sum_{i=1}^{n} \log[f(x_i, \boldsymbol{\eta})] + (a-1) \sum_{i=1}^{n} \log[F(x_i, \boldsymbol{\eta})] + \beta \sum_{i=1}^{n} \log[\log\{1 - F^a(x_i, \boldsymbol{\eta})\}]$$

$$+ \sum_{i=1}^{n} \log[1 - [\log\{1 - F^a(x_i, \boldsymbol{\eta})\}]^{-1}] - (\beta+1) \sum_{i=1}^{n} \log[F^a(x_i, \boldsymbol{\eta}) - 1] - \sum_{i=1}^{n} \left[\alpha \left\{ \frac{-\log[1 - F^a(x_i, \boldsymbol{\eta})]}{1 - F^a(x_i, \boldsymbol{\eta})} \right\}^{\beta} \right]$$

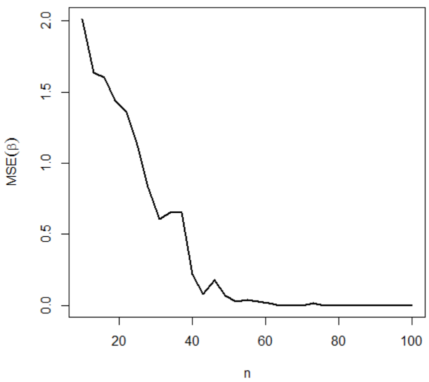
The MLEs are computed using quasi-Newton code for bound constrained optimization (L-BFGS-B) and the log-likelihood function is evaluated.

4.1. Monte Carlo simulation

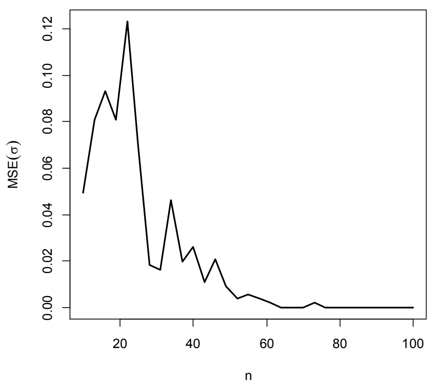
In this section, a graphical Monte Carlo simulation study is conducted to compare the performance of the different estimators of the unknown parameters for the JW-Lx$(a, \sigma, \alpha, \beta, \theta)$ distribution. All the computations in this section are done by R program. We generate N = 3000 samples of size n = 20, 30, …, 100 from JW-Lx distribution with true parameters values $a = 0.5$, $\sigma = 1.5$, $\alpha = 0.5$, $\beta = 2$ and $\theta = 3$. We also calculate the mean square error (MSE) of the MLEs empirically. The MSE are calculated by $\widehat{MSE}_h = \frac{1}{N} \sum_{i=1}^{N} (\hat{h}_i - h)^2$ (for $h = a, \sigma, \alpha, \beta, \theta$). The results of this simulation are illustrated in Figure 2. We observe that when the sample size increases, the empirical MSEs approach to 0 in all cases.



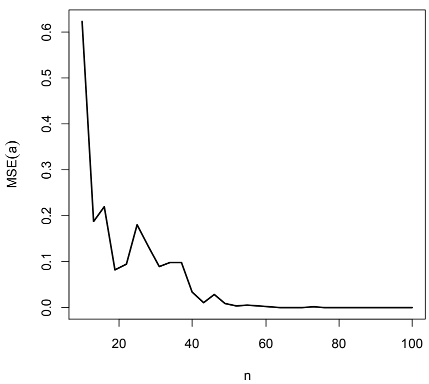
(a) The plot of MSE for parameter α



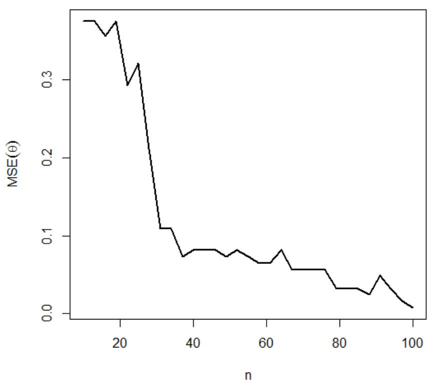
(b) The plots of MSE for parameter β



(c) The plots of MSE for parameter σ



(d) The plots of MSE for parameter a



(e) The plots of MSE for parameter θ

Figure 2 The plots of MSE for the parameters (a) α (b) β (c) σ (d) a and (e) θ for JW-Lx distribution

