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The New Extension of Odd Log-Logistic Chen Distribution: Mathematical Properties and Applications

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Abstract

A new lifetime model called the odd log-logistic Chen distribution is being introduced in this paper. We provide a comprehensive account of the mathematical properties of the proposed family including the hazard rate function, moments, conditional moments, coefficient of skewness, coefficient of kurtosis, entropy and order statistics. The parameters of this distribution are estimated by several methods of estimation. A simulation study is performed in order to investigate the properties of the proposed estimators. Finally, in order to show the NOLL-Ch distribution flexibility, two applications using real data sets are presented.

Keywords: Hazard rate function, moments, skewness, kurtosis, entropy.

1. Introduction

Chen (2000) proposed a new two parameter lifetime distribution with bathtub shaped or increasing hazard rate function. A new generalization of this distribution was recently defined by Chaubey and Zhang (2015) which called the Extended Chen (EC) family. The cumulative distribution function (cdf) and probability density function (pdf) are given by the following:

$$
\begin{aligned}
F_{EC}(x;\lambda,\beta,\alpha) &= \left(1-e^{\lambda(1-e^{x^{\beta}})}\right)^{\alpha}, \; x>0, \\
f_{EC}(x;\lambda,\beta,\alpha) &= \alpha\beta\lambda x^{\beta-1}e^{x^{\beta}}e^{\lambda(1-e^{x^{\beta}})}\left(1-e^{\lambda(1-e^{x^{\beta}})}\right)^{\alpha-1}, \; x>0,
\end{aligned}
$$

where $\lambda,\beta,\alpha>0$ and denoted by $X\sim \mathrm{EC}(\lambda,\beta,\alpha)$. Dey et al. (2017) studied further various properties and estimation methods for the EC distribution. In literature, there exist many generalized (G-) classes of distributions where one or more parameter(s) are added to the baseline distribution. Gleaton and Lynch (2006) introduced a new class of distribution which called Generalized Logistic family (GLL-G). The cdf and pdf of this family for any baseline cdf $G(x;\boldsymbol{\theta})$ are given by the following:

$$\begin{aligned} F(x;\alpha,\boldsymbol{\theta}) &= \int_0^{\frac{G(x;\boldsymbol{\theta})}{\bar{G}(x;\boldsymbol{\theta})}} \frac{\alpha t^{\alpha-1}}{(1+t^\alpha)^2}dt = \frac{G(x;\boldsymbol{\theta})^\alpha}{G(x;\boldsymbol{\theta})^\alpha+\bar{G}(x;\boldsymbol{\theta})^\alpha},\ \ x>0,\\ f(x;\alpha,\boldsymbol{\theta}) &= \frac{\alpha g(x;\boldsymbol{\theta})G(x;\boldsymbol{\theta})^{\alpha-1}\bar{G}(x;\boldsymbol{\theta})^{\alpha-1}}{[G(x;\boldsymbol{\theta})^\alpha+\bar{G}(x;\boldsymbol{\theta})^\alpha]^2},\ \ x>0. \end{aligned}$$

Alizadeh et al. (2015) and Cordeiro et al. (2016) used odd log-logistic (OLL-G) instead of GLL-G, since we can obtain this family using odd transform from log-logistic distribution. The cdf and pdf of this family are obtained as follows:

$$\begin{aligned} F(x;\gamma,\delta,\boldsymbol{\theta}) &= \int_0^{\frac{G(x;\boldsymbol{\theta})^\gamma}{\bar{G}(x;\boldsymbol{\theta})^\delta}} \frac{dt}{(1+t)^2} = \frac{G(x;\boldsymbol{\theta})^\gamma}{G(x;\boldsymbol{\theta})^\gamma+\bar{G}(x;\boldsymbol{\theta})^\delta},\ \ x>0, \qquad (1)\\ f(x;\gamma,\delta,\boldsymbol{\theta}) &= \frac{g(x;\boldsymbol{\theta})G(x;\boldsymbol{\theta})^{\gamma-1}\bar{G}(x;\boldsymbol{\theta})^{\delta-1}\left[\gamma+(\delta-\gamma)G(x;\boldsymbol{\theta})\right]}{\left[G(x;\boldsymbol{\theta})^\gamma+\bar{G}(x;\boldsymbol{\theta})^\delta\right]^2},\ \ x>0, \end{aligned}$$

where γ and δ are two shape parameters, $\boldsymbol{\theta}$ is the vector of parameters for baseline cdf G and $g(x;\boldsymbol{\theta}) = \frac{dG(x;\boldsymbol{\theta})}{dx}$. By inserting $G(x;\boldsymbol{\theta}) = 1-e^{\lambda(1-e^{x^\beta})}$ as Chen cdf for any $x>0$ and $\lambda,\beta>0$ in (1), we propose a new distribution called new odd log-logistic Chen distribution. The cdf of this distribution is equal to

$$F(x;\gamma,\delta,\lambda,\beta) = \frac{\left[1-e^{\lambda(1-e^{x^\beta})}\right]^\gamma}{\left[1-e^{\lambda(1-e^{x^\beta})}\right]^\gamma + e^{\delta\lambda(1-e^{x^\beta})}},\ \ x>0, \qquad (2)$$

where $x>0$ and $\gamma,\delta,\lambda,\beta>0$. A random variable X with the cdf (2), is denoted by $X\sim$ NOLL-Ch$(\gamma,\delta,\lambda,\beta)$. The corresponding pdf of NOLL-Ch distribution is given by

$$\begin{aligned} f(x;\gamma,\delta,\lambda,\beta) &= \frac{\lambda\beta x^{\beta-1}e^{x^\beta}e^{\lambda(1-e^{x^\beta})}\left[1-e^{\lambda(1-e^{x^\beta})}\right]^{\gamma-1}e^{(\delta-1)\lambda(1-e^{x^\beta})}}{\left\{\left[1-e^{\lambda(1-e^{x^\beta})}\right]^\gamma+e^{\delta\lambda(1-e^{x^\beta})}\right\}^2}\\ &\quad\times\left\{\gamma+(\delta-\gamma)\left[1-e^{\lambda(1-e^{x^\beta})}\right]\right\}\ \ x>0. \qquad (3) \end{aligned}$$

A significant amount of researches have been attributed towards developing the Odd Log-Logistic family of lifetime distributions. In 2015 and 2016, Cordeiro et al. introduced the Zografos-Balakrishnan odd log-logistic and the beta odd log-logistic generalized distributions and studied some mathematical properties of these distributions. Cordeiro et al. (2017a,b) defined two new classes of continuous distributions named the odd log-logistic generalized half-normal distribution and the generalized odd log-logistic family of distributions, respectively with a discussion on some properties of these families. A new three parameters model called the odd loglogistic normal (OLLN) distribution defined and studied by Da Silva et al. (2016). Ozel et al. (2017) defined two lifetime models called the odd log-logistic Lindley (OLL-L) and odd log-logistic Lindley Poisson (OLL-LP) distributions with various hazard rate shapes. Haghbin et al. (2016) introduced and derived general mathematical properties of a new generator of continuous distributions, called the new generalized odd log-logistic family of distributions.

In this paper, we propose an extension of the EC distribution and discuss distributional properties of this distribution, including survival and hazard rate functions, moments, moment generating function and order statistics. We estimate the model parameters by different estimation procedures. Real data sets are used to illustrate the potentiality of the proposed family.

The paper is organized as follows: In Section 2, Shape characteristics of pdf and hazard rate function (hrf) of the new distribution are investigated. Also, statistical properties of the proposed distribution include moments, conditional moments, skewness, kurtosis, entropy and order statistics are derived and studied. In Section 3, estimation of the model parameters by maximum likelihood, least square, Cram'er-von Mises, Anderson Darling and right-tailed Anderson Darling estimators are presented. Simulation study is investigated in Section 4. In Section 5, applications to real data sets illustrate the performance of the new family. The paper is concluded in Section 6.

2. Statistical Properties

In this section, we introduce the new distribution and present shapes of the pdf and hrf. We also derive some useful expansions of this distribution and discuss some structural properties.

