Thailand Statistician
July 2021; 19(3): 437-449
http://statassoc.or.th
Contributed paper

Negative Binomial-Reciprocal Inverse Gaussian Distribution: Statistical Properties with Applications in Count Data

Anwar Hassan [a], Ishfaq Shah*[a] and Bilal Peer [b]
[a] Department of Statistics, University of Kashmir, Srinagar, Jammu&Kashmir, India.
[b] Department of Mathematical Sciences, Islamic University of Science and Technology, Pulwama, Jammu&Kashmir, India.
*Corresponding author; e-mail: peerishfaq007@gmail.com

Received: 3 August 2019
Revised: 27 January 2020
Accepted: 10 February 2020

Abstract

In this paper, a new count distribution has been introduced by mixing negative binomial with reciprocal inverse Guassian distribution. This model is tractable with some important properties not only limited to actuarial science but in other fields as well where over-dispersion pattern is seen. A recurrence relation for the probabilities of the new distribution and an integral equation for the probability density function of the compound version, when the claim severities are absolutely continuous, are derived. Brief idea about its respective multivariate version are also given. Parameters involved in the proposed model have been estimated by maximum likelihood estimation technique. Finally, applications of the model to real data sets are presented and compared with the fit attained by some other well-known one and two-parameter distributions.

Keywords: Over-dispersion, goodness of fit, aggregate loss, maximum likelihood estimation.

1. Introduction

The classical Poisson distribution is one of the prominent distribution to model count data. However, due to the presence of over-dispersion phenomenon in count data, one has to look for such models which relaxes over-dispersion restriction of the Poisson distribution. Negative binomial (NB) model takes care of the over-dispersion pattern. Keeping the wide applications of NB in context, there has been significant development in the extension of NB distribution, like the negative binomial-inverse Gaussian distributions (Déniz et al. 2008), the negative binomial-Beta exponential distribution (Pudprommarat et al. 2012) and the negative binomial-Erlang distribution (Kongrod et al. 2014).

Mixture approach is one of the prominent method of obtaining new probability distributions in the applied field of probability and statistics, mainly because of its simplicity and unambiguous interpretation of the unobserved heterogeneity that is likely to occur in most of practical situations. In this article a NB mixture model that includes as mixing distribution the reciprocal inverse Gaussian ($\mathcal{RIG}$) distribution is proposed by taking $\theta = exp(-\omega)$, (where θ is negative binomial parameter) assuming that ω is distributed according to a $\mathcal{RIG}$ distribution, obtaining the negative binomial-reciprocal inverse Gaussian distribution, denoted by $\mathcal{NBRIG}$, which can be viewed as a competitive model to Poisson-reciprocal inverse Gaussian ($\mathcal{PRIG}$), NB and Poisson distributions.

The new distribution is unimodal, having thick tails, positively or negatively skewed and posses over-dispersion character. Recursive expressions of probabilities are also obtained which are an important component in compound distributions particularly in collective risk model. Basically there are three parameters involved in the new distribution which have been estimated by using an important technique namely maximum likelihood estimation (MLE) and goodness of fit has been checked by using chi-square criterion.

The main contents of the paper are: In Section 2, we study some basic characteristics of the distribution like probability mass function (pmf), factorial moments and over-dispersion property. In Section 3, we study $\mathcal{NBRIG}$ as compound distribution and recurrence relation of probabilities are being discussed to compute successive probabilities. Extension of univariate to multivariate version have been discussed briefly in Section 4. Section 5 contains information about estimation of parameters by MLE. Two numerical illustrations have been discussed in Section 6 followed by conclusion in last section.

2. Basic Results

In this section we will start with classical negative binomial distribution (Johnson et al. 2005) denoted as $Y \sim \mathcal{NB}(r, \theta)$ whose pmf is:

$$P(Y = y) = \binom{r+y-1}{y}\theta^r(1-\theta)^y; \quad y = 0, 1, \ldots, \tag{1}$$

with $r > 0$ and $\theta \in (0, 1)$. Since its usage is important later, so we will discuss some important characteristics of this distribution. The first three moments about zero of $\mathcal{NB}(r, \theta)$ distribution (Balakrishnan et al., 2003) are given by:

$$\begin{aligned}\mathbb{E}(Y) &= \frac{r(1-\theta)}{\theta},\\ \mathbb{E}(Y^2) &= \frac{r(1-\theta)\left[1 + r(1-\theta)\right]}{\theta^2},\\ \mathbb{E}(Y^3) &= \frac{r(1-\theta)}{\theta^3}\left[1 + (3r+1)(1-\theta) + r^2(1-\theta)^2\right].\end{aligned}$$

Also the factorial moment of $\mathcal{NB}(r, \theta)$ of order k is:

$$\begin{aligned}\mu_{[k]}(Y) &= \mathbb{E}\left[Y(Y-1)\ldots(Y-k+1)\right]\\ &= \frac{\Gamma(r+k)}{\Gamma(r)}\frac{(1-\theta)^k}{\theta^k}, \quad k = 1, 2, \ldots,\end{aligned} \tag{2}$$

where $\Gamma(r) = (r-1)!$.