5. Real Life Applications

In this section, we provide two applications to real data sets to illustrate the applicability of the JW-Lx distribution. We see that it can provide better model than the corresponding distributions, Lomax (Lx), The beta Lomax (BLx) and Kumaraswamy Lomax (KwLx) distributions are intruded by Lemonte and Cordeiro (2013), exponentiated Lomax (ELx) by El-Bassiouny et al. (2015), Topp-Leone Lomax (TLLx) by Al-Shomrani et al. (2016), Topp-Leone exponential (TLE) by Al-Shomrani et al. (2016) and Odd-Lindley-Lomax (OLLx) by Frank Gomes-Silva et al. (2017). The first data set consists of 20 observation of ordered failure of components is given as follows:

Data I: 0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.This data set is obtained from Nigm et al. (2003).

The second data set consists of 63 observations of the strength of 1.5 cm glass fiber is given as follows:

Data II: 0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11,1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. This data set has also been studied by Reyad and Othman (2017).

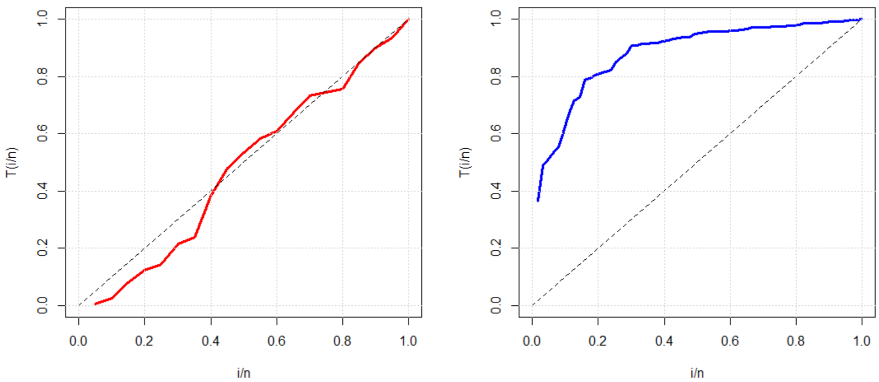
We have considered some well-known goodness of fit measures to compare the fitted models Anderson-Darling (A^*) and Cramer von-Mises (W^*) test statistic. The value for the Kolmogorov-Smirnov (KS) statistic and its p-value are also provided. The fitted densities, fitted cdf's and PP plots of data set I and data set II are presented in Figures 4 and 5. These plots reveal that the proposed distributions provide a good fit to these data.

5.1. TTT plots and descriptive statistics for the data sets

In data modelling applications, information about the shape of the hazard function can help us in deciding a particular model. To meet this objective, the concept of total time on test (TTT) plot was proposed by Aarset (1987). The TTT is drawn by plotting $T(i/n) = \left\{ \left(\sum_{r=1}^{i} y_{(r)} \right) + (n-i) y_{(i)} \right\} \Big/ \sum_{r=1}^{n} y_{(r)}$ where, $i = 1, 2, ..., n$ and $y_{(r)} (r = 1, 2, ..., n)$ are the order statistics of the sample, against i/n. The hazard of the given data set is constant, decreasing and increasing depending on the shape of the TTT plot being a straight diagonal line, is of convex shape and concave shape respectively. The TTT plots for the data sets considered here are presented in Figure 3, indicating that the first data set have decreasing as well as constant hazard and the 2nd data set have increasing hazard rate. The descriptive statistics of the data set I and data set II are given in Table 3. The findings of the data fitting of data set I and data set II are given in Tables 4, 5, 6 and 7, respectively.