2.1. Survival and hazard rate functions

The survival function (sf) and the hrf of NOLL-Ch distribution are equal as follows:

$$S(x;\gamma,\delta,\lambda,\beta) = \frac{e^{\delta\lambda(1-e^{x^\beta})}}{\left[1-e^{\lambda(1-e^{x^\beta})}\right]^\gamma + e^{\delta\lambda(1-e^{x^\beta})}},$$

$$h(x;\gamma,\delta,\lambda,\beta) = \frac{\lambda\beta x^{\beta-1}e^{x^\beta}\left[1-e^{\lambda(1-e^{x^\beta})}\right]^{\gamma-1}\left\{\gamma+(\delta-\gamma)\left[1-e^{\lambda(1-e^{x^\beta})}\right]\right\}}{\left\{\left[1-e^{\lambda(1-e^{x^\beta})}\right]^\gamma + e^{\delta\lambda(1-e^{x^\beta})}\right\}}.$$

Figure 1 and Figure 2 provide the pdf and the hrf of NOLL-Ch$(\gamma,\delta,\lambda,\beta)$ for different parameter values.

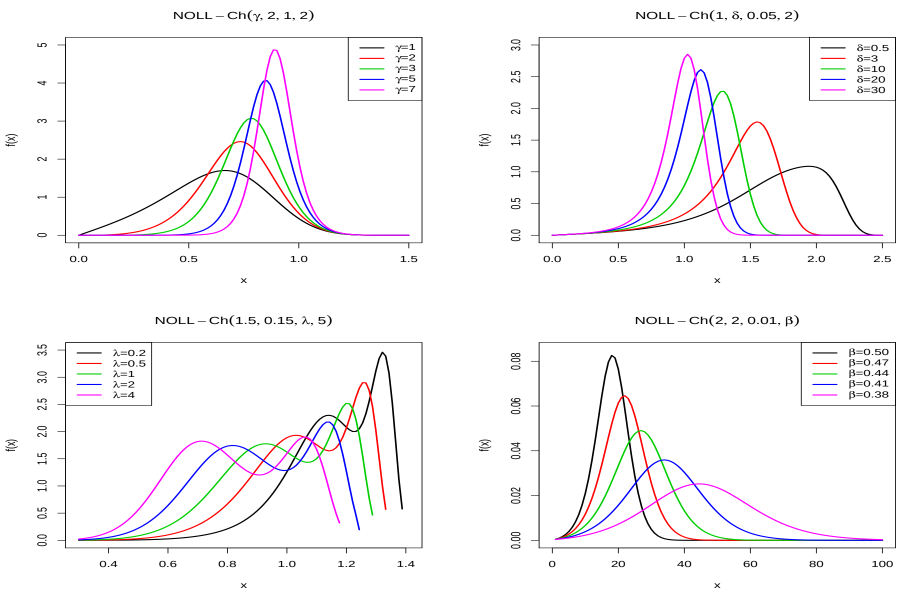

Figure 1 The sample curves of pdf of NOLL-Ch

Special cases: The NOLL-Ch$(\gamma,\delta,\lambda,\beta)$ distribution contains as special sub-models the following well-known distributions:

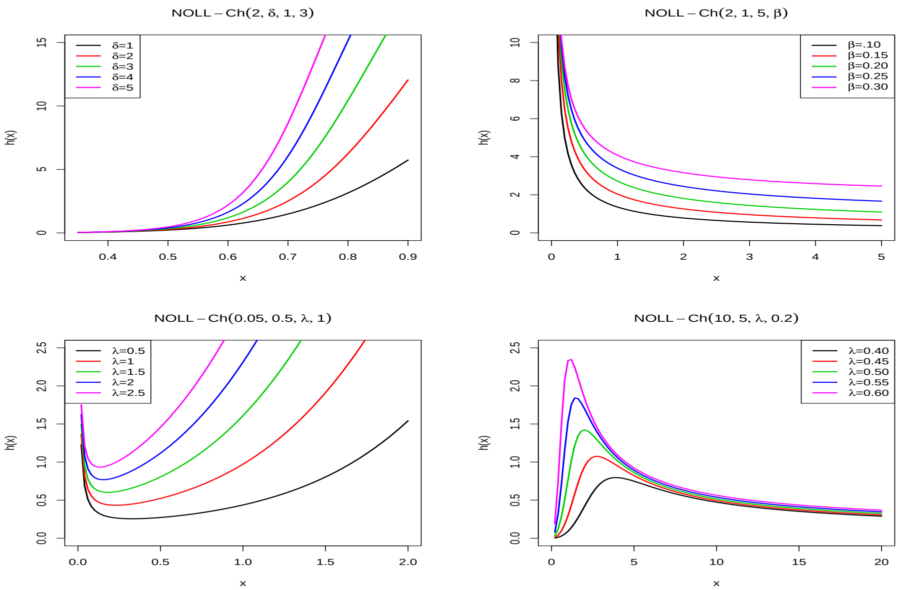

Figure 2 The sample curves of hrf function of NOLL-Ch

- For $\gamma = \delta$, we obtain odd log-logistic Chen (OLL-Ch).
- For $\gamma = \delta = 1$, we obtain Chen distribution.

In Figure 3 some pdfs for above special cases of NOLL-Ch have been drown.

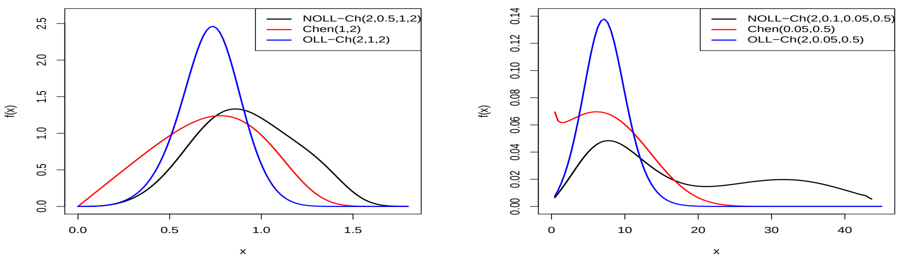

Figure 3 The sample curves of pdf in special cases

Table 1 illustrates that additional parameters add more right tail probability for NOLL-Ch distribution with respect to Chen and OLL-Ch distributions with same λ and β parameters for special cases in Figure 3.

Table 1 Right tail probabilities of Chen, NOLL-Ch and OLL-Ch distributions

Model	$P(X > 20)$	Model	$P(X > 1.25)$
NOLL-Ch(2,0.1,0.05,0.5)	0.39984	NOLL-Ch(2,0.5,1,2)	0.13720
Chen(0.05,0.5)	0.01320	Chen(1,2)	0.02304
OLL-Ch(2,0.05,0.5)	0.00018	OLL-Ch(2,1,2)	0.00056

2.2. Asymptotic

The following results can be easily obtained from the cdf properties. The asymptotic of cdf, pdf, and hrf of NOLL-Ch as $x \to 0$ are given by

$$\begin{aligned} F(x) &\sim (\lambda x^{\beta})^{\gamma}, && as\ x \to 0, \\ f(x) &\sim \gamma\beta\lambda^{\gamma} x^{\beta\gamma-1}, && as\ x \to 0, \\ h(x) &\sim \gamma\beta\lambda^{\gamma} x^{\beta\gamma-1}, && as\ x \to 0. \end{aligned}$$

The asymptotic of cdf, pdf, and hrf of NOLL-Ch as $x \to \infty$ are given by

$$\begin{aligned} 1 - F(x) &\sim e^{-\delta\lambda e^{x^{\beta}}}, && as\ x \to \infty, \\ f(x) &\sim \beta\delta\lambda x^{\beta-1} e^{x^{\beta}} e^{-\delta\lambda e^{x^{\beta}}}, && as\ x \to \infty, \\ h(x) &\sim \beta\delta\lambda x^{\beta-1} e^{x^{\beta}}, && as\ x \to \infty. \end{aligned}$$

2.3. Mixture representations for pdf and cdf

We show that the NOLL-Ch distribution can be viewed as a mixture of EC distributions. First using Generalized binomial expansion for any $|u| < 1$ and $\gamma > 0$, we can write as the following:

$$\begin{aligned} u^{\gamma} = [1 - (1-u)]^{\gamma} &= \sum_{i=0}^{\infty} (-1)^i \binom{\gamma}{i} (1-u)^i \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} (-1)^{i+k} \binom{\gamma}{i} \binom{i}{k} u^k \\ &= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} (-1)^{i+k} \binom{\gamma}{i} \binom{i}{k} u^k = \sum_{k=0}^{\infty} a_k u^k \end{aligned}$$

where $a_k = \sum_{i=k}^{\infty} (-1)^{i+k} \binom{\gamma}{i} \binom{i}{k}$. Therefore

$$\begin{aligned} \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^{\gamma} &= \sum_{k=0}^{\infty} a_k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k, \\ \left[e^{\lambda(1-e^{x^{\beta}})}\right]^{\delta} &= \left[1 - \left(1 - e^{\lambda(1-e^{x^{\beta}})}\right)\right]^{\delta} = \sum_{k=0}^{\infty} \binom{\delta}{k} (-1)^k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k. \end{aligned}$$

So, we have

$$\left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^{\gamma} + e^{\delta\lambda(1-e^{x^{b}})} = \sum_{k=0}^{\infty} b_k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k$$

where $b_k = a_k + \binom{\delta}{k}(-1)^k$. By using the ratio of two power series, we obtain as follows:

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k}{\sum_{k=0}^{\infty} b_k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k} = \sum_{k=0}^{\infty} c_k \left[1 - e^{\lambda(1-e^{x^{\beta}})}\right]^k,$$

where $c_0 = \frac{a_0}{b_0}$ and for any $k \geq 1$,

$$c_k = \frac{1}{b_0} \left[a_k - \frac{1}{b_0} \sum_{r=1}^{k} b_r c_{k-r}\right]. \tag{4}$$

So we have as the following:

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} c_k\, F_{EC}(x;\lambda,\beta,k), \\ f(x) &= \sum_{k=0}^{\infty} c_{k+1}\, f_{EC}(x;\lambda,\beta,k+1), \end{aligned}$$

where $F_{EC}(x;\lambda,\beta,k)$ and $f_{EC}(x;\lambda,\beta,k)$ denote the cdf and the pdf of EC distribution with parameters λ, β and k.