Let random variable Z has reciprocal inverse Gaussian distribution (Déniz et al., 2017) denoted as $Z \sim \mathcal{RIG}(\alpha, m)$ whose probability density function (pdf) is given by

$$g(z, \alpha, m) = \sqrt{\frac{\alpha}{2\pi z}}e^{-\frac{\alpha}{2m}\left(zm-2+\frac{1}{zm}\right)}, \quad z > 0, \tag{3}$$

where $\alpha, m > 0$. The moment generating function (mgf) of $\mathcal{RIG}(\alpha, m)$ is given by:

$$M_Z(t) = \sqrt{\frac{\alpha}{\alpha - 2t}}exp\left\{\frac{\alpha}{m^2}\left[m - \frac{m}{\sqrt{\alpha}}\sqrt{\alpha - 2t}\right]\right\}, \quad |z| < 1. \tag{4}$$

Definition 1 *A random variable Y is said to have negative binomial-reciprocal inverse Gaussian distribution if it follows the stochastic representation as:*

$$\begin{aligned}Y|\omega &\sim \mathcal{NB}(r, \theta = e^{-\omega}),\\ \omega &\sim \mathcal{RIG}(\alpha, m),\end{aligned} \tag{5}$$

where $r, \alpha, m > 0$ *and we can write* $Y \sim \mathcal{NBRIG}(r, \alpha, m)$ *and which is shown in Theorem 1.*

Theorem 1 *Let* $Y \sim \mathcal{NBRIG}(r, \alpha, m)$ *as defined in* (5) *then pmf is given by*

$$p(y) = \binom{r+y-1}{y} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \sqrt{\frac{\alpha}{\alpha + 2(r+j)}} exp \left\{ \frac{\alpha}{m^2} \left[m - \frac{m}{\sqrt{\alpha}} \sqrt{\alpha + 2(r+j)} \right] \right\}, \quad (6)$$

with $y = 0, 1, \ldots$ *and* $r, \alpha, m > 0$.

Proof: Since $Y|\omega \sim \mathcal{NB}(r, \theta = e^{-\omega})$ and $\omega \sim \mathcal{RIG}(\alpha, m)$. Then unconditional pmf of Y is given by

$$p\,(Y = y) = \int_0^\infty h_1(y|\omega) h_2(\omega; \alpha, m) d\omega \quad (7)$$

where

$$h_1(y|\omega) = \binom{r+y-1}{y} e^{-\omega r} (1 - e^{-\omega})^x$$

$$= \binom{r+y-1}{y} \sum_{j=0}^{y} \binom{y}{j} (-1)^j e^{-(r+j)} \quad (8)$$

and $h_2(\omega; \alpha, m)$ is the pdf of $\mathcal{RIG}(\alpha, m)$.

Put (8) in Equation (7), we get

$$\begin{aligned} p\,(Y = y) &= \binom{r+y-1}{y} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \int_0^\infty e^{-(r+j)} h_2(\beta; \alpha, m) d\omega \\ &= \binom{r+y-1}{y} \sum_{j=0}^{y} \binom{y}{j} (-1)^j M_\omega \left(-(r+j) \right). \quad (9) \end{aligned}$$

Use (4) in Equation (9) to get the required pmf.

