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On Chung's Law of Large Numbers for Arrays of Extended Negatively Dependent Random Variables

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Abstract

In this article, we present some sharp convergence results for partial sums of arrays of rowwise extended negatively dependent random variables. These results are established without assumptions of the identical distribution and stochastic domination. The results generalize and improve the corresponding results of Hu and Taylor (1997), Wu and Zhu (2010), and Wu et al. (2014).

Keywords: Arrays of rowwise extended negatively dependent random variables, complete convergence, complete q^{th} moment convergence, L_q convergence.

1. Introduction

By weakening the assumptions of validity of the law of large numbers, we provide an extension for possible applications of the probability theory to various fields, especially to the field of statistics. In many theoretical statistical frameworks, we assume that variables are independent. However, in real studies, this assumption is not plausible. Therefore, many statisticians have revised this assumption in order to consider dependent cases, such as negatively associated random variables, positively associated random variables, negatively orthant dependent random variables, extended negatively dependent random variables (END), and many others. In this article, we consider the END structure, which includes independent random variables, negatively associated random variables and negatively orthant dependent random variables as special cases, and present some sharp results on complete convergence, complete q^{th} moment convergence and L_q convergence for END random variables.

1.1. Extended negative dependence

The concept of extended negatively dependent random variables was introduced by Liu (2009) as follows.

Definition 1 *A finite collection of random variables $X_1, X_2, ..., X_n$ is said to be extended negatively dependent (END) if there exists a constant $C > 0$ such that both inequalities*

$$P(X_1 > x_1, X_2 > x_2, ..., X_n > x_n) \le C\prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P\left(X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\right) \le C\prod_{i=1}^{n} P\left(X_i \le x_i\right)$$

hold for all real numbers $x_1, x_2, ..., x_n$. An infinite sequence $\{X_n; n \ge 1\}$ is said to be END if every finite subset is END.

Let $\{k_n, n \geq 1\}$ be a sequence of natural numbers such that $k_n \to \infty$ as $n \to \infty$. An array of random variables $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ is called rowwise END random variables if for every $n \geq 1$, $\{X_{ni}; 1 \leq i \leq k_n\}$ are END random variables.

Obviously, the negatively orthant dependent structure is a special case of the END structure with $C = 1$. The END structure is superordinate to the negatively orthant dependent structure which was introduced by Lehmann (1966) and later developed by Ebrahimi and Ghosh (1981) (cf. also Joag-Dev and Proschan (1983). The END structure can reflect not only a negative dependence structure but also a positive one to some extent. Liu (2009) pointed out that some sequences of END random variables obey both negatively and positively dependent properties, and provided interesting examples to support this idea.

Joag-Dev and Proschan (1983) proved that negatively associated random variables must be negatively orthant dependent (but negatively orthant dependent is not necessarily negatively associated); thus negatively associated random variables are also END.

Since the paper of Ebrahimi and Ghosh (1981) appeared, the convergence properties of negatively orthant dependent random variables have been studied in various aspects by many authors. For example, Volodin (2002) established the Kolmogorov exponential inequality; Ko and Kim (2005), Ko et al. (2006) investigated the strong laws of large numbers for weighted sums; Amini and Bozorgnia (2003) studied the complete convergence; Asadian et al. (2006) obtained the Rosenthal type inequality; Qiu et al. (2011) investigated the strong convergence rate and complete convergence, and so forth.

Some probability limit properties and applications for END random variable sequences have also been obtained in literature. We mention a few here. Liu (2010) studied the sufficient and necessary conditions of moderate deviations for END random variables with heavy tails; Chen et al. (2010) established the strong law of large numbers for END random variables and showed applications to risk theory and renewal theory; Shen (2011) presented some probability inequalities for END random variables and gave some applications; Wang and Wang (2012) investigated the extended precise large deviations of random sums in the presence of END structure and consistent variation; Wu and Guan (2012) presented some convergence properties for the partial sums of END random variables; Wang and Wang (2013) investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. (2011), Wang et al. (2013a, 2013b, 2014) and Hu et al. (2015) provided results on complete convergence for END random variables; Wu et al. (2014) established the complete moment convergence for arrays of rowwise END random variables; Wang et al. (2015) studied the complete consistency for the estimator of nonparametric regression models based on END errors.