2.4. Moments

Let X be a random variable following NOLL-Ch distribution with parameters γ, δ, λ and β. We define and compute

$$A(a,b,c,r;\theta) = \int_0^{\infty} x^r x^{\theta-1} e^{ax^\theta} e^{b(1-e^{ax^\theta})} \left(1-e^{b(1-e^{ax^\theta})}\right)^c dx,$$

for $a, b > 0$ and $c > -1$. By substituting $x^r = \left(x^\theta\right)^{\frac{r}{\theta}}$ and $u = 1 - e^{b(1-e^{ax^\theta})}$ and using the expanding of function $\log\left[1 - \frac{1}{b}\log(1-u)\right]$, we have

$$A(a,b,c,r;\theta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i\left(\frac{r}{\theta}\right) a_j\left(\frac{r}{\theta}+i\right) \frac{(-1)^{\frac{2r}{\theta}+i}}{a^{\frac{r}{\theta}+1} b^{\frac{r}{\theta}+i+1}[(i+j+c+1)\theta+r]},$$

where $a_i\left(\frac{r}{\theta}\right)$ is the coefficient of $\left[\frac{1}{b}\log(1-u)\right]^{\frac{r}{\theta}+i}$ in the expansion of $\left[\sum_{l=1}^{\infty}\frac{(\frac{1}{b}\log(1-u))^l}{l}\right]^{\frac{r}{\theta}}$ and $a_j\left(\frac{r}{\theta}+i\right)$ is the coefficient of $u^{i+j+\frac{r}{\theta}}$ in the expansion of $\left(\sum_{k=1}^{\infty}\frac{u^k}{k}\right)^{\frac{r}{\theta}+i}$. (For more details see Dey et al. 2017).
Next, the n-th moment of the NOLL-Ch distribution will be

$$E(X^n) = \lambda\beta \sum_{k=0}^{\infty}(k+1)c_{k+1} A(1,\lambda,k,n;\beta).$$

The following theorem give the required condition for convergence of series.

Theorem 1 If distribution $G(x)$ has a moment generating function, then distribution function $F(x) = \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\beta}$ has a moment generating function.

Proof: Let $m = \inf\{x | G(x) \geq 0.5\}$, then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}}\,\mathrm{f(x)dx} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\mathrm{g(x)G(x)}^{\alpha-1}\bar{\mathrm{G}}(\mathrm{x})^{\beta-1}[\alpha+(\beta-\alpha)\mathrm{G(x)}]}{[\mathrm{G(x)}^\alpha + \bar{\mathrm{G}}(\mathrm{x})^\beta]^2}\,\mathrm{dx} \\ &\leq \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\,\mathrm{g(x)}}{[\mathrm{G(x)}^\alpha + \bar{\mathrm{G}}(\mathrm{x})^\beta]^2}\,\mathrm{dx} \\ &= \int_{-\infty}^{m} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\,\mathrm{g(x)}}{[\mathrm{G(x)}^\alpha + \bar{\mathrm{G}}(\mathrm{x})^\beta]^2}\,\mathrm{dx} + \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\,\mathrm{g(x)}}{[\mathrm{G(x)}^\alpha + \bar{\mathrm{G}}(\mathrm{x})^\beta]^2}\,\mathrm{dx} \end{aligned}$$

The first integral in last line is finite, the second integral is no greater than

$$\int_m^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\,\mathrm{g(x)}}{[\mathrm{G(x)}^\alpha]^2}\,\mathrm{dx}.$$

For $x > m$, we have $G(x) \geq 0.5$, so that

$$\int_{m}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\, \mathrm{g(x)}}{[\mathrm{G(x)}^{\alpha}]^{2}}\, \mathrm{dx} < \alpha\, 2^{2\alpha} \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{t\,x}}\, \mathrm{g(x)dx} < \infty.$$

Then $M_X(t) < \infty$.

Corollary 1 Every distribution in NOLL-G class has exactly the same number of moments of $G(x)$.

The measures of skewness and kurtosis of the NOLL-Ch distribution can be obtained as follows:

$$Skewness(X) = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(\mu_2' - (\mu_1')^2)^{\frac{3}{2}}},$$

$$Kurtosis(X) = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_3' - 3(\mu_1')^4}{\mu_2' - (\mu_1')^2}.$$

respectively, where $\mu_n' = E(X^n)$ for integer values of n. Moreover, the moment generating function of NOLL-Ch distribution is equal to

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \lambda\beta \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^r}{r!}(k+1)\, c_{k+1}\, A(1, \lambda, k, r; \beta).$$

In Figure 4 some skewness and kurtosis for NOLL-Ch have been drown.

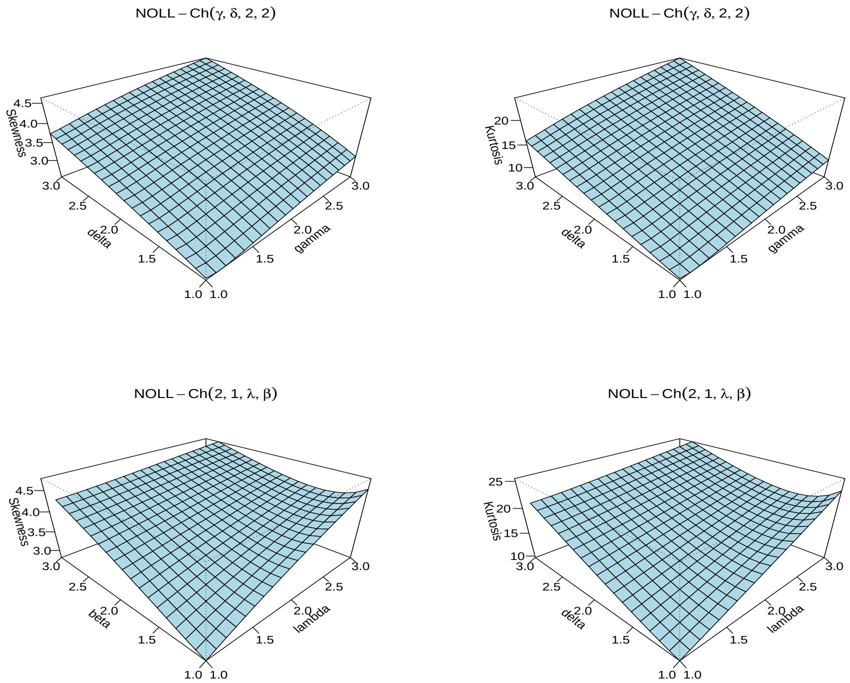

Figure 4 Skewness and Kurtosis for NOLL-Ch

2.5. Conditional moments

Here, we define and compute the following equation for the conditional moments

$$B(a,b,c,r,t;\theta)=\int_t^{\infty} x^r x^{\theta-1}e^{ax^\theta}e^{b(1-e^{ax^\theta})}\left(1-e^{b(1-e^{ax^\theta})}\right)^c dx,$$

for $a,b>0$ and $c>-1$. Then, one can obtain

$$B(a,b,c,r,t;\theta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i\binom{r}{\theta}a_j\binom{\frac{r}{\theta}+i}{}\frac{(-1)^{\frac{2r}{\theta}+i}\left\{1-\left[1-e^{b(1-e^{at^\theta})}\right]^{\frac{(i+j+c+1)\theta+r}{\theta}}\right\}}{a^{\frac{r}{\theta}+1}b^{\frac{r}{\theta}+i+1}[(i+j+c+1)\theta+r]}.$$

So, the n-th conditional moments of X obtained as follows:

$$E(X^n|X>x)=\frac{\lambda\beta\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,B(1,\lambda,k,n,x;\beta)}{1-\sum_{k=0}^{\infty}c_k\,V^k(x)},$$

where $V(x)=1-e^{\lambda(1-e^{x^\beta})}$.