Table 3 Descriptive Statistics for the data set I and II

Data Sets	Minimum	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Maximum
I	0.001	0.161	0.132	0.157	1.231	1.074	0.037	0.211	0.614
II	0.550	1.507	1.590	0.324	−0.879	0.800	1.375	1.685	2.240



(a) TTT plot for data set I (b) TTT plot for data set II

Figure 3 TTT-plots for the (a) data set I and (b) data set II

Table 4 The MLEs for the data set I

Distribution	Estimates with standard error in parenthesis				
	$\hat{a}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
JW-Lx	1.6469 (0.7014)	20.2750 (1.2923)	0.8329 (1.1999)	0.7912 (0.7865)	12.5808 (2.4272)
OLLx	11.4080 (54.9220)	5.2808 (79.2656)	2.9327 (32.0138)	–	–
TLLx	13.7262 (54.6939)	37.0004 (145.1542)	0.7958 (0.2233)	–	–
BLx	0.8075 (0.2211)	10.8542 (139.0261)	–	26.0677 (148.8267)	12.1996 (153.1953)
KwLx	46.4929 (127.0901)	3.9739 (12.8003)	0.7426 (0.4514)	0.4890 (1.9023)	–
ELx	8.1186 (24.9598)	44.1626 (132.4419)	0.7986 (0.2249)	–	–
ETE	0.7925 (0.2208)	2.6644 (0.8240)	–	–	–
Lx	27.4680 (250.6248)	171.2326 (155.7832)	–	–	–

6. Conclusions

In the present study, we computed the mathematical properties of JW-X family such as probability weighted moment, order statistics, moment-generating function. The linear expansions for the cumulative distribution and probability density function are also expressed as a new exponentiated X family. Tables 4-7 contains the MLEs with standard errors of the parameters for all the fitted models along with their A^*, $W^*$and KS statistic with p-value for the data sets I and II are presented. It is evident from the results of the Tables 4-7 that both the data sets, the JW-Lx distribution with lowest A^*, W^* and highest p-value for the KS statistic is not only a better model than the sub models but also better than the most of the recently introduced extensions of Lomax distribution. The findings of the study also correspond to the plots of fitted densities, fitted cdfs and P-P plot given in Figures 4 and 5 for the data sets I and II, respectively. It is evident from the plots that the proposed distribution provides closest fit to the both data sets.

Table 5 Some statistics for the models fitted to the data set I

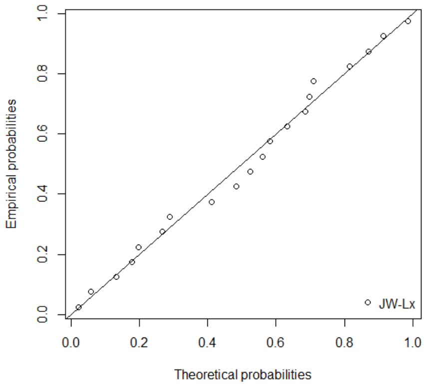
Distribution	Estimates with standard error in parenthesis				
	A^*	W^*	L	KS	p-value
JW-Lx	0.1373	0.0217	−17.2738	0.0906	0.9913
OLLx	0.1905	0.0335	−16.5229	0.1082	0.9533
TLLx	0.2177	0.0384	−16.8290	0.1234	0.8847
BLx	0.2206	0.0389	−16.8132	0.1229	0.8876
KwLx	0.2111	0.0373	−16.8355	0.1234	0.8844
ELx	0.2213	0.0391	−16.8072	0.1241	0.8806
ETE	0.2124	0.0375	−16.8608	0.1225	0.8899
Lx	0.2184	0.0386	−16.4889	0.1099	0.9065

Table 6 The MLEs for the data set II

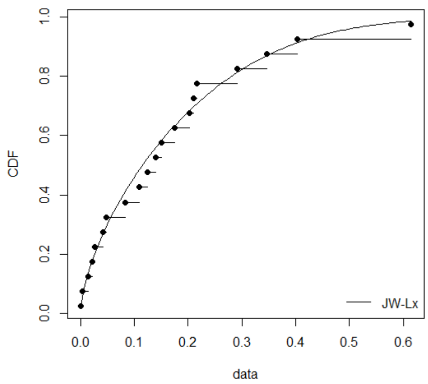
Distribution	Estimates with standard error in parenthesis				
	$\hat{a}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
JW-Lx	0.0301	14.54263	0.9826	2.9615	21.2903
	(0.0021)	(0.0801)	(0.1199)	(0.9753)	(3.5134)
OLLx	1.8079	79.9901	29.5177	−	−
	(0.1482)	(22.3521)	(8.8291)		
TLLx	44.4017	59.6955	32.8566	−	−
	(54.9352)	(72.1453)	(10.1778)		
BLx	17.8478	40.1913	−	77.5816	19.3027
	(3.4499)	(60.2638)		(222.1545)	(41.0907)
KwLx	27.1531	46.1506	9.0710	91.7567	−
	(67.0525)	(125.0250)	(2.3638)	(89.5981)	
ELx	43.2618	116.1829	32.7701	−	−
	(53.4628)	(139.3477)	(10.3498)		
ETE	31.3565	1.3057	−	−	−
	(9.5283)	(0.1190)			
Lx	211.2298	140.5679	−	−	−
	(322.7742)	(214.5268)			

