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Transmuted Mixture Exponential Distribution: Properties and Applications

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Abstract

Transmutation method is a useful technique to generate flexibility distributions in modeling a particular data set. In this paper, the transmuted mixture exponential (TME) distribution is proposed and studied. We provide a description of the mathematical properties of the subject distribution along with its reliability aspects. Also, we develop the maximum likelihood estimators of the unknown parameters and their corresponding asymptotic confidence intervals. The usefulness of the proposed distribution for modeling data is illustrated using two real data sets.

Keywords: Transmuted distribution; reliability probability; maximum likelihood estimation.

1. Introduction

The transmutation method is known and used in theory and applications of statistical methods in analysing the empirical data. By transmutation one obtain, from the given distribution (of random variable), the whole 1-parametric family of distributions. Thus, this procedure (transmutation), obviously, give more flexibility of possible fitting parameters of the distribution, to a given set of empirical data. In the paper, we consider the so-called Transmuted Mixture Exponential (TME) distribution, which is the effect of applying the transmutation procedure to the Mixture Exponential (ME) distribution. The family of ME distributions depends on two parameters, which implies that the family of TME is a 3-parameter family of distributions. The ME distribution exhibits monotone hazard rate and is a competitor to the families of two parameter gamma and Weibull distributions. Various statistical and reliability aspects of this model is explored by Mirhossaini and Dolati (2008). Several numerical examples based on real data have been showed the flexibility of the new distribution for modeling proposes.

Transmutation method first introduced by Shaw and Buckley (2009) has been used by many authors to generate new distributions starting with suitable continuous distributions (see Oguntunde and Adejumo 2015) for details). According to the Quadratic Rank Transmutation Map, (QRTM), approach the cumulative distribution function (CDF) satisfy the relationship

$$F_2(x) = (1+m)F_1(x) - m(F_1(x))^2,$$

which on differentiation yields

$$f_2(x) = f_1(x)\left(1 + m - 2mF_1(x)\right),$$

where $f_1(x)$ and $f_2(x)$ are the corresponding probability density function (pdf) associated with $F_1(x)$ and $F_2(x)$ respectively and $-1 \le m \le 1$. One can use the above formulation for a pair of distributions $F(x)$ and $G(x)$ where $G(x)$ is a submodel of $F(x)$.

We shall use the technique of transmutation for introducing a new transmuted mixture exponential distribution (TMED). The motive behind the construction of this distribution is to assess its potentiality and flexibility in modeling two particular data sets.

Definition 1 A *random variable* X *is said to have a transmuted distribution if its cumulative distribution function (cdf) is given by*

$$F(x) = (1+m)G(x) - m(G(x))^2,\ -1 \le m \le 1,$$

where $G(x)$ *is the cdf of the base distribution.*

Observe that at $m = 0$ we have the distribution of the base random variable. Some authors have studied transmuted versions of so many already existing distributions. Ahmad et al. (2014) studied the transmuted model of the inverse Rayleigh distribution. Aryal and Tsokos (2009) studied the transmuted extreme value distribution. Hussian (2014) studied the transmuted exponentiated gamma distribution. Merovci (2013) studied the transmuted Lindley distribution. Dar et al. (2017) investigated different characteristic as well as structural properties of transmuted weighted exponential distribution.

The rest of the paper is as follows. In Section 2, we introduce the transmuted mixture exponential distribution. In Section 3, we give the moment generating function, mean, variance, coefficients of skewness and kurtosis and higher moments of TME distribution. In Section 4, we compute the reliability probability for different values of m, the mean deviations, mean residual life and Bonferroni and Lorenz curves. The interesting result is that the reliability probability is not depend on the basic distribution parameters a and b and transmuted parameters. In Section 5, we give the pdf of k^{th} order statistic. In Section 6, the maximum likelihood estimators of the all unknown parameters and their corresponding asymptotic confidence sets are developed. The usefulness of the transmuted distribution for modeling data is illustrated using two real data sets in Section 7.