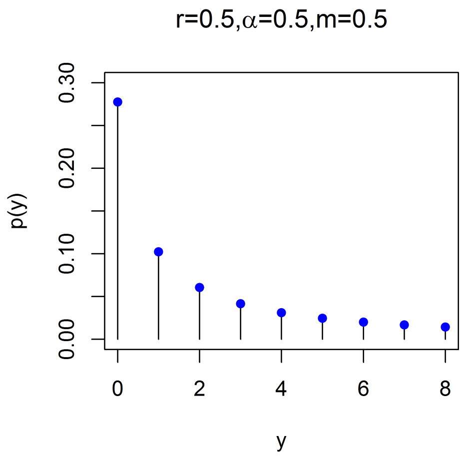

Figure 1 pmf plot for different value of parameters

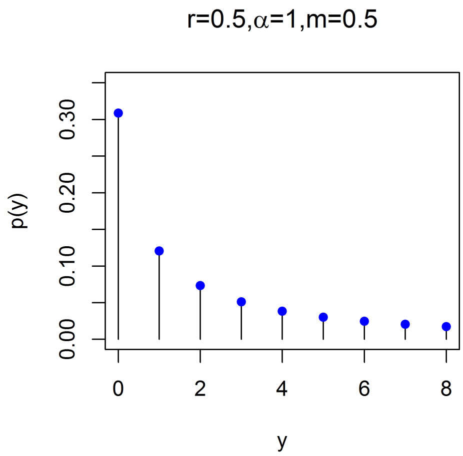

Figure 2 pmf plot for different value of parameters

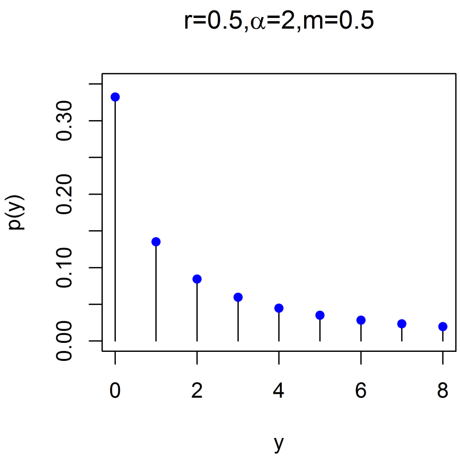

Figure 3 pmf plot for different value of parameters

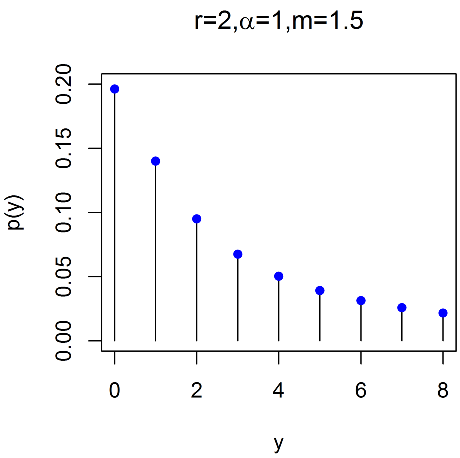

Figure 4 pmf plot for different value of parameters

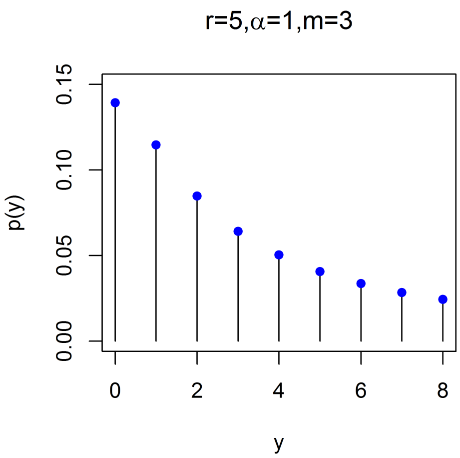

Figure 5 pmf plot for different value of parameters

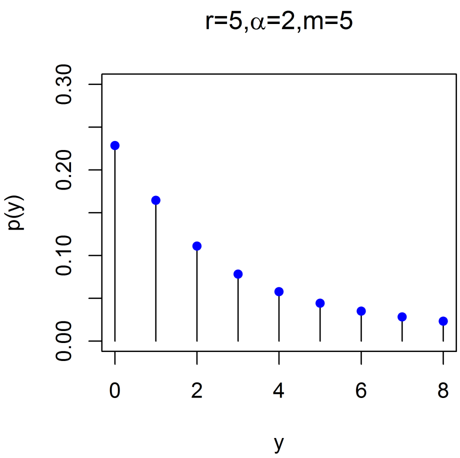

Figure 6 pmf plot for different value of parameters

Theorem 2 *Let* $Y \sim \mathcal{NBRIG}(r, \alpha, m)$ *as defined in* (5)*, then its factorial moment of order* k *is given by*

$$\mu_{[k]}(Y) = \frac{\Gamma(r+k)}{\Gamma(r)} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \sqrt{\frac{\alpha}{\alpha - 2(k-j)}} exp\left\{ \frac{\alpha}{m^2} \left[m - \frac{m}{\sqrt{\alpha}} \sqrt{\alpha - 2(k-j)} \right] \right\}. \tag{10}$$

Proof: If $Y|\omega \sim \mathcal{NB}(r, \theta = e^{-\omega})$ and $\omega \sim \mathcal{RIG}(\alpha, m)$, then factorial moment of order k can be find out as:

$$\mu_{[k]}(y) = E_\omega \left[\mu_{[k]}(y|\omega) \right].$$

Using the factorial moment of order k of $\mathcal{NB}(r, \theta)$, then

$$\mu_{[k]}(y) = E_\omega \left[\frac{\Gamma(r+k)}{\Gamma(r)} (e^\omega - 1)^k \right] = \frac{\Gamma(r+k)}{\Gamma(r)} E_\omega (e^\omega - 1)^k.$$

Using the binomial expansion of $(e^\omega - 1)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{\omega(k-j)}$, we have

$$\begin{aligned} \mu_{[k]}(y) =& \frac{\Gamma(r+k)}{\Gamma(r)} \sum_{j=0}^{k} \binom{k}{j} (-1)^j E_\omega (e^{\omega(k-j)}) \\ =& \frac{\Gamma(r+k)}{\Gamma(r)} \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_\omega (k-j). \end{aligned}$$

From the mgf of $\mathcal{RIG}(\alpha, m)$ given in Equation (4) with $t = k - j$, we get finally the required result as:

$$\frac{\Gamma(r+k)}{\Gamma(r)} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \sqrt{\frac{\alpha}{\alpha - 2(k-j)}} exp\left\{ \frac{\alpha}{m^2} \left[m - \frac{m}{\sqrt{\alpha}} \sqrt{\alpha - 2(k-j)} \right] \right\},$$

which proves the theorem.

The mean, second order moment and variance can be obtained directly from (10) which are given by

$$E(Y) = r\left[M_\omega(1) - 1\right], \tag{11}$$

$$E\left(Y^2\right) = (r + r^2) M_\omega(2) - (r + 2r^2) M_\omega(1) + r^2, \tag{12}$$

$$V(Y) = (r + r^2) M_\omega(2) - r M_\omega(1) - r^2 M_\omega^2(1), \tag{13}$$

where $M_\omega(v)$ is the mgf of $\mathcal{RIG}(\alpha, m)$ defined in (4).

Index of Dispersion (ID), which is actually ratio of variance to mean can be calculated from (11) and (13). Table 1 shows that the distribution is over-dispersed as $ID > 1$ for all combinations of parameters. Further more, the next result which is in the form of a theorem establishes that the $\mathcal{NBRIG}(r, \alpha, m)$ distribution is over-dispersed as compared to the NB distribution with the same mean.