1.2. Complete convergence

The concept of complete convergence was first introduced by Hsu and Robbins (1947) as follows: a sequence $\{X_n; n \geq 1\}$ of random variables is said to converge completely to a constant λ if for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|X_n - \lambda| > \varepsilon) < \infty$.

In view of the Borel-Cantelli lemma, this implies that $X_n \to \lambda$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables.

Let $\{k_n, n \geq 1\}$ be a sequence of natural numbers such that $k_n \to \infty$ as $n \to \infty$. For an array of random variables $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$, let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. We say that the array $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ is centered if $EX_{ni} = 0$, for all $1 \leq i \leq k_n, n \geq 1$.

Suppose that $\{g_n(t); n \geq 1\}$ is a sequence of positive, even functions such that

$$\frac{g_n(|t|)}{|t|} \uparrow \text{ and } \frac{g_n(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow, \tag{1}$$

for some real number $p > 1$.

In the following, we will use the assumptions as follows

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{Eg_i(X_{ni})}{g_i(a_n)} < \infty; \tag{2}$$

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k_n}\frac{E|X_{ni}|^r}{a_n^r}\right)^s<\infty, \tag{3}$$

where $0<r\le 2$, $s>0$;

$$\sum_{i=1}^{k_n}\frac{Eg_i(X_{ni})}{g_i(a_n)}\to 0 \text{ as } n\to\infty. \tag{4}$$

$$\frac{1}{a_n^r}\sum_{i=1}^{n}E|X_{ni}|^r\to 0 \text{ as } n\to\infty, \tag{5}$$

where $0<r\le 2$.

There are two exceptionally important manuscripts for the investigation presented in this article: Wu and Zhu (2010) and Wu et al. (2014). To explain their results and point out what is new in our article, we need to introduce some notation.

Let $\{X_{ni}; 1\le i\le n, n\ge 1\}$ be a centered array of rowwise negatively orthant dependent random variables, and $\{a_n; n\ge 1\}$ be a sequence of positive real numbers with $a_n\uparrow\infty$, $\{g_n(t); n\ge 1\}$ being a sequence of nonnegative even functions such that (1) holds.

Wu and Zhu (2010) discussed the convergence properties of partial sums for arrays of rowwise negatively orthant dependent random variables and established the following three theorems, which extend and improve the corresponding results of Hu and Taylor (1997) for independent random variables.

Theorem 1 *If* $1<p\le 2$, *then assumption* (2) *implies*

$$\sum_{n=1}^{\infty}P\left(\left|\sum_{i=1}^{n}X_{ni}\right|>\varepsilon a_n\right)<\infty \tag{6}$$

for all $\varepsilon>0$. *In addition, if* $p>2$, *then assumptions* (2) *and* (3) *also imply* (6).

Theorem 2 *If* $1<p\le 2$, *then assumption* (2) *implies*

$$\sum_{n=1}^{\infty}a_n^{-1}E\left(\left|\sum_{i=1}^{n}X_{ni}\right|-\varepsilon a_n\right)_+<\infty \tag{7}$$

for all $\varepsilon>0$. *In addition, if* $p>2$, *then assumptions* (2) *and* (3) *imply* (7).

Theorem 3 (1) *If* $1<p\le 2$, *then* (4) *implies*

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni}\xrightarrow{L_1}0. \tag{8}$$

(2) *If* $p>2$ *then* (4) *and* (5) *with* $0<r\le 2$ *imply that* (8) *still holds.*

Wu et al. (2014) discussed the convergence properties of partial sums for arrays of rowwise END random variables and established the following two theorems, which extend and improve the corresponding results of Hu and Taylor (1997) for independent random variables.

Let $\{X_{ni}; 1\le i\le n, n\ge 1\}$ be a centered array of rowwise END random variables, and $\{a_n; n\ge 1\}$ be a sequence of positive real numbers with $a_n\uparrow\infty$.

Theorem 4 *Let* $g_n(t)=\Psi(t)$ *for all* $n\ge 1$ *and all* t. *If* $1<p\le 2$, *then condition* (2) *implies* q^{th} *moment convergence*

$$\sum_{n=1}^{\infty}a_n^{-q}E\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|-\varepsilon a_n\right)_+^q<\infty \tag{9}$$

for all $\varepsilon>0$.