In a real life situation, we sometimes need the conditional moments of kind $E(X^n|X\le x)$. Therefore, in the following we give an expression for these moments. For $a,b>0$ and $c>-1$, we define

$$C(a,b,c,r,t;\theta)=\int_0^{t} x^r x^{\theta-1}e^{ax^\theta}e^{b(1-e^{ax^\theta})}\left(1-e^{b(1-e^{ax^\theta})}\right)^c dx.$$

Then we have

$$C(a,b,c,r,t;\theta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i\binom{r}{\theta}a_j\binom{\frac{r}{\theta}+i}{}\frac{(-1)^{\frac{2r}{\theta}+i}\left\{1-e^{b(1-e^{at^\theta})}\right\}^{\frac{(i+j+c+1)\theta+r}{\theta}}}{a^{\frac{r}{\theta}+1}\,b^{\frac{r}{\theta}+i+1}\,[(i+j+c+1)\theta+r]}.$$

Therefore we can write

$$E(X^n|X\le x)=\frac{\lambda\beta\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,C(1,\lambda,k,n,x;\beta)}{\sum_{k=0}^{\infty}c_k\,V^k(x)}.$$

2.6. Mean deviations

The mean deviations can be used as a measure of spread in a population. The mean deviations about the mean and about the median are given by the following:

$$\begin{aligned}D(\mu)&=\int_0^{\infty}|x-\mu|f(x)dx=2\mu F(\mu)-2\int_0^{\mu}xf(x)dx,\\ D(m)&=\int_0^{\infty}|x-m|f(x)dx=\mu-2\int_0^{m}xf(x)dx,\end{aligned}$$

respectively, where $\mu=E(X)$ and $m=median(X)$. These quantities can be calculated as

$$\begin{aligned}D(\mu)&=2\mu F(\mu)-2\lambda\beta\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,C(1,\lambda,k,1,\mu;\beta),\\ D(m)&=\mu-2\lambda\beta\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,C(1,\lambda,k,1,m;\beta).\end{aligned}$$

2.7. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves have applications in several fields such as economics, reliability, demography, insurance and medicine. If $X \sim$NOLL-Ch$(\gamma, \delta, \lambda, \beta)$, then one can obtain

$$\begin{aligned} B(F(x)) &= \frac{1}{\mu F(x)} \int_0^x t f(t) dt = \frac{\lambda \beta}{\mu F(x)} \sum_{k=0}^{\infty} (k+1)\, c_{k+1}\, C(1, \lambda, k, 1, x; \beta), \\ L(F(x)) &= \frac{1}{\mu} \int_0^x t f(t) dt = \frac{\lambda \beta}{\mu} \sum_{k=0}^{\infty} (k+1)\, c_{k+1}\, C(1, \lambda, k, 1, x; \beta). \end{aligned}$$

2.8. Entropy

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are due to Renyi (1961) and Shannon (1948). The Renyi entropy of a random variable with pdf $f(x)$ is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left(\int_0^{\infty} f^{\gamma}(x) dx \right),$$

for $\gamma > 0$ and $\gamma \neq 1$. In Figure 5 one can see some curves of the Renyi entropy function of the NOLL-Ch distribution for some parameters.

The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the special case

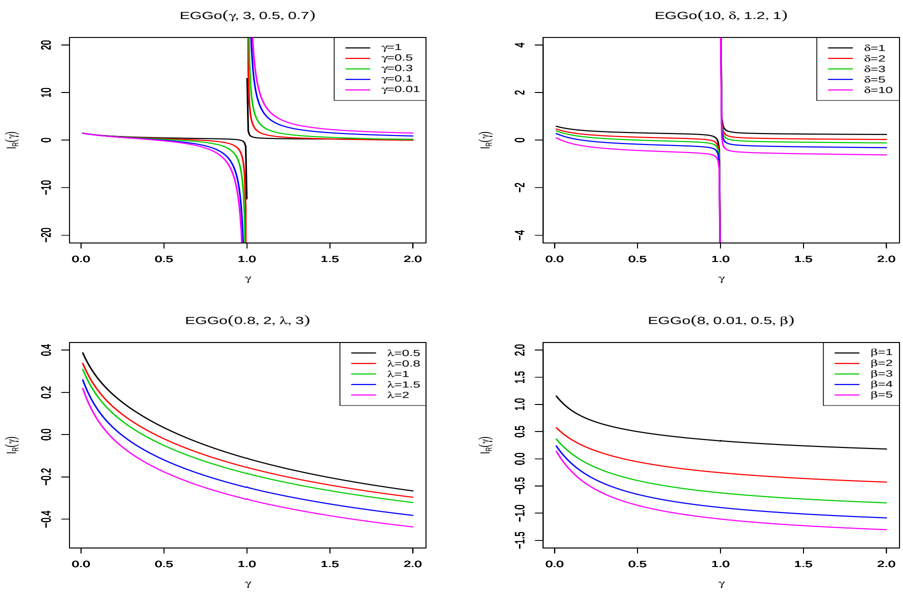

Figure 5 Plots of Renyi entropy of NOLL-Ch distribution for parameters

of the Renyi entropy when $\gamma \uparrow 1$.

We tend to derive an expression for the Shannon entropy of the NOLL-Ch distribution. If X is a non-negative continuous random variable with pdf $f(x)$, then Shannon's entropy of X is defined as

$$H(X) = E[-\ln f(X)] = -\int_0^{\infty} f(x) \ln(f(x)) dx.$$

Here, we consider the Shannon's entropy for NOLL-Ch distribution as follows:

$$\begin{aligned} H(X) &= -E\{\log g(X)]\} + (1-\gamma)E\{\log G(X)\} + (1-\delta)E\{\log \bar{G}(X)\} \\ &- E\{\log[\gamma + (\delta-\gamma)G(X)]\} + 2E\{\log[G(X)^{\gamma} + \bar{G}(X)^{\delta}]\}. \end{aligned}$$

Let $$D(a_1, a_2, a_3, a_4; \gamma, \delta) = \int_0^1 \frac{u^{a_1}(1-u)^{a_2}[\gamma + (\delta-\gamma)u]^{a_3}}{[u^{\gamma} + (1-u)^{\delta}]^{a_4}} du.$$

Using binomial expansion, one can obtain

$$\begin{aligned} D(a_1, a_2, a_3, a_4; \gamma, \delta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{-a_4}{i}\binom{a_3}{j}\binom{a_2 + i\delta}{k} \\ &\times \frac{(-1)^k \gamma^{a_3-j}(\delta-\gamma)^j}{a_1 + j - \gamma(i + a_4) + k + 1}. \end{aligned} \tag{5}$$

Using (5), we obtain the following theorem.

Theorem 2 Suppose $X \sim NOLL - Ch(\gamma, \delta, \lambda, \beta)$. Then

$$\begin{aligned} E\{\log G(X)\} &= \frac{\partial}{\partial t} D(\gamma - 1 + t, \delta - 1, 1, 2; \gamma, \delta)\Big|_{t=0}, \\ E\{\log \bar{G}(X)\} &= \frac{\partial}{\partial t} D(\gamma - 1, \delta - 1 + t, 1, 2; \gamma, \delta)\Big|_{t=0}, \\ E\{\log[\gamma + (\delta-\gamma)G(X)]\} &= \frac{\partial}{\partial t} D(\gamma - 1, \delta - 1, 1 + t, 2; \gamma, \delta)\Big|_{t=0}, \\ E\{\log[G(X)^{\gamma} + \bar{G}(X)^{\delta}]\} &= \frac{\partial}{\partial t} D(\alpha - 1, \delta - 1, 1, -t + 2; \gamma, \delta)\Big|_{t=0}, \\ E\{\log g(X)\} &= \log(\lambda\beta) + (\beta - 1)E(\log X) + E(X^{\beta}) + \lambda E(1 - e^{X^{\beta}}), \end{aligned}$$

where

$$\begin{aligned} E(\log X) &= \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k(j)a_l(k+j)\binom{i}{j}\frac{(-1)^{3j+k+1}}{i\lambda^{k+j}} \\ &\times D(j + k + l + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta), \\ E(X^{\beta}) &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_j(i)\frac{(-1)^{i+1}}{i\lambda^i} D(i + j + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta), \\ E(1 - e^{X^{\beta}}) &= -\frac{1}{\lambda}\sum_{i=1}^{\infty}\frac{1}{i} D(i + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta). \end{aligned}$$