Table 1 ID for different value of parameters

		$r=1$			
α	m	0.5	1.5	2.5	3.5
5.0	ID	85.12	10.37	7.19	6.26
8.0	ID	22.56	4.09	2.98	2.63
11.0	ID	15.46	3.15	2.33	2.06
		$r=5$			
5.0	ID	230.76	25.63	17.51	15.23
8.0	ID	48.06	7.56	5.40	4.75
11.0	ID	28.31	5.00	3.62	3.20
		$r=10$			
5.0	ID	230.76	25.63	17.51	15.23
8.0	ID	48.06	7.56	5.40	4.75
11.0	ID	28.30	5.00	3.63	3.20

Theorem 3 *Let $\omega \sim \mathcal{RIG}(\alpha, m)$ whose pdf is given in* (3) *and $\tilde{Y}$ is another random variable following negative binomial distribution i.e.,$\tilde{Y} \sim \mathcal{NB}(r, \theta = [E(e^{\omega})]^{-1})$. Suppose consider another random variable $Y \sim \mathcal{NBRIG}$ which is defined in* (5)*. Then we have:*

(i) $E(\tilde{Y})$=$E(Y)$ & $Var(Y) > Var(\tilde{Y})$,

(ii) $Var(Y) > E(Y)$.

Proof: We have $E(e^{\omega}) = M_{\omega}(1) > 1$, then $\theta = \frac{1}{E(e^{\omega})}$ is well defined. Using the definition of conditional expectation, it follows that

$$\begin{aligned}
E(Y) &= E_{\omega}\left(E(Y|\omega)\right) = r\left(M_{\omega}(1) - 1\right) = r\left[E(e^{\omega}) - 1\right], \\
Var(Y) &= E_{\omega}\left[V(Y|\omega)\right] + V_{\omega}\left[E(Y|\omega)\right] \\
&= (r + r^2)M_{\omega}(2) + rM_{\omega}(1) - r^2 M_{\omega}^2(1) \\
&= rM_{\omega}(2) + r^2 M_{\omega}(2) - rM_{\omega}(1) - r^2 M_{\omega}^2(1) \\
&= rE\left[e^{2\omega}\right] + r^2 E\left[e^{2\omega}\right] - rE\left[e^{\omega}\right] - r^2\left(E\left[e^{\omega}\right]\right)^2 \\
&= rE\left[e^{2\omega}\right] + r^2 V(e^{\omega}) - rE\left[e^{\omega}\right] \\
Var(Y) &= r\left[E(e^{2\omega}) - E(e^{\omega})\right] + r^2 V(e^{\omega}). \qquad (14)
\end{aligned}$$

Also, since $\tilde{Y} \sim \mathcal{NB}(r, \theta = [E(e^{\omega})]^{-1})$, we have

$$E(\tilde{Y}) = r\left[E(e^{\omega}) - 1\right] = E(Y),$$

and

$$Var(\tilde{Y}) = r\left[E(e^{\omega}) - 1\right]E(e^{\omega}).$$

Now, using Equation (14), we have

$$\begin{aligned}
Var(Y) - Var(\tilde{Y}) &= r\left[E(e^{2\omega}) - E(e^{\omega})\right] + r^2 V(e^{\omega}) - Var(\tilde{Y}) \\
&= r\left[E(e^{2\omega}) - E(e^{\omega})\right] + r^2 V(e^{\omega}) - r\left[E(e^{\omega}) - 1\right]E(e^{\omega}) \\
&= rE(e^{2\omega}) - rE(e^{\omega}) + r^2 V(e^{\omega}) - r\left(E(e^{\omega})\right)^2 + rE(e^{\omega}) \\
&= (r + r^2)V(e^{\omega}) > 0.
\end{aligned}$$

It follows that

$$Var(Y) > Var(\tilde{Y}) \qquad (15)$$

(ii) Since $\tilde{Y} \sim \mathcal{NB}(r, \theta = [E(e^{\omega})]^{-1})$

$$\begin{aligned}\Rightarrow & Var(\tilde{Y}) > E(\tilde{Y}), \text{ but} E(\tilde{Y}) = E(Y) \\ \Rightarrow & Var(\tilde{Y}) > E(Y)\end{aligned} \tag{16}$$

Combining (15) and (16), it follows that $Var(Y) > E(Y)$.

3. Collective Risk Model under Negative Binomial-Reciprocal Inverse Gaussian Distribution

In non-life Insurance portfolio, the aggregate loss (S) is a random variable defined as the sum of claims occurred in a certain period of time. Let us consider

$$S = Y_1 + Y_2 + \ldots + Y_N, \tag{17}$$

where S denote aggregate losses associated with a set of N observed claims, $Y_1, Y_2, \ldots, Y_N$ satisfying independent assumptions:

1. The $Y_{j's}(j = 1, 2, \ldots, N)$ are independent and identically distributed (i.i.d) random variables with cumulative distribution function (cdf) $F_Y(y)$ and pdf $f_Y(y)$.
2. The random variables $N, Y_1, Y_2, \ldots$ are mutually independent.