2. Transmuted Distribution

Definition 2 *The probability density function (pdf) of a mixture exponential distribution is*

$$g_X(x \mid a,b) = be^{-bx}\left(1 + a(2e^{-bx} - 1)\right), \quad x,b > 0, -1 \le a \le 1$$

and the respective cdf is

$$G_X(x \mid a,b) = \left(1 - e^{-bx}\right)\left(1 + ae^{-bx}\right).$$

The transmuted cumulative distribution function (cdf) is $(TME(a,b,m))$:

$$F_X(x \mid a,b,m) = e^{-4bx}(e^{bx} - 1)(a + e^{bx})\left(e^{2bx} + am - (a-1)e^{bx}m\right), \tag{1}$$

with transmuted pdf

$$f_X(x \mid a,b,m) = be^{-3bx}(e^{bx} - 1)\left(-(a-1)me^{bx} + am + e^{2bx}\right) + be^{-3bx}(a + e^{bx})\left(-(a-1)me^{bx} + am + e^{2bx}\right)$$

$$-4be^{-4bx}(e^{bx}-1)(a+e^{bx})\left(-(a-1)me^{bx}+am+e^{2bx}\right)$$
$$+e^{-4bx}(e^{bx}-1)(a+e^{bx})\left(2be^{2bx}-(a-1)bme^{bx}\right), \tag{2}$$

where $-1 \le a, m \le 1$ and $b > 0$.

We can write the probability density function (pdf) as the following product form to simplify in some subsequent calculations:

$$f_X(x \mid a,b,m) = be^{-4bx}\left(a(e^{bx}-2)-e^{bx}\right)\left(e^{2bx}(m-1)-2am+2(a-1)e^{bx}m\right). \tag{3}$$

The density of the TME distribution for different values of the parameters $b > 0$, $a = -0.5, +0.5, -1, +1$ and $m = -1, +1$ are plotted in Figure 1, indicates that the TME distribution can be a suitable candidate for a vast class of positively skewed data.

3. Moment Generating Function and Moments

In the next theorem we derive the moment generating function of the TME distribution.

Theorem 1 *Suppose that $X \sim TME(a,b,m)$. Then the moment generating function (mgf) of the random variable X is*

$$M_X(t) = b\left(\frac{(a-1)(m-1)}{b-t} + \frac{2\left(a+m+(a-3)am\right)}{2b-t} - \frac{4a^2m}{t-4b} + \frac{6(a-1)am}{t-3b}\right).$$

Proof: From (2),

$$M_X(t) = \int_0^\infty e^{tx} f_X(x \mid a,b,m)dx = A + B - C + D,$$

where

$$A = \int_0^\infty e^{tx} be^{-3bx}(e^{bx}-1)\left(-(a-1)me^{bx}+am+e^{2bx}\right)dx$$
$$= b\left(\frac{am}{t-3b} + \frac{-am+m-1}{b-t} + \frac{m-2am}{t-2b} - \frac{1}{t}\right),$$

$$B = \int_0^\infty e^{tx} be^{-3bx}(a+e^{bx})\left(-(a-1)me^{bx}+am+e^{2bx}\right)dx$$
$$= b\left(\frac{a^2m}{3b-t} + \frac{(a-2)am}{t-2b} + \frac{a(-m)+a+m}{b-t} - \frac{1}{t}\right),$$

$$C = \int_0^\infty e^{tx} 4be^{-4bx}\left(e^{bx}-1\right)\left(a+e^{bx}\right)\left(-(a-1)me^{bx}+am+e^{2bx}\right)dx$$
$$= 4b\left(\frac{a^2m}{t-4b} - \frac{2(a-1)am}{t-3b} + \frac{(a-3)am+a+m}{t-2b} + \frac{(a-1)(m-1)}{t-b} - \frac{1}{t}\right),$$

$$D = \int_0^\infty e^{tx} e^{-4bx}\left(e^{bx}-1\right)\left(a+e^{bx}\right)\left(2be^{2bx}-(a-1)bme^{bx}\right)dx$$
$$= b\left(\frac{(a-1)(m-2)}{t-b} + \frac{(a-1)am}{3b-t} + \frac{(a-1)^2m+2a}{t-2b} - \frac{2}{t}\right).$$

Now proof is completed.