Theorem 5 *Let* $1\le q<p$.

(1) If $1<p\le 2$, *then assumption* (4) *implies*

$$\frac{1}{a_n}\sum_{i=1}^{k_n} X_{ni} \xrightarrow{L_q} 0. \tag{10}$$

(2) If $p > 2$, *then assumptions* (4) *and* (5) *imply* (10).

The following are the main differences between the results presented in Wu and Zhu (2010) and Wu et al. (2014) and our results

1. In both manuscripts, Wu and Zhu (2010) and Wu et al. (2014) only consider triangular arrays, that is, $k_n = n$ for all $n \geq 1$ in (2) and (3).

2. Wu and Zhu (2010) considered negatively orthant dependent random variables, whereas we consider a more general case of END random variables.

3. Wu et al. (2014) considered the case of all the same functions $g_n, n \geq 1$, that is, $g_n(t) = \Psi(t)$ for all $n \geq 1$.

4. Assumption (3) is more general than a similar assumption used in Wu and Zhu (2010) and Wu et al. (2014). Their assumption is a special case of (3) with $r = 2$ and $s = 2k$, where k is a positive integer.

5. Our proofs are based on the results obtained in Wu et al. (2019); they are very different from, and much simpler than corresponding proofs from Wu and Zhu (2010) and Wu et al. (2014).

In this article, inspired by the aforementioned results by Wu and Zhu (2010) and Wu et al. (2014), we investigate the complete convergence, the complete q^{th} moment convergence and the L_q convergence properties of partial sums for arrays of rowwise END random variables under some more general conditions, and obtain some improved theorems without assumptions of identical distribution and stochastic domination.

Throughout this article, $I(A)$ denotes the indicator function of the set A. As usual, the symbol C denotes a positive constant, which may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$.

2. Main Results

Now we present the main, completely theoretical, results of this work. The proofs of the following five theorems will be detailed in next section. In these five theorems, we let $\{k_n, n \geq 1\}$ be a sequence of natural numbers such that $k_n \to \infty$ as $n \to \infty$; $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ be a centered array of rowwise END random variables, $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$ and $\{g_n(t); n \geq 1\}$ be a sequence of nonnegative even functions such that

$$\frac{g_n(|t|)}{|t|^q} \uparrow \text{ and } \frac{g_n(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow, \tag{11}$$

for some q and p to be specified in each theorem separately (of course, $q < p$).

Theorem 6 *If* $1 \leq q < p \leq 2$, *then assumption* (2) *implies*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon a_n\right) < \infty, \tag{12}$$

for all $\varepsilon > 0$.

Theorem 7 *If* $1 \leq q < p$ *and* $p > 2$, *then assumptions* (2) *and* (3) *imply* (12) *for all* $\varepsilon > 0$.

Theorem 8 *If* $1 \leq q < p \leq 2$, *then assumption* (2) *implies* q *-moment convergence* (9) *for all* $\varepsilon > 0$.

Theorem 9 *If* $1 \leq q < p$ *and* $p > 2$, *then assumptions* (2) *and* (3) *imply* (9) *for all* $\varepsilon > 0$.

Theorem 10 *Let* $1 \leq q < p$.

(1) If $1 < p \leq 2$, *then assumption* (4) *implies* (10).

(2) If $p > 2$, *then assumptions* (4) *and* (5) *imply* (10).

The following important remarks discuss how our main results generalize and simplify some known ones.

Remark 1 Since a sequence of independent random variables is a special case of END sequence, Theorems 7 and 8 hold for arrays of rowwise independent random variables. Therefore, Theorems 7 and 8 are extensions and improvements of the corresponding theorems of Hu and Taylor (1997) for independent random variables.

Remark 2 Take $q=1$ and $k_n = n, n \geq 1,$ in the above theorems, then the conditions and the conclusions of the main results are the same as those of Wu and Zhu (2010). Compared with the corresponding conslusions of Wu and Zhu (2010), our conclusions are stronger and the assumptions are more general. In addition, it is worthy to point out that the methods applied in this paper are different from those of Wu and Zhu (2010).