Therefore, the Shannon entropy is obtained as follows:

$$\begin{aligned} H(X) &= -\log(\lambda\beta) + \frac{\beta - 1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k(j)a_l(k+j)\binom{i}{j}\frac{(-1)^{3j+k+1}}{i\lambda^{k+j}} \\ &\quad D(j + k + l + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta) + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a_j(i)\frac{(-1)^{i+1}}{i\lambda^i} \\ &\quad D(i + j + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta) - \sum_{i=1}^{\infty}\frac{1}{i} D(i + \gamma - 1, \delta - 1, 1, 2; \gamma, \delta) \\ &\quad + (1-\gamma)\frac{\partial}{\partial t} D(\gamma - 1 + t, \delta - 1, 1, 2; \gamma, \delta)\Big|_{t=0} \end{aligned}$$

$$+(1-\delta)\frac{\partial}{\partial t}D(\gamma-1,\delta-1+t,1,2;\gamma,\delta)\Big|_{t=0}$$
$$-\frac{\partial}{\partial t}D(\gamma-1,\delta-1,1+t,2;\gamma,\delta)\Big|_{t=0}$$
$$+2\frac{\partial}{\partial t}D(\gamma-1,\delta-1,1,-t+2;\gamma,\delta)\Big|_{t=0}.$$

2.9. Order statistics

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from (3). The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}F^{i-1}(x)(1-F(x))^{n-i}f(x) \\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F^{j+i-1}(x)f(x) \end{aligned}$$

where $B(i,n-i+1)$ is the Beta function. We use an equation by Gradshteyn and Ryzhik (2007), page 17, for a power series raised to a positive integer n, $(n \geq 1)$

$$\left(\sum_{i=0}^{\infty}a_iu^i\right)^n = \sum_{i=0}^{\infty}d_{n,i}u^i, \tag{6}$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \cdots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (ia_0)^{-1}\sum_{m=1}^{i}\left[m(n+1)-i\right]a_m d_{n,i-m}.$$

Using (6), the pdf of the i-th order statistic of any NOLL-Ch distribution is obtained as follows:

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} e_{r,k}\, f_{EC}(x;\lambda,\beta,r+k+1)$$

where $f_{EC}(x;\lambda,\beta,r+k+1)$ denotes the pdf of EC distribution with parameters λ, β and $r+k+1$ and

$$e_{r,k} = \frac{n!(r+1)(i-1)!\,c_{r+1}}{(r+k+1)}\sum_{j=0}^{n-i}\frac{(-1)^j w_{j+i-1,k}}{(n-i-j)!j!}.$$

Here the quantities $w_{j+i-1,k}$ can be determined given that $w_{j+i-1,0} = c_0^{j+i-1}$ and recursively we have

$$w_{j+i-1,k} = (kc_0)^{-1}\sum_{m=1}^{k}\left[m(j+i)-k\right]c_m\, w_{j+i-1,k-m},\ k \geq 1,$$

and c_r is given by (4). Therefore the pdf of the i^{th} order statistic from NOLL-Ch is a linear combination of EPL distributions. So, some of mathematical quantities of these order statistics such as moments, moment generating function, mean deviations and so on can be derived using this result.

The m-th moment of $X_{i:n}$ can be written as

$$E(X_{i:n}^m) = \lambda\beta\sum_{r,k=0}^{\infty}(r+k+1)\,e_{r,k}\,A(1,\lambda,r+k,m;\beta).$$

3. Estimation

There are several approaches to estimate the parameters of distributions that each of them has characteristic features and benefits. In this section, four of these methods are briefly introduced and will be numerically investigated in the simulation study. For more details, interested readers can refer to Dey et al. (2017).

3.1. Maximum-likelihood estimators

Here, the maximum likelihood estimates (MLEs) of the parameters of NOLL-Ch distribution are determined. Let $X_1, \cdots, X_n$ is a random sample from NOLL-Ch model with unknown parameters γ, δ, λ and β. The log likelihood function based on observed random sample of size n is given by

$$\begin{aligned} l(\gamma,\delta,\lambda,\beta;x) &= n\log(\lambda\beta)+(\beta-1)\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}x_i^{\beta}+\lambda(n-\sum_{i=1}^{n}e^{x_i^{\beta}}) \\ &\quad +(\gamma-1)\sum_{i=1}^{n}\log t_i+(\delta-1)\sum_{i=1}^{n}\log(1-t_i) \\ &\quad +\sum_{i=1}^{n}\log[\gamma+(\delta-\gamma)t_i]-2\sum_{i=1}^{n}\log[t_i^{\gamma}+(1-t_i)^{\delta}], \end{aligned}$$

where $t_i = G(x_i) = 1 - e^{\lambda(1-e^{x_i^{\beta}})}$. Therefore

$$\begin{aligned} \frac{\partial l}{\partial\gamma} &= \sum_{i=1}^{n}\log t_i+\sum_{i=1}^{n}\frac{1-t_i}{\gamma+(\delta-\gamma)t_i}-2\sum_{i=1}^{n}\frac{(\log t_i)t_i^{\gamma}}{t_i^{\gamma}+(1-t_i)^{\delta}}=0, \\ \frac{\partial l}{\partial\delta} &= \sum_{i=1}^{n}\log(1-t_i)+\sum_{i=1}^{n}\frac{t_i}{\gamma+(\delta-\gamma)t_i}-2\sum_{i=1}^{n}\frac{(\log(1-t_i))(1-t_i)^{\delta}}{t_i^{\gamma}+(1-t_i)^{\delta}}=0, \\ \frac{\partial l}{\partial\lambda} &= \frac{n}{\lambda}+\left(n-\sum_{i=1}^{n}e^{x_i^{\beta}}\right)+\sum_{i=1}^{n}\frac{t_i^{(\lambda)}[\gamma-1-(\gamma+\delta-2)t_i]}{t_i(1-t_i)} \\ &\quad +(\delta-\gamma)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{\gamma+(\delta-\gamma)t_i}-2\sum_{i=1}^{n}\frac{t_i^{(\lambda)}\left[\gamma t_i^{\gamma-1}-\delta(1-t_i)^{\delta-1}\right]}{t_i^{\gamma}+(1-t_i)^{\delta}}=0, \\ \frac{\partial l}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^{n}\log x_i\left[1+x_i^{\beta}(1-\lambda e^{x_i^{\beta}})\right]+\sum_{i=1}^{n}\frac{t_i^{(\beta)}[\gamma-1-(\gamma+\delta-2)t_i]}{t_i(1-t_i)} \\ &\quad +(\delta-\gamma)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{\gamma+(\delta-\gamma)t_i}-2\sum_{i=1}^{n}\frac{t_i^{(\beta)}\left[\gamma t_i^{\gamma-1}-\delta(1-t_i)^{\delta-1}\right]}{t_i^{\gamma}+(1-t_i)^{\delta}}=0, \end{aligned}$$

where
$$t_i^{(\lambda)} = \frac{\partial t_i}{\partial\lambda} = -\left(1-e^{x_i^{\beta}}\right)e^{\lambda(1-e^{x_i^{\beta}})} \quad (7)$$
$$t_i^{(\beta)} = \frac{\partial t_i}{\partial\beta} = \lambda\,(\log x_i)\,x_i^{\beta}\,e^{x_i^{\beta}}\,e^{\lambda(1-e^{x_i^{\beta}})}. \quad (8)$$

The maximum likelihood estimates $\hat{\gamma}$, $\hat{\delta}$, $\hat{\lambda}$ and $\hat{\beta}$ of γ, δ, λ and β are obtained by solving these nonlinear system of equations.