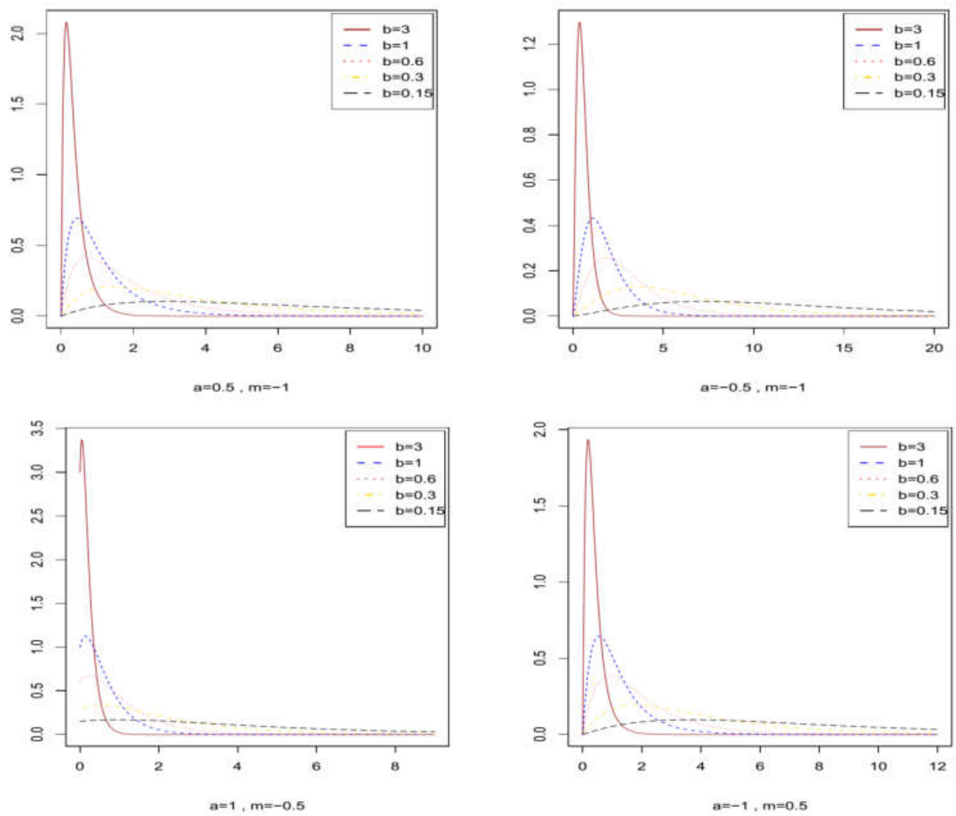

Figure 1 The density plot of the TME distribution for different values of the parameters a,b and m

Theorem 2 *The* r^{th} *moment* $\mu_r' = \mathbb{E}(X^r)$ *of a random variable* X *is given as*

$$\mu_r' = \left(\frac{1}{(12b)^r} \left(-2^{1+2r} + 3^r + 6^r\right) a^2 m + 6^r \left(2^r + m - 2^r m\right) + 2^r a \left(3^r - 6^r + \left(2^{1+r} - 3^{1+r} + 6^r\right) m\right) \right) \times \Gamma\left(1+r\right),$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof: By definition,

$$\mu_r' = \int_0^\infty x f_X(x \mid a,b,m) dx.$$

We have,

$$\int_0^\infty x^{r-1} e^{-cx} dx = c^{-r} \times r, \quad c,r \in \mathbb{R}.$$