Remark 3 Take $g_n(t) = \Psi(t)$ for all $n \geq 1$ and t, and $k_n = n, n \geq 1,$ in the above theorems, then the conditions and the conclusions of the main results are the same as those of Wu, Song, and Wang (2014). Compared with the corresponding conclusions of Wu, Song, and Wang (2014), our conclusions are stronger and the assumptions are more general. In addition, it is worthy to point out that the methods applied in this paper are different from those of Wu, Song, and Wang (2014).

Remark 4 Note that q^{th} moment complete convergence implies complete convergence. This fact is mentioned in Wu et al. (2018) in the much more general situation of f -moment complete convergence. The argument is as follows.

For any random variable S and any $\varepsilon > 0,$ we have

$$\begin{aligned} E(\{|S|-\varepsilon\}_+)^q &= \int_0^\infty P(\{|S|-\varepsilon\}_+^q > t)dt \\ &= \int_0^\infty P(|S| > \varepsilon + t^{1/q})dt \\ &\geq \int_0^{\varepsilon^q} P(|S| > \varepsilon + t^{1/q})dt \\ &\geq \varepsilon^q P(|S| > \varepsilon + (\varepsilon^q)^{1/q}) \\ &= \varepsilon^q P(|S| > 2\varepsilon). \end{aligned}$$

If we take $S = \frac{1}{a_n}\sum_{i=1}^{k_n} X_{ni}$, then the last argument implies that

$$\sum_{n=1}^{\infty} a_n^{-q} E\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| - \varepsilon a_n\right)_+^q \geq C\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > 2\varepsilon a_n\right),$$

which implies that q^{th} moment complete convergence is stronger than complete convergence.

In connection with this, Theorems 6 and 7 can be considered as corollaries of Theorems 8 and 9, respectively. We present different, more direct and elegant proofs of Theorems 6 and 7.

3. Lemmata

To prove the main results, we need the following already known lemmas.

Lemma 1 (Liu 2010) *Let $\{X_n; n \geq 1\}$ be a sequence of END random variables, and $\{f_n; n \geq 1\}$ be a sequence of Borel functions, all of which are monotone increasing. Then $\{f_n(X_n); n \geq 1\}$ is a sequence of END random variables.*

Lemma 2 (Shen, 2011, Corollary 3.2) *Let $r \geq 2$ and $\{X_n; n \geq 1\}$ be a sequence of END mean zero random variables with $E|X_n|^r < \infty$ for all $n \geq 1$. Then there exists a positive constant $C = C(r)$ depending only on r, such that for all $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq C\left(\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} EX_i^2\right)^{r/2}\right).$$

Remark 5 For the case of negatively orthant dependent random variables, Lemmas 1 and 2 have been established in Ebrahimi and Ghosh (1981) and Asadian et al. (2006), respectively.

Lemma 3 *Let* $q \geq 1, \{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ *be a centered array of rowwise END random variables and* $\{a_n; n \geq 1\}$ *be a sequence of positive real numbers. Suppose that the following conditions hold*

(a) $\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{k_n} E \mid X_{ni} \mid^q I(\mid X_{ni} \mid > a_n \varepsilon) < \infty$ for any $\varepsilon > 0$;

(b) there exist some constants $s > q$, $0 < r \leq 2$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} \left(a_n^{-r} \sum_{i=1}^{k_n} E \left| X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r \right)^s < \infty .$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n^{-q} E \left\{ \left| \sum_{i=1}^{k_n} X_{ni} \right| - a_n \varepsilon \right\}_+^q < \infty .$$

Proof: In Corollary 4.3 of Wu et al. (2018), the following statement has been proved.

Let $q \geq 0, \{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ be a centered array of rowwise END random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers. Suppose that the following conditions hold:

(i) $\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{k_n} E \mid X_{ni} \mid^q I(\mid X_{ni} \mid > a_n \varepsilon) < \infty$ for any $\varepsilon > 0$;

(ii) there exist some constants $s > \max(1, q)$, $0 < r \leq 2$ and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \left(a_n^{-r} \sum_{i=1}^{k_n} E \left| X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) - E X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r \right)^s < \infty;$$

(iii) $a_n^{-1} \sum_{i=1}^{k_n} E \left| X_{ni} \right| I(\left| X_{ni} \right| > a_n \delta 16s) \to 0$, as $n \to \infty$.