Table 7 Some statistics for the models fitted to the data set II

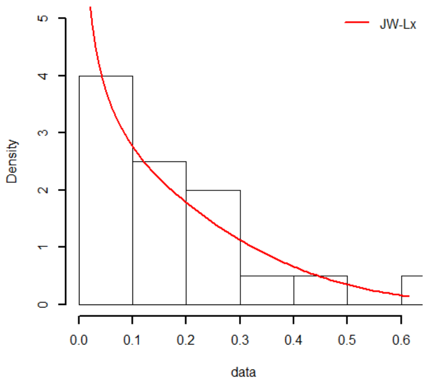
Distribution	Estimates with standard error in parenthesis				
	A^*	W^*	L	KS	p-value
JW-Lx	1.0141	0.1815	14.3771	0.1390	0.1751
OLLx	2.4574	0.4481	69.9755	0.3724	0.0000
TLLx	4.3452	0.7972	31.7284	0.2300	0.0025
BLx	3.1526	0.5748	24.1456	0.2165	0.0054
KwLx	1.7966	0.3281	17.2688	0.1731	0.0457
ELx	4.3439	0.7969	31.7373	0.2285	0.0027
ETE	4.2870	0.7861	31.3834	0.2295	0.0027
Lx	3.1354	0.5717	89.0439	0.4179	0.0000



(a) P-P plot of the JW-Lx for data set I

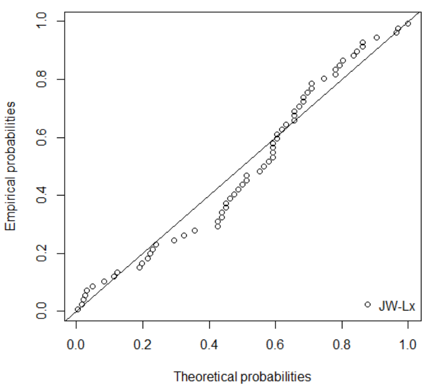


(b) cdf plot of the JW-Lx for data set I

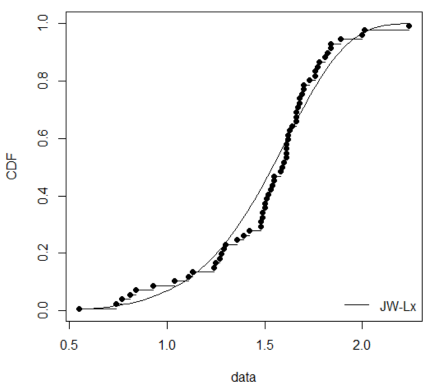


(c) density plot of the JW-Lx for data set I

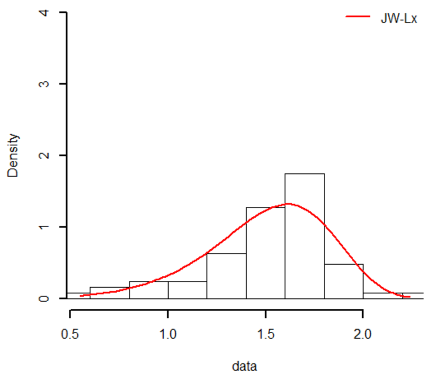
Figure 4 Various fits of the JW-Lx P-P plot, empirical and theoretical cdf and densities for data set I



(a) P-P plot of the JW-Lx for data set II



(b) cdf plot of the JW-Lx for data set II



(c) density plot of the JW-Lx for data set II

Figure 5 Various fits of the JW-Lx P-P plot, empirical and theoretical cdf and densities for data set II

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