Here N be the claim count variable representing number of claims in certain time period and $Y_j : j = 1, 2, \ldots$ be the amount of jth claim (or claim severity). When $\mathcal{NBRIG}(r, \alpha, m)$ is chosen as primary distribution (N), the distribution of aggregate claim S is called compound negative binomial-reciprocal inverse Gaussian distribution $(\mathcal{CNBRIG})$ whose cdf is as:

$$\begin{aligned}F_S(y) =& P(S \leq y) \\ =& \sum_{n=0}^{\infty} p_n P(S \leq y | N = n) \\ =& \sum_{n=0}^{\infty} p_n F_Y^{\star n}(y)\end{aligned}$$

where $F_Y(y) = P(Y \leq y)$ is the common distribution of $Y_j's$ and $p_n = P(N = n)$ is given by (6). $F_Y^{\star}(n)$ is the n-fold convolution of the cdf of Y. It can be obtained as

$$F_Y^{\star 0}(y) = \begin{cases} 0, & y < 0, \\ 1, & y \geq 0. \end{cases}$$

Next, we will obtain the recursive formula for the pmf of $\mathcal{NBRIG}(r, \alpha, m)$ in the form of a theorem.

Theorem 4 *Let $p(k; r)$ denote the pmf of $\mathcal{NBRIG}(r, \alpha, m)$ and for $r = 1, 2, \ldots$, the expression for recursive formula is:*

$$p(k; r) = \frac{r + k - 1}{k} \left[p(k-1; r) - \frac{r}{r + k - 1} p(k-1; r+1) \right], \quad \text{for } k = 1, 2, \ldots. \tag{18}$$

Proof: The pmf of $\mathcal{NB}$ can be written as

$$p(k|\omega) = \binom{r + k - 1}{k} e^{-\omega r} (1 - e^{-\omega})^k; \quad k = 0, 1, \ldots.$$

Now,
$$\begin{aligned}\frac{p(k|\omega)}{p(k-1|\omega)} &= \frac{\binom{r+k-1}{k}e^{-\omega r}(1-e^{-\omega})^{k}}{\binom{r+k-2}{k-1}e^{-\omega r}(1-e^{-\omega})^{k-1}}\\ &= \frac{r+k-1}{k}(1-e^{-\omega})\\ \frac{p(z=k|\omega)}{p(z=k-1|\omega)} &= \frac{r+k-1}{k}(1-e^{-\omega}), \quad ;k=1,2,\ldots.\\ p(z=k|\omega) = p(z=k-1|\omega)&\frac{r+k-1}{k}(1-e^{-\omega}), \quad k=1,2,\ldots. \end{aligned} \tag{19}$$

Using the definition of $\mathcal{NBRIG}(r,\alpha,m)$ and (19), we have:

$$\begin{aligned}p(k|r) &= \int_0^\infty p(z=k|\omega)h_1(\omega)d\omega\\ &= \int_0^\infty \frac{r+k-1}{k}(1-e^{-\omega})p(z=k-1|\omega)h_1(\omega)d\omega\\ &= \int_0^\infty \frac{r+k-1}{k}p(z=k-1|\omega)h_1(\omega)d\omega - \int_0^\infty \frac{r+k-1}{k}p(z=k-1|\omega)e^{-\omega}h_1(\omega)d\omega\\ &= \frac{r+k-1}{k}p(k-1;r) - \frac{r+k-1}{k}\int_0^\infty e^{-\omega}p(z=k-1|\omega)h_1(\omega)d\omega.\end{aligned}$$

Also, we obtain now

$$\begin{aligned}\int_0^\infty e^{-\omega}p(z=k-1|\omega)h_1(\omega)d\omega &= \int_0^\infty e^{-\omega}\binom{r+k-2}{k-1}e^{-\omega r}(1-e^{-\omega})^{k-1}h_1(\omega)d\omega\\ &= \frac{r}{r+k-1}\int_0^\infty \binom{r+1+k-2}{k-1}e^{-\omega(r+1)}(1-e^{-\omega})^{k-1}h_1(\omega)d\omega\\ &= \frac{r}{r+k-1}p(k-1;r+1),\end{aligned}$$

and thus (18) is obtained.

Theorem 5 *If Y_j have pdf $f_Y(y)$ for $y>0$, then the pdf $g_s(y;r)$ of the $(\mathcal{CNBRIG})$ satisfies:*

$$\begin{aligned}g_s(y;r) &= p(0;r) + \int_0^y \frac{rz+y-z}{y}g_s(y-z;r)h_1(z)dz\\ &\quad - \int_0^y \frac{rz}{y}g_s(y-z;r+1)h_1(z)dz.\end{aligned} \tag{20}$$

Proof: The aggregate claim distribution is given by

$$\begin{aligned}g_s(y;r) &= \sum_{k=0}^{\infty} p(k;r)h_1^{k\star}(y)\\ &= p(0;r)h_1^{0\star}(y) + \sum_{k=1}^{\infty} p(k;r)h_1^{k\star}(x).\end{aligned}$$

Using (18), we get:

$$g_s(y;r) = p(0;r) + \sum_{k=1}^{\infty} h_1^{k\star}(y)\left[\frac{r+k-1}{k}\left(p(k-1;r) - \frac{r}{r+k-1}p(k-1;r+1)\right)\right]$$
$$= p(0;r) + \sum_{k=1}^{\infty}\frac{r-1}{k}p(k-1;r)h_1^{k\star}(y) + \sum_{k=1}^{\infty}p(k-1;r)h_1^{k\star}(y)$$
$$+ \sum_{k=1}^{\infty}\frac{r}{k}p(k-1;r+1)h_1^{k\star}(y). \tag{21}$$