3.2. Least-Square estimators

Suppose $F(X_{i:n})$ denotes the cdf of i-th order statistic of a random sample $X_1, \cdots, X_n$. The least square estimators (LSE) of γ, δ, λ and β are obtained by minimizing the following function

$$S(\gamma,\delta,\lambda,\beta)=\sum_{i=1}^{n}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]^2,$$

with respect to γ, δ, λ and β, where $F(.)$ is defined in (2). So, the estimators obtained by solving

$$\begin{aligned}
\sum_{i=1}^{n}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]\phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]\phi_2(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]\phi_3(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]\phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,
\end{aligned}$$

$$\begin{aligned}
\text{where}\qquad \phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta) &= \frac{(\log t_i)t_i^{\gamma}(1-t_i)^{\delta}}{\left[t_i^{\gamma}+(1-t_i)^{\delta}\right]^2},\\
\phi_2(x_{i:n}|\gamma,\delta,\lambda,\beta) &= \frac{-(\log(1-t_i))t_i^{\gamma}(1-t_i)^{\delta}}{\left[t_i^{\gamma}+(1-t_i)^{\delta}\right]^2},\\
\phi_3(x_{i:n}|\gamma,\delta,\lambda,\beta) &= \frac{t_i^{(\lambda)}t_i^{\gamma-1}(1-t_i)^{\delta-1}\left[\gamma+(\delta-\gamma)t_i\right]}{\left[t_i^{\gamma}+(1-t_i)^{\delta}\right]^2},\\
\phi_4(x_{i:n}|\gamma,\delta,\lambda,\beta) &= \frac{t_i^{(\beta)}t_i^{\gamma-1}(1-t_i)^{\delta-1}\left[\gamma+(\delta-\gamma)t_i\right]}{\left[t_i^{\gamma}+(1-t_i)^{\delta}\right]^2}.
\end{aligned}\tag{9}$$

Here, $t_i^{(\lambda)}$ and $t_i^{(\beta)}$ are defined in (7) and (8).
The weighted least square estimators (WLSE) can be obtained by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{i}{n+1}\right]^2.$$

These estimators obtained by similar manner to least square estimators.

3.3. Cram'er-von-Mises estimators

The Cram'er-von Mises estimators $\hat{\gamma}_{CME}$, $\hat{\delta}_{CME}$, $\hat{\lambda}_{CME}$ and $\hat{\beta}_{CME}$ are obtained by minimizing,

$$C(\gamma,\delta,\lambda,\beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{2i-1}{2n}\right)^2,$$

with respect to the parameters γ, δ, λ and β, respectively, which is equivalent to solving the following equations

$$\sum_{i=1}^{n}\left(F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{2i-1}{2n}\right)\phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta) = 0,$$

$$\begin{aligned}
\sum_{i=1}^{n}\left(F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{2i-1}{2n}\right)\phi_2(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,\\
\sum_{i=1}^{n}\left(F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{2i-1}{2n}\right)\phi_3(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,\\
\sum_{i=1}^{n}\left(F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{2i-1}{2n}\right)\phi_4(x_{i:n}|\gamma,\delta,\lambda,\beta) &= 0,
\end{aligned}$$

where $\phi_1(.)$, $\phi_2(.)$, $\phi_3(.)$ and $\phi_4(.)$ are defined in (9).

3.4. Anderson–Darling and right-tailed Anderson–Darling estimators

The Anderson–Darling (ADE) and right-tailed Anderson–Darling estimators (RTADE) are obtained by minimizing the following functions

$$\begin{aligned}
A(\gamma,\delta,\lambda,\beta) &= -n-\frac{1}{n}\sum_{i=1}^{n}\left\{\log F(x_{i:n}|\gamma,\delta,\lambda,\beta)+\log S(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta)\right\},\\
R(\gamma,\delta,\lambda,\beta) &= \frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n}|\gamma,\delta,\lambda,\beta)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta),
\end{aligned}$$

respectively. These estimators for γ are denoted by $\hat{\gamma}_{ADE}$ and $\hat{\gamma}_{RTADE}$ and obtained by solving the following equations

$$\begin{aligned}
\sum_{i=1}^{n}(2i-1)\left[\frac{\phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta)}{F(x_{i:n}|\gamma,\delta,\lambda,\beta)}-\frac{\phi_1(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta)}{S(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta)}\right] &= 0,\\
-2\sum_{i=1}^{n}\phi_1(x_{i:n}|\gamma,\delta,\lambda,\beta)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\phi_1(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta)}{S(x_{n+1-i:n}|\gamma,\delta,\lambda,\beta)} &= 0.
\end{aligned}$$

The estimators $\hat{\delta}_{ADE}$ ($\hat{\delta}_{RTADE}$), $\hat{\lambda}_{ADE}$ ($\hat{\lambda}_{RTADE}$) and $\hat{\beta}_{ADE}$ ($\hat{\beta}_{RTADE}$), are similarly obtained.

4. Simulation Study

Here, we evaluate the performance of the different parameter estimators in terms of bias and mean square error.

4.1. The maximum likelihood estimators

In this subsection, the maximum likelihood estimators of parameters of purpose density function has been assessed by simulating: $(\gamma,\delta,\lambda,\beta)=(2,0.5,1,1)$. The pdf has been indicated in Figure 6.

To verify the validity of the maximum likelihood estimator, the bias of MLE and the mean square error of MLE have been used. For example, as described in Subsection 3.1, for $(\gamma,\delta,\lambda,\beta)=(2,0.5,1,1)$, $r=1000$ times have been simulated samples of $n=50,51,...,110$ of NOLL-Ch$(2,0.5,1,1)$. To estimate the numerical value of the maximum likelihood, the $optim$ function (in the $stat$ package) and Nelder-Mead method in R software has been used. If $\boldsymbol{\xi}=(\gamma,\delta,\lambda,\beta)$, for any simulation by n volumes and $i=1,2,...,r$, the maximum likelihood estimates are obtained as $\widehat{\boldsymbol{\xi}}_i=(\widehat{\gamma}_i,\widehat{\delta}_i,\widehat{\lambda}_i,\widehat{\beta}_i)$.

To examine the performance of the MLE's for the NOLL-Ch distribution, we perform a simulation study:

1. Generate r samples of size n from (3).

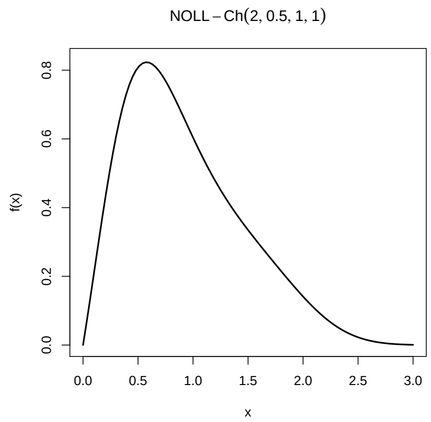

Figure 6 The pdf for simulation study

2. Compute the MLE's for the r samples, say $(\hat{\gamma}, \hat{\delta}, \hat{\lambda}, \hat{\beta})$ for $i = 1, 2, \ldots, r$.
3. Compute the standard errors of the MLE's for r samples, say $(s_{\hat{\gamma}}, s_{\hat{\delta}}, s_{\hat{\lambda}}, s_{\hat{\beta}})$ for $i = 1, 2, \ldots, r$.
4. Compute the biases and mean squared errors given by the following:

$$Bias_{\widehat{\boldsymbol{\xi}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\xi}_i} - \boldsymbol{\xi}_i),$$
$$MSE_{\widehat{\boldsymbol{\xi}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\xi}_i} - \boldsymbol{\xi}_i)^2,$$

for $\boldsymbol{\xi} = (\gamma, \delta, \lambda, \beta)$.

We repeat these steps for $r = 1000$ and $n = 50, 51, \ldots, n^*$ (n^* is different in each issue) with different values of $(\gamma, \delta, \lambda, \beta)$, so computing $Bias_{\widehat{\boldsymbol{\xi}}}(n)$ and $MSE_{\widehat{\boldsymbol{\xi}}}(n)$.

Figures 7 and 8, respectively reveals how the four biases, mean squared errors vary with respect to n. As expected, the Biases and MSEs of estimated parameters converges to zero while n growing.

4.2. The other estimation methods

In order to explore the efficiency of the estimators introduced in the previous section, we consider one model that have been used above, and investigate MSE of these estimators for different samples. For instance according to what has been mentioned, for $(\gamma, \delta, \lambda, \beta) = (2, 0.5, 1, 1)$, we have simulated r = 1000 times with sample size of the $n = 50, 55, 60, \cdots 550$. Then the MSE formula that are mentioned in the subsection 4.1 are calculated for them. To obtain the value of the estimators, we have used the optima function and Nelder-Mead method in R.

The result of the simulations of this subsection is shown in Figure 9. As it is clear from the MSE plot for two parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the parameters of NOLL-Ch distribution.