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n^{-q} E \left\{ \left| \sum_{i=1}^{k_n} X_{ni} \right| - a_n \varepsilon \right\}_+^q < \infty .$$

First of all, we note that the assumption (iii) is a redundant assumption for $q \geq 1$ because its validity follows from (i) by the following arguments

$$\begin{aligned} a_n^{-1} E \left| X_{ni} \right| I(\left| X_{ni} \right| > a_n \delta 16s) &\leq \frac{1}{a_n} E \mid X_{ni} \mid^q \left(\frac{a_n \delta}{16s} \right)^{1-q} I(\left| X_{ni} \right| > a_n \delta 16s) \\ &\leq \left(\frac{16s}{\delta} \right)^{q-1} a_n^{-q} E \mid X_{ni} \mid^q I(\left| X_{ni} \right| > \frac{a_n \delta}{16s}). \end{aligned}$$

Therefore,

$$a_n^{-1} \sum_{i=1}^{k_n} (E \left| X_{ni} \right| I \left| X_{ni} \right| > \frac{a_n \delta}{16s}) \leq C a_n^{-q} \sum_{i=1}^{k_n} \left| E \ \ X_{ni} \right|^q I(\left| X_{ni} \right| > \frac{a_n \delta}{16s}) \to 0$$

as the n^{th} term of the convergent series (i) with $\varepsilon = \frac{\delta}{16s}$.

Next, assumption (ii) can be written as (b) by the famous c_r-inequality, which can be formulated in the following way. For any two random variables X and Y and $r \geq 1$ such that their r^{th} absolute moments exist $E \left| X + Y \right|^r \leq 2^{r-1} \left(E \left| X \right|^r + E \left| Y \right|^r \right)$. Hence,

$$\begin{aligned} & E \left| X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) - E X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r \\ & \leq 2^{r-1} E \left| X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r + \left| E X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r \\ & \leq 2^r E \left| X_{ni} I\left(\left| X_{ni} \right| \leq a_n \delta \right) \right|^r . \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\left(a_n^{-r}\sum_{i=1}^{k_n}E\left|X_{ni}I\left(|X_{ni}|\le a_n\delta\right)-EX_{ni}I\left(|X_{ni}|\le a_n\delta\right)\right|^r\right)^s$$
$$\le C\sum_{n=1}^{\infty}\left(a_n^{-r}\sum_{i=1}^{k_n}E\left|X_{ni}I\left(|X_{ni}|\le a_n\delta\right)\right|^r\right)^s$$

The proof of Lemma 3 is completed.

Remark 6 Note that Corollary 4.3 of Wu et al. (2018) considers the triangular array $\{X_{ni}; 1\le i\le n, n\ge 1\}$ of random variables. A careful analysis of the proof shows that this corollary is valid for a general array $\{X_{nk}; 1\le k\le k_n, n\ge 1\}$ of random variables with no changes.

4. Proofs

Proof of Theorem 6: For any $1\le i\le k_n$ and $n\ge 1$, define the so-called monotone truncation:

$$Y_{ni}=-a_nI\left(X_{ni}<-a_n\right)+X_{ni}I\left(|X_{ni}|\le a_n\right)+a_nI\left(X_{ni}>a_n\right), Z_{ni}=X_{ni}-Y_{ni}.$$

Then for all $\varepsilon>0$,

$$\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon a_n\right)=\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon a_n,\bigcap_{i=1}^{k_n}\left(X_{ni}=Y_{ni}\right)\right)\bigcup\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon a_n,\bigcup_{i=1}^{k_n}\left(X_{ni}\ne Y_{ni}\right)\right)$$
$$\subset\left(\left|\sum_{i=1}^{k_n}Y_{ni}\right|>\varepsilon a_n\right)\bigcup\left(\bigcup_{i=1}^{k_n}\left(|X_{ni}|>a_n\right)\right)$$
$$\subset\left(\left|\sum_{i=1}^{k_n}\left(Y_{ni}-EY_{ni}\right)\right|>\varepsilon a_n-\left|\sum_{i=1}^{k_n}EY_{ni}\right|\right)\bigcup\left(\bigcup_{i=1}^{k_n}\left(|X_{ni}|>a_n\right)\right),$$

which implies that

$$P\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon a_n\right)\le P\left(\left|\sum_{i=1}^{k_n}\left(Y_{ni}-EY_{ni}\right)\right|>\varepsilon a_n-\left|\sum_{i=1}^{k_n}EY_{ni}\right|\right)+P\left(\bigcup_{i=1}^{k_n}\left(|X_{ni}|>a_n\right)\right). \tag{13}$$