Using the identities:

$$f^{k\star}(y) = \int_0^y h_1^{(k-1)\star}(y-z)f(z)dz, \quad k = 1,2,\ldots, \tag{22}$$
$$\frac{h_1^{k\star}(y)}{k} = \int_0^y \frac{z}{y}h_1^{(k-1)\star}(y-z)f(z)dz, \quad k = 1,2,\ldots. \tag{23}$$

Therefore, now (21) can be written as:

$$\sum_{k=1}^{\infty}(r-1)p(k-1;r)\int_0^y \frac{z}{y}f^{(k-1)\star}(y-z)h_1(z)dz$$
$$+\sum_{k=1}^{\infty}p(k-1;r)\int_0^y f^{(k-1)\star}(y-z)h_1(z)dz$$
$$-\sum_{k=1}^{\infty}rp(k-1;r+1)\int_0^y \frac{z}{y}h_1^{(k-1)\star}(y-z)h_1(z)dz$$
$$= \int_0^y \frac{rz+y-z}{y}h_1^{(k-1)\star}(y-z)h_1(z)dz\sum_{k=1}^{\infty}p(k-1;r)$$
$$-\int_0^y \frac{rz}{y}h_1^{(k-1)\star}(y-z)h_1(z)dz\sum_{k=1}^{\infty}p(k-1;r+1). \tag{24}$$

Also, we can write:

$$g_s(y,r) = \sum_{k=1}^{\infty}p(k-1;r)h_1^{(k-1)\star}(y), \quad k = 1,2,\ldots,$$
$$g_s(y-z,r) = \sum_{k=1}^{\infty}p(k-1;r)h_1^{(k-1)\star}(y-z),$$
$$g_s(y-z,r+1) = \sum_{k=1}^{\infty}p(k-1;r+1)h_1^{(k-1)\star}(y-z),$$

thus (24) becomes:

$$\int_0^y \frac{rz+y-z}{y}h_1(z)dzg_s(y-z,r) - \int_0^y \frac{rz}{z}h_1(z)dzg_s(y-z,r+1).$$

Therefore we finally get:

$$g_s(y;r) = p(0;r) + \int_0^y \frac{rz+y-z}{y}g_s(y-z;r)h_1(z)dz$$
$$-\int_0^y \frac{rz}{y}g_s(y-z;r+1)h_1(z)dz.$$

Hence proved.

The Integral equation obtained in above theorem can be solved numerically in practice and the discrete version of it can be obtained in a similar fashion by interchanging $\int_0^y$ to $\sum_{z=1}^{y}$ in expressions (22) and (23) (Rolski et al., 1999). So its discrete version obtained are as

$$\begin{aligned} g_s(y;r) &= p(0;r) + \sum_{z=1}^{y} \frac{rz+y-z}{y} g_s(y-z;r)h_1(y) \\ &\quad - \sum_{z=1}^{y} \frac{rz}{y} g_s(y-z;r+1)h_1(z). \end{aligned}$$

4. Multivariate Version of $\mathcal{NBRIG}$

Here we propose the multivariate version of $\mathcal{NBRIG}(r,\alpha,m)$ which is actually extension of definition (5). The multivariate version of $\mathcal{NBRIG}$ can be considered as a mixture of independent $\mathcal{NB}(r_i,\theta=e^{-\omega}), i=1,2,\ldots,d$ combined with a $\mathcal{RIG}(\alpha,m)$ (Déniz et al., 2008).

Definition 2 *A multivariate negative binomial-reciprocal inverse Gaussian distribution* $(Y_1,Y_2,\ldots,Y_d)$ *is defined by stochastic representation:*

$$\begin{aligned} Y_i|\omega &\sim \mathcal{NB}(r_i,e^{-\omega}), \quad i=1,2,\ldots,d \textit{ are independent}, \\ \omega &\sim \mathcal{RIG}(\alpha,m). \end{aligned}$$

Using the same arguments as mentioned in Section 2, the joint pmf obtained is:

$$\begin{aligned} P(Y_1=y_1,Y_2=y_2,\ldots,Y_d=y_d) &= \prod_{i=1}^{d} \binom{r_i+y_i-1}{y_i} \sum_{j=0}^{\tilde{y}} (-1)^j \binom{\tilde{y}}{j} \\ &\quad \times \sqrt{\frac{\alpha}{\alpha+2(\tilde{r}+j)}} exp\left\{\frac{\alpha}{m^2}\left[m - \frac{m}{\sqrt{\alpha}}\sqrt{\alpha+2(\tilde{r}+j)}\right]\right\}, \end{aligned} \tag{25}$$

where $y_1,y_2,\ldots,y_d=0,1,2,\ldots;\alpha,m,r_1,r_2,\ldots,r_d>0$ and

$$\tilde{r} = r_1+r_2+\ldots+r_d, \tag{26}$$

$$\tilde{y} = y_1+y_2+\ldots+y_d. \tag{27}$$

The above joint pmf can be written in a more convenient form for the purpose of computing multivariate probabilities. Let $\tilde{Y} \sim \mathcal{NBRIG}(\tilde{r},\alpha,m)$, where $\tilde{r}$ is given in (4.2), an alternative structure for (25) with $d \geq 2$ is given by:

$$P(Y_1=y_1,Y_2=y_2,\ldots,Y_d=y_d) = \frac{\prod_{i=1}^{d}\binom{r_i+y_i-1}{y_i}}{\binom{\tilde{r}+\tilde{y}-1}{\tilde{y}}} \cdot P(\tilde{Y}=\tilde{y}), \tag{28}$$