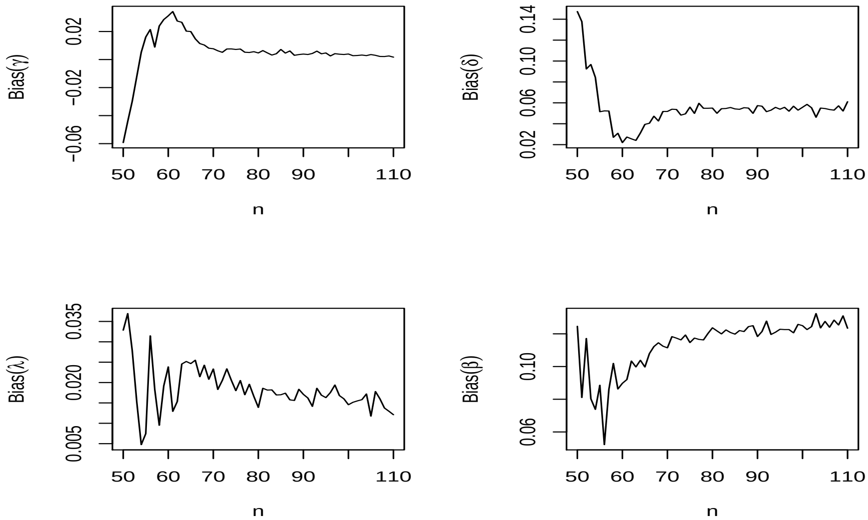

Figure 7 Bias of $(\hat{\gamma}, \hat{\delta}, \hat{\lambda}, \hat{\beta})$ for $i = 1, 2, \ldots, r$ versus n when $(\gamma, \delta, \lambda, \beta) = (2, 0.5, 1, 1)$

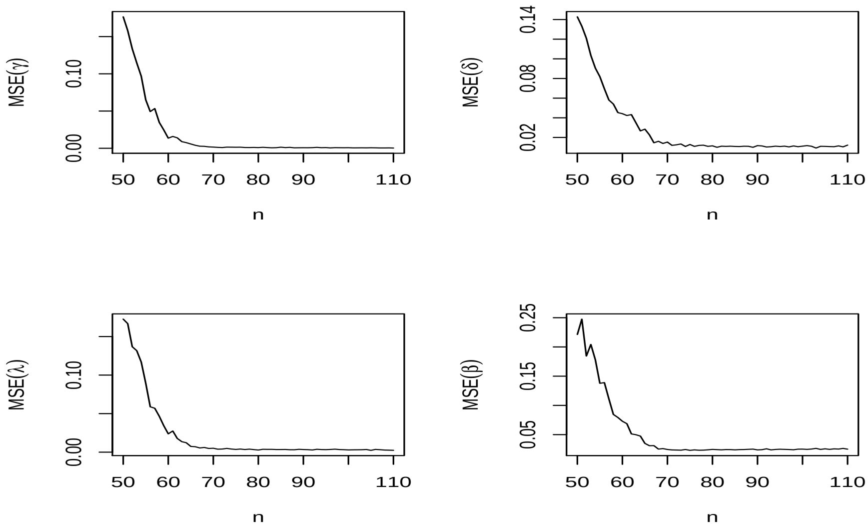

Figure 8 MSE of $(\hat{\gamma}, \hat{\delta}, \hat{\lambda}, \hat{\beta})$ for $i = 1, 2, \ldots, r$ versus n when $(\gamma, \delta, \lambda, \beta) = (2, 0.5, 1, 1)$

5. Applications

In this section, we present two applications by fitting the NOLL-Ch model and some famous models. The Akaike information criterion (AIC), Bayesian information criterion (BIC), Cramér–von Mises (W^*), Anderson-Darling (A^*), Kolmogorov Smirnov (K.S) and the P-Value of K.S test, have been chosen for comparison of models for the first two examples. For the two applications, we adopt

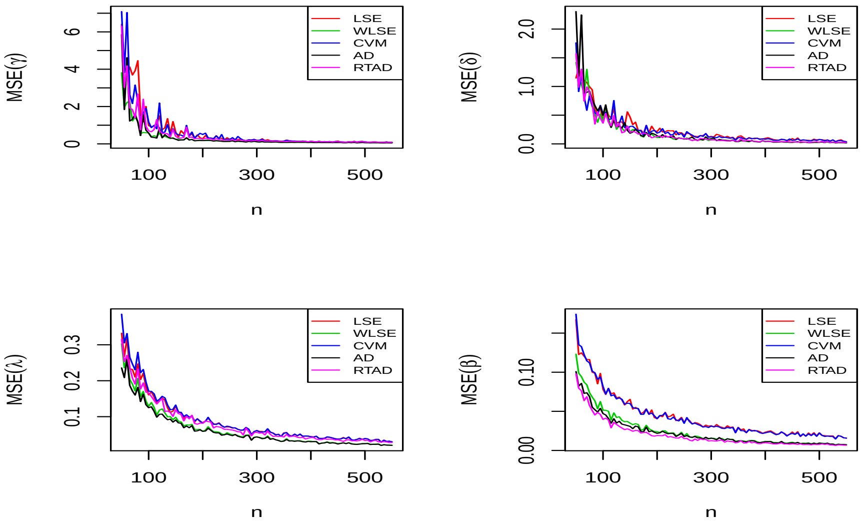

Figure 9 MSE of $(\hat{\gamma}, \hat{\delta}, \hat{\lambda}, \hat{\beta})$ versus n when $(\gamma, \delta, \lambda, \beta) = (2, 0.5, 1, 1)$

only the A* statistics.

The Gamma-Chen distribution (GaC) (Alzaatreh et al. 2014), The Beta-Chen distribution (BC) (Euguene et al. 2002), Marshall-Olkin Normal distribution (MOC) (Jose 2011), The Kumaraswamy Chen distribution (KwC) (Cordeiro and De Castro (2011)), The Transmuted Chen (TC) (Khan et al. 2013), The Transmuted Exponentiated Chen (TEC) (Khan et al. 2016), The Extended Chen (EC) (Chaubey and Zhang 2015), Odd Log-Logistic Chen (OLL-C) and Chen distribution have been selected for comparison in the two examples. The parameters of models have been estimated by the MLE method.

5.1. The relief times of twenty patients data

This subsection is related to study of the data set (Gross and Clark 1975, p. 105) on the relief times of twenty patients receiving an analgesic is 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

In the Tables 2 and 3, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. Models have been sorted from the lowest to the highest value of A*. As you see, the NOLL-Ch is selected as the best model with more criteria $(W^*, A^*, K.S, p$-value). The histogram of the relief times of twenty patients data and the plots of fitted pdf are displayed in Figure 10. In Figure 11, the plot of uni-modality of profile likelihood functions of parameters of NOLL-Ch is shown.

Table 2 The relief times of twenty patients data

Model	AIC	BIC	W^*	A^*	$K.S$	p-value
NOLL-Ch	38.32	42.31	0.02	0.14	0.09	0.996
GaC	46.35	50.33	0.03	0.20	0.99	0
TEC	39.56	43.55	0.04	0.23	0.12	0.949
EC	38.14	41.13	0.05	0.30	0.13	0.864
KwC	40.02	44.00	0.05	0.30	0.14	0.820
OLL-C	39.10	42.08	0.05	0.32	0.11	0.963
BC	40.51	44.49	0.06	0.34	0.15	0.769
MOC	44.88	47.87	0.14	0.84	0.15	0.774
TC	53.63	56.62	0.27	1.57	0.23	0.243
Chen	53.14	55.13	0.29	1.66	0.24	0.206

Table 3 Estimated MLE's and Standard errors for the relief times of twenty patients data

Model	MLE	Standard errors
NOLL-Ch($\gamma, \delta, \lambda, \beta$)	$(31.40, 0.34, 1.06, 0.65)$	$(36.25, 0.24, 0.51, 0.25)$
GaC($\gamma, \delta, \lambda, \beta$)	$(7.59, 1.99, 5.00, 0.53)$	$(2.09, 0.46, 1.07, 0.003)$
TEC($\gamma, \delta, \lambda, \beta$)	$(300.01, 0.50, 2.43, 0.34)$	$(587.04, 0.56, 1.08, 0.11)$
EC(γ, λ, β)	$(250.01, 2.40, 0.37)$	$(407.52, 0.89, 0.10)$
KwC($\gamma, \delta, \lambda, \beta$)	$(160.07, 0.49, 2.21, 0.52)$	$(222.41, 0.51, 0.75, 0.21)$
OLL-C(γ, λ, β)	$(58.59, 0.39, 0.05)$	$(120.36, 0.03, 0.09)$
BC($\gamma, \delta, \lambda, \beta$)	$(85.87, 0.48, 2.01, 0.55)$	$(103.13, 0.51, 0.69, 0.20)$
MOC(γ, λ, β)	$(400.01, 2.32, 0.43)$	$(488.06, 0.64, 0.08)$
TC(γ, λ, β)	$(0.75, 0.07, 1.02)$	$(0.28, 0.03, 0.09)$
Chen(λ, β)	$(0.14, 0.95)$	$(0.05, 0.09)$