It is simple to see that

$$\frac{1}{a_n}\left|\sum_{i=1}^{k_n}EY_{ni}\right|\to 0 \quad \text{as} \quad n\to\infty. \tag{14}$$

Really, for $1\le i\le k_n, n\ge 1$, note that $|Z_{ni}|\le|X_{ni}|I\left(|X_{ni}|>a_n\right)$ and $EY_{ni}=-EZ_{ni}$ because $EX_{ni}=0$. By (2), we have

$$\frac{1}{a_n}\left|\sum_{i=1}^{k_n}EY_{ni}\right|=\frac{1}{a_n}\left|\sum_{i=1}^{k_n}EZ_{ni}\right|$$
$$\le\frac{1}{a_n}\sum_{i=1}^{k_n}E|Z_{ni}|$$
$$\le C\sum_{i=1}^{k_n}\frac{E|X_{ni}|I\left(|X_{ni}|>a_n\right)}{a_n}$$
$$\le C\sum_{i=1}^{k_n}\frac{E|X_{ni}|^qI\left(|X_{ni}|>a_n\right)}{a_n^q}$$
$$\le C\sum_{i=1}^{k_n}\frac{Eg_i\left(X_{ni}\right)}{g_i\left(a_n\right)}\to 0 \quad \text{as} \quad n\to\infty. \tag{15}$$

Hence for n large enough, by (13) we have

$$P\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon a_n\right)\le P\left(\left|\sum_{i=1}^{k_n}\left(Y_{ni}-EY_{ni}\right)\right|>\frac{\varepsilon a_n}{2}\right)+\sum_{i=1}^{k_n}P\left(|X_{ni}|>a_n\right).$$

Therefore, to prove (12), it needs only to show that

$$J_1 \doteq \sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{k_n} \left(Y_{ni} - EY_{ni} \right) \right| > \frac{\varepsilon a_n}{2} \right) < \infty, \tag{16}$$

$$J_2 \doteq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P\left(\left| X_{ni} \right| > a_n \right) < \infty . \tag{17}$$

For J_2, we have

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} EI\left(\left| X_{ni} \right| > a_n \right) \\ &\le \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\left| X_{ni} \right|^q I\left(\left| X_{ni} \right| > a_n \right)}{a_n^q} \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E g_i\left(X_{ni} \right)}{g_i\left(a_n \right)} < \infty. \end{aligned} \tag{18}$$

For J_1, by Lemma 1, $\left\{ Y_{ni} - EY_{ni}; 1 \le i \le k_n, n \ge 1 \right\}$ is still a centered array of rowwise END random variables. Hence, for $1 \le q < p \le 2$, by the Markov inequality, Lemma 2 with $r = 2$ and (2), we have that

$$\begin{aligned} J_1 &\le C \sum_{n=1}^{\infty} \frac{1}{a_n^2} E\left(\left| \sum_{i=1}^{k_n} \left(Y_{ni} - EY_{ni} \right) \right|^2 \right) \\ &\le C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{i=1}^{k_n} E\left| Y_{ni} - EY_{ni} \right|^2 \\ &\le C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{i=1}^{k_n} EY_{ni}^2 \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\left| Y_{ni} \right|^p}{a_n^p} \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E g_i\left(\left| Y_{ni} \right| \right)}{g_i\left(a_n \right)} \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E g_i\left(X_{ni} \right)}{g_i\left(a_n \right)} < \infty . \end{aligned} \tag{19}$$

The proof of Theorem 6 is completed.

Proof of Theorem 7: Following the notations and methods from the proof in Theorem 6, we have that (13), (14) and $J_2 < \infty$ hold. It suffices to prove $J_1 < \infty$ for $1 \le q < p$ and $p > 2$.