where $\tilde{y}$ is defined in equation (4.3). The marginal distribution will be obviously as $\tilde{Y} \sim \mathcal{NBRIG}(r_i,\alpha,m)$, $i=1,2,\ldots,d$ and any subvector $(Y_1,Y_2,\ldots,Y_s)$ with $s<d$ is again a multivariate $\mathcal{NBRIG}(r,\alpha,m)$ distribution of dimension s. Using (11) and (13), the following expressions for moments can be obtained as:

$$E(Y) = r_i\left[M_\omega(1)-1\right], \quad i=1,2,\ldots,r \tag{29}$$

$$V(Y) = (r_i+r_i^2)M_\omega(2) - r_iM_\omega(1) - r_i^2M_\omega^2(1), \quad i=1,2,\ldots,r \tag{30}$$

$$Cov(Y_i,Y_j) = r_ir_j\left[M_\omega(2)-M_\omega^2(1)\right], \quad i\neq j. \tag{31}$$

Since $M_\omega(2)=E\left[e^{2\omega}\right]$ and $M_\omega(1)=E\left[e^{\omega}\right] \Rightarrow V(e^\omega)=M_\omega(2)-M_\omega^2(1)$.

Therefore $Cov(Y_i,Y_j)=r_ir_jV(e^\omega); i\neq j$. Now $\rho(Y_i,Y_j)=\frac{Cov(Y_i,Y_j)}{\sigma_{y_i,y_j}}=\frac{r_ir_jV(e^\omega)}{\sigma_{y_i,y_j}}>0$, thus, it follows $\rho(Y_i,Y_j)>0$.

5. Maximum Likelihood Estimation

Suppose $\underline{\mathbf{y}} = \{y_1, y_2, \ldots, y_n\}$ be a random sample of size n from the $\mathcal{NBRIG}(r, \alpha, m)$ with pmf given in (6). The log-likelihood function is:

$$\log L(m, \alpha, r|\underline{\mathbf{y}}) = \sum_{i=1}^{n} \log \binom{r + y_i - 1}{y_i} + \sum_{i=1}^{n} \log \left[\sum_{j=1}^{y_i} \binom{y_i}{j} (-1)^j \sqrt{\frac{\alpha}{\Lambda}} \right] + \sum_{i=1}^{n} \frac{\alpha}{m^2} \left[m - m\sqrt{\frac{\Lambda}{\alpha}} \right]. \quad (32)$$

For simplification point of view, it is assumed that $\alpha + 2(r + j) = \Lambda$. The ML estimates $\hat{m}$ of m, $\hat{\alpha}$ of α and $\hat{r}$ of r, respectively, can be obtained by solving equations

$$\frac{\partial \log L}{\partial m} = 0, \quad \frac{\partial \log L}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \log L}{\partial r} = 0,$$

where

$$\frac{\partial \log L}{\partial m} = \frac{n\alpha \left(\sqrt{\frac{\Lambda}{\alpha}} - 1 \right)}{m^2}, \quad (33)$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{\sum_{j=0}^{y_i} \frac{(-1)^j \binom{y_i}{j} \left(\frac{1}{\Lambda} - \frac{\alpha}{\Lambda^2} \right)}{2\sqrt{\frac{\alpha}{\Lambda}}}}{\sum_{j=0}^{y_i} (-1)^j \binom{y_i}{j} \sqrt{\frac{\alpha}{\Lambda}}}, \quad (34)$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n} \frac{\sum_{j=0}^{y_i} - \frac{\alpha (-1)^j \binom{y_i}{j}}{\sqrt{\frac{\alpha}{\Lambda}} \Lambda^2}}{\sum_{j=0}^{y_i} (-1)^j \binom{y_i}{j} \sqrt{\frac{\alpha}{\Lambda}}} + n \left(\psi^{(0)}(r + y_i) - \psi^{(0)}(r) \right), \quad (35)$$

with $\psi(k) = \frac{d}{dk}\Gamma(k)$ is a digamma function (Abramowitz et al. 1972). Since the above three normal equations are in implicit form and are complex to be solved numerically, so we make use of Mathematica Software 9.0 to find the estimates numerically by using ”NMaximize” function.

6. Numerical Illustrations

To explore the potential of the proposed model, two data sets have been taken into consideration from actuarial literature.

Illustration 1: The first data set is about automobile liability policies in Switzerland (see Klugman et al. (2008), pp.488-489). Models like Poisson ($\mathcal{P}$), NB, $\mathcal{PRIG}$ and $\mathcal{NBRIG}$ distribution are being fitted to the given data set and parameters of each model were estimated by MLE. In order to test the goodness of fit, chi-square test criterion has been employed. It is pertinent to mention that expected frequencies have been grouped into classes for getting cell frequencies greater than five. Based on the results like log-likelihood, Chi-square (χ^2) value, p-value and Akaike's information criterion (AIC) that there exists enough statistical evidence that the $\mathcal{NBRIG}$ distribution fits the data very well (Table 2).

Illustration 2: The second data set is about about 23,589 automobile drivers where number of accidents per driver in one year is mentioned (Klugman et al. 2008, pp. 249-250). Again, $\mathcal{P}$, NB, $\mathcal{PRIG}$ and $\mathcal{NBRIG}$ distributions have been fitted to data by MLE. Observed and expected values together with parameter estimates including log-likelihood, χ^2 value, p-value and AIC are exhibited in Table 3 . Based on the results, it clearly suggests that the $\mathcal{NBRIG}$ distribution outperforms other three competing models.