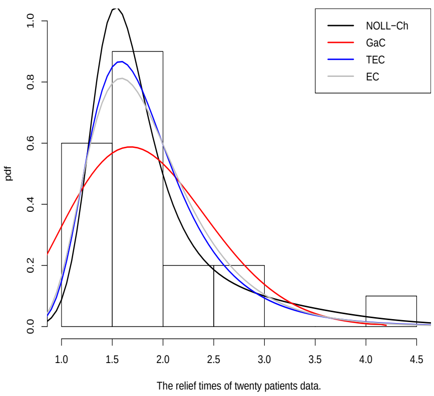

Figure 10 Histogram for the relief times of twenty patients data

5.2. Time to failure $(10^3$ h) of turbocharger

This subsection is related to study of the time to failure (turbocharger of one type of engine which presented by Xu et al. 2003 that include 40 observations. The data are 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7,

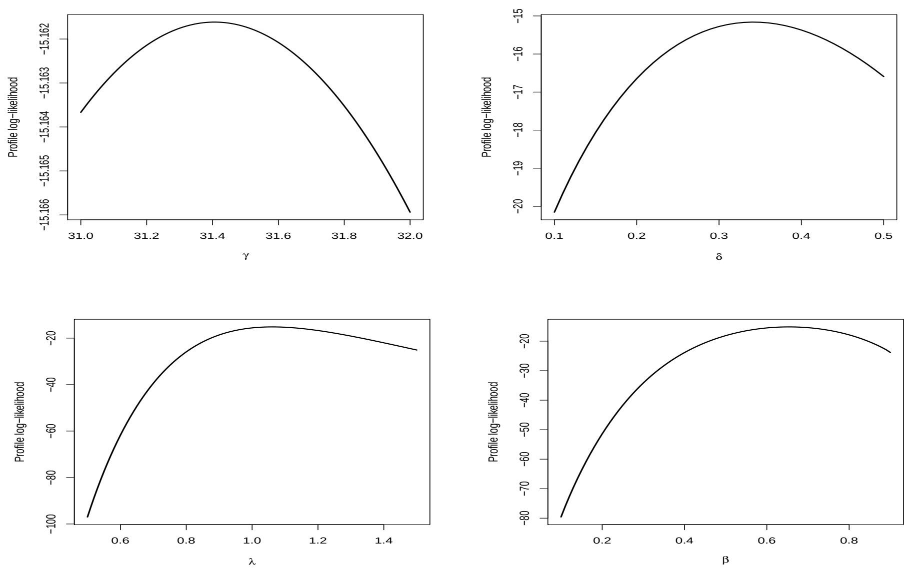

Figure 11 Uni-modality of profile likelihood functions of parameters of NOLL-Ch for the relief times of twenty patients data

7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0. Similar to the previous application example, we have Tables 4 and 5. As it is clear, the NOLL-Ch is selected as the best model with more criteria. The histogram of the Time to failure (10^3 h) of turbocharger data and the plots of fitted pdf are displayed in Figure 12. In Figure 13, the plot of uni-modality of profile likelihood functions of parameters of NOLL-Ch is shown.

Table 4 Time to failure (10^3 h) of turbocharger data

Model	**AIC**	**BIC**	**W***	A^*	$K.S$	p**-value**
NOLL-Ch	165.51	172.27	0.01	0.13	0.080	0.969
OLL-C	165.34	170.40	0.03	0.20	0.090	0.893
BC	167.51	174.27	0.03	0.22	0.080	0.964
MOC	166.16	171.23	0.03	0.24	0.070	0.923
KwC	168.14	174.89	0.03	0.24	0.090	0.920
TEC	168.20	174.95	0.03	0.24	0.092	0.891
EC	166.27	171.33	0.03	0.24	0.090	0.881
TC	166.31	171.38	0.03	0.25	0.091	0.895
Chen	164.29	167.66	0.03	0.25	0.090	0.880
GaC	168.14	174.90	0.04	0.31	0.070	0.990

Table 5 Estimated MLE's and Standard errors for the time to failure (10^3 h) of turbocharger data

Model	MLE	Standard errors
NOLL-Ch($\gamma, \delta, \lambda, \beta$)	$(0.89, 0.23, 0.003, 0.98)$	$(0.21, 0.21, 0.001, 0.06)$
OLL-C(γ, λ, β)	$(0.82, 0.004, 0.89)$	$(0.19, 0.002, 0.04)$
BC($\gamma, \delta, \lambda, \beta$)	$(0.71, 0.32, 0.01, 0.91)$	$(0.30, 0.31, 0.01, 0.07)$
MOC(γ, λ, β)	$(1.49, 0.01, 0.82)$	$(1.03, 0.01, 0.05)$
KwC($\gamma, \delta, \lambda, \beta$)	$(0.77, 0.11, 0.04, 0.87)$	$(0.07, 0.10, 0.04, 0.03)$
TEC($\gamma, \delta, \lambda, \beta$)	$(0.93, -0.25, 0.01, 0.84)$	$(0.35, 0.55, 0.01, 0.05)$
EC(γ, λ, β)	$(0.94, 0.01, 0.85)$	$(0.23, 0.003, 0.04)$
TC(γ, λ, β)	$(-0.22, 0.01, 0.83)$	$(0.44, 0.01, 0.05)$
Chen(λ, β)	$(0.01, 0.84)$	$(0.002, 0.03)$
GaC($\gamma, \delta, \lambda, \beta$)	$(0.96, 1.62, 0.01, 0.86)$	$(0.82, 5.11, 0.01, 0.20)$

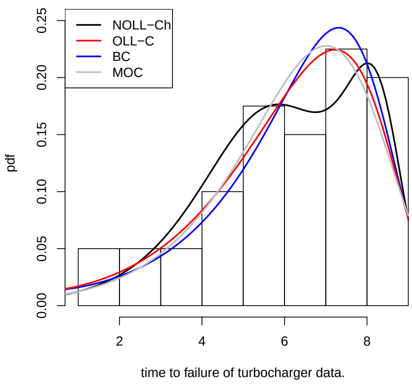

Figure 12 Histogram for time to failure (10^3 h) of turbocharger data

6. Conclusions

A new four-parameter lifetime distribution is introduced in this paper. We studied the properties of the pdf and hrf of this distribution. Several structural properties of it are discussed in details include the general n-th moments, conditional moments, mean deviations and order statistics. Moreover, we derived several estimation techniques for estimating the unknown parameters of the NOLL-Ch distribution and presented an extensive simulation study in order to compare the efficiency of these estimators. Finally, applications to real data are given to demonstrate the applicability of the new distribution in practical situations.

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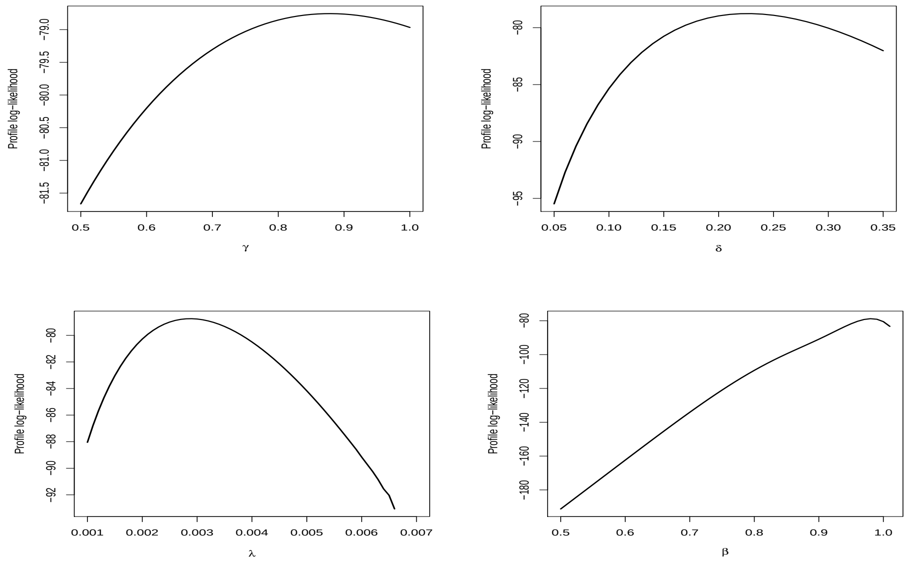

Figure 13 Uni-modality of profile likelihood functions of parameters of NOLL-Ch fortime to failure (10^3 h) of turbocharger data

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