Note that $\left| Y_{ni} \right| \le \left| X_{ni} \right| I\left(\left| X_{ni} \right| \le a_n \right)$. By the Markov inequality, Lemma 2 with $r > p > 2$, the c_r inequality, (2) and (3) for some $0 < u \le 2$ and $s > 0$, we have that

$$\begin{aligned} J_1 &\le C \sum_{n=1}^{\infty} \frac{1}{a_n^r} E\left(\left| \sum_{i=1}^{k_n} \left(Y_{ni} - EY_{ni} \right) \right|^r \right) \\ &\le C \sum_{n=1}^{\infty} \frac{1}{a_n^r} \left(\sum_{i=1}^{k_n} E\left| Y_{ni} - EY_{ni} \right|^r + \left(\sum_{i=1}^{k_n} E\left(Y_{ni} - EY_{ni} \right)^2 \right)^{r/2} \right) \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\left| Y_{ni} \right|^r}{a_n^r} + C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k_n} \frac{E\left| Y_{ni} \right|^2}{a_n^2} \right)^{r/2} \\ &\le C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\left| X_{ni} \right|^p I\left(\left| X_{ni} \right| \le a_n \right)}{a_n^p} + C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k_n} \frac{E\left| X_{ni} \right|^u I\left(\left| X_{ni} \right| \le a_n \right)}{a_n^u} \right)^{r/2} \end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{Eg_i\left(X_{ni}\right)}{g_i\left(a_n\right)}+C\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k_n}\frac{E\left|X_{ni}\right|^u}{a_n^u}\right)^s\right)^{r/2s}<\infty. \tag{20}$$

The proof of Theorem 7 is completed.

Proof of Theorem 8: To prove (2), we just need to check that the assumptions (a) and (b) of Lemma 3 are true.

(a) For any $\varepsilon>0,$ we have by (11) and (2) that

$$\begin{aligned}\sum_{n=1}^{\infty}a_n^{-q}\sum_{k=1}^{k_n}\left|E\,X_{nk}\right|^q I(\left|X_{nk}\right|>a_n\varepsilon)&\leq\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\frac{E\mid X_{nk}\mid^q}{a_n^q}\\&\leq\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{Eg_i(X_{ni})}{g_i(a_n)}<\infty.\end{aligned} \tag{21}$$

(b) Note that for any sequence of positive numbers $\left\{c_n;n\geq1\right\},$ if $s>1,$ then $\left(\sum_{n=1}^{\infty}c_n^s\right)^{1/s}\leq\sum_{n=1}^{\infty}c_n.$ Hence, it is enough to prove that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\frac{E\left|X_{nk}I\left(\left|X_{nk}\right|\leq a_n\delta\right)\right|^p}{a_n^p}<\infty.$$

By (11) and (2), we have that

$$\delta^p\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\frac{E\left|X_{nk}I\left(\left|X_{nk}\right|\leq a_n\delta\right)\right|^p}{(a_n\delta)^p}\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{Eg_i\left(X_{ni}\right)}{g_i\left(a_n\right)}<\infty.$$

The proof of Theorem 8 is completed.

Proof of Theorem 9: We will need again to check that the assumptions of Lemma 3 are satisfied. Assumption (a) is already checked in (21). Assumption (b) follows easily from (3). The proof of Theorem 10 is completed.

Proof of Theorem 10: In Remark 3, we showed that the q-moment complete convergence implies the usual complete convergence. Now we show that the q-moment convergence is stronger than convergence in q-means. Really, by c_r-inequality, for any number a and any $\varepsilon>0,$ we have that

$$\mid a\mid^q\leq c_q\left\{(\left|a\right|-\varepsilon)_+^q+\varepsilon^q\right\}.$$

Hence for any random variable $S,$

$$E\mid S\mid^q\leq c_q\left\{E(\left|S\right|-\varepsilon)_+^q+\varepsilon^q\right\}.$$

Take $S=\frac{1}{a_n}\sum_{i=1}^{k_n}X_{ni},$ then by Theorems 8 and 9, for any $\varepsilon>0,$ there exists $n(\varepsilon)$ such that for all $n\geq n(\varepsilon)$ we have that $E(\left|S\right|-\varepsilon)_+^q<\varepsilon^q.$ And then $E\left|\frac{1}{a_n}\sum_{i=1}^{k_n}X_{ni}\right|^q\leq2c_q\varepsilon^q.$ Therefore, $E\left|\frac{1}{a_n}\sum_{i=1}^{k_n}X_{ni}\right|^q\to0$ as $n\to\infty.$ It is obvious that the conclusion of Theorem 10 follows now from Theorems 8 and 9. The proof of Theorem 10 is completed.

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