Table 2 Number of automobile liability policies in Switzerland

Count	Observed Frequency	Expected frequency $\mathcal{P}$	$\mathcal{NB}$	$\mathcal{PRIG}$	$\mathcal{NBRIG}$
0	103704	102630	103724	102643.9	103710
1	14075	15921.9	13989.9	15909.81	14054.8
2	1766	1235.07	1857.07	1233.01	1787.35
3	255	63.87*	245.20	63.71*	251.93
4	45	2.48*	32.29*	2.47*	40.22
5	6	0.08*	4.24*	0.08*	7.22*
6	2	0.01*	0.56*	0.01*	1.44*
Total	119853	119853	119853	119853	119853
	Estimated parameter	$\hat{\lambda} = 0.155$	$\hat{r} = 1.03$, $\hat{\theta} = 0.87$	$\hat{\phi} = 1.572 \times 10^7$, $\hat{\mu} = 0.16$	$\hat{r} = 3.40$, $\hat{m} = 35.90$, $\hat{\alpha} = 61.50$
	log-likelihood	-55108.5	-54615.3	-55108.5	-54609
	χ^2(d.f)	1332.130(2)	12.205(2)	1334.856(1)	0.940(2)
	p-value	0.0001	0.0022	0.0001	0.6240
	AIC	110219	109234.6	110221	109224

*Expected frequencies have been combined for the calculation of χ^2.

Table 3 Number of accidents per driver

Count	Observed Frequency	Expected frequency $\mathcal{P}$	$\mathcal{NB}$	$\mathcal{PRIG}$	$\mathcal{NBRIG}$
0	20592	20420.90	20596.80	20420.95	20595.30
1	2651	2945.11	2631.03	2945.09	2637.36
2	297	212.37	318.37	212.37	311.71
3	41	10.21*	37.81	10.21*	38.72
4	7	0.37*	4.45*	0.37*	5.10*
5	0	0.02*	0.52*	0.02*	0.71*
6	1	0.01*	0.06*	0.01*	0.10*
Total	23589	23589	23589	23589	23589
	Estimated parameter	$\hat{\lambda} = 0.144$	$\hat{r} = 1.118$, $\hat{\theta} = 0.886$	$\hat{\phi} = 3.24 \times 10^8$, $\hat{\mu} = 0.144$	$\hat{r} = 21.427$, $\hat{m} = 317.520$, $\hat{\alpha} = 28.492$
	log likelihood	-10297.8	-10223.4	-10297.8	-10222.3
	χ^2(d.f)	203.63(2)	3.59(2)	203.63(1)	1.63(1)
	p-value	0.0001	0.1650	0.0001	0.2010
	AIC	20597.6	20450.8	20450.6	20450

*Expected frequencies have been combined for the calculation of χ^2.

7. Conclusions

In this paper, we introduce a new three-parameter $\mathcal{NBRIG}(r, \alpha, m)$ distribution including its multivariate extension as well. This model is obtained by mixing the NB with $\mathcal{RIG}(\alpha, m)$ distribution. In addition, the moments of the $\mathcal{NBRIG}(r, \alpha, m)$ distribution which includes the factorial moments, mean, variance, are derived. Moreover, the parameters have been estimated by MLE. The superior fit of the proposed model as compared to Poisson, NB and $\mathcal{PRIG}$ have been illustrated on two real data sets containing extra proportion of zeros. We are hopeful that $\mathcal{NBRIG}(r, \alpha, m)$ distribution may attract wider applications in analyzing count data.

Acknowledgments

The authors are sincerely indebted to thank Assistant Prof. Ms. Reeta Chauhan, Mathematics, Prestige Institute of Management, Gwalior-India for providing access to Mathematica software. The authors are also grateful to the Editor-in-Chief for encouragement and anonymous referees for a very careful reading of the manuscript and suggestions which have improved the quality of the paper.

References

Abramowitz M, Stegun I. Handbook of mathematical function. 2nd ed., New York, Dover; 1972.

Balakrishnan N and Nevzorov V. A primer on statistical distributions. New York: John Wiley & Sons; 2003.

Déniz EG, Sarabia JM, Ojeda EC. Univariate and multivariate versions of the negative binomial-inverse Gaussian distributions with applications. Insur Math Econ. 2008; 42(1): 39–49.

Déniz EG, Sarabia JM, Ojeda EC. Properties and applications of the Poisson reciprocal inverse Gaussian distribution. J Stat Comput Sim. 2017; 88(1): 1–21.

Johnson NL, Kemp AW, Kotz S. Univariate discrete distributions, 3rd ed. New York: John Wiley & Sons; 2005.

Klugman SA, Panjer HH, Willmot GE. Loss models: from data to decisions, 3rd ed. New York: John Wiley & Sons; 2008.

Kongrod S, Bodhisuwan W, Payakkapong P. The negative binomial-Erlang distribution with applications. Int. J. Pure Appl. Math. 2014; 92(3): 389–401.

Pudprommarat C, Bodhisuwan W, Zeephongsekul P. A new mixed negative binomial distribution. J. Appl. Sci.(Faisalabad). 2012; 12(17): 1853–1858.

Rolski T, Schmidli H, Schmidth V, Teugels J. Stochastic process for insurance and finance. New York: John Wiley & Sons; 1999.