Again, with simple but long calculations, proof is completed. Thus,

$$\mathbb{E}(X) = \frac{\left(a(2+a)-6\right)m - 6(a-2)}{12b},$$

$$\mathbb{E}(X^2) = \frac{36(4-3m) + a\left((68+13a)m - 108\right)}{72b^2},$$

$$\mathbb{E}(X^3)=\frac{\big(a(1208+115a)-1512\big)m-216(7a-8)}{288b^3},$$

$$\mathbb{E}(X^4)=\frac{1296(16-15a)+5\big(a(3472+173a)-3888\big)m}{864b^4}.$$

and

$$\mathbb{V}ar(X)=-\frac{\big(-6(a-2)+\big(a(2+a)-6\big)m\big)^2}{144b^2}-\frac{2\big(36(4-3m)+a\big((68+13a)m\big)-108\big)}{144b^2}.$$

Using the above results, the measures of skewness and kurtosis of the TME distribution can be obtained as

$$Skewness(X)=\frac{\mathbb{E}(X^3)-3\mathbb{E}(X^2)\mathbb{E}(X)+2\mathbb{E}^3(X)}{\mathbb{V}ar(X)^{3/2}},$$

$$Kurtosis(X)=\frac{\mathbb{E}(X^4)-4\mathbb{E}(X^3)\mathbb{E}(X)+6\mathbb{E}(X^3)\mathbb{E}^2(X)-3\mathbb{E}^4(X)}{\mathbb{V}ar\big(X\big)},$$

respectively. We can also find the r^{th} central moment of the TME distribution as $(\mu=\mu_1'=\mathbb{E}(X))$:

$$\mathbb{E}(X-\mu)^r=\sum_{j=0}^{r}\binom{r}{j}\mu'_j(-\mu)^{r-j}.$$

Corollary 1 *Set $m=0$ in Theorem 1 and Theorem 2. Thus, the results reduce to the mixture exponential distribution introduced by Mirhossaini and Dolati (2008).*

Now, we state and prove a theorem which characterizes the distribution.

Theorem 3 *The random variable X follows a TME distribution with parameters a,b and m if and only if the density function f satisfies the homogeneous differential equation of the form:*

$$K(a,b,m;x)f_X'(x\,|\,a,b,m)+N(a,b,m;x)f_X(x\,|\,a,b,m)=0, \tag{4}$$

where prime denotes first-order differentiation,

$$K(a,b,m;x)=(ae^{bx}-e^{bx}-2a)\big(2ae^{bx}m+e^{2bx}m-e^{2bx}-2am-2e^{bx}m\big),$$

and

$$N(a,b,m;x)=b\big((a-1)(m-1)e^{3bx}+16a^2m-18(a-1)ame^{bx}+4e^{2bx}\left(a+m+(a-3)am\right)\big).$$

Proof: Suppose X is a TME distribution random variable. We have

$$\begin{aligned}f_X'(x\,|\,a,b,m)&=be^{-4bx}(abe^{bx}-be^{bx})\big(e^{2bx}(m-1)-2am+2(a-1)e^{bx}m\big)\\&-4b^2e^{-4bx}\big(a(e^{bx}-2)-e^{bx}\big)\big(e^{2bx}(m-1)-2am+2(a-1)e^{bx}m\big)\\&+be^{-4bx}\big(a(e^{bx}-2)-e^{bx}\big)\big(2be^{2bx}(m-1)+2(a-1)be^{bx}m\big).\end{aligned}$$

By substituting $f_X(x\,|\,a,b,m)$ and $f_X'(x\,|\,a,b,m)$ in the differential Equation (4), the equation is satisfied. Conversely, suppose equation (4) holds. i.e.,

$$\frac{f_X'(x \mid a,b,m)}{f_X(x \mid a,b,m)} + \frac{N(a,b,m;x)}{K(a,b,m;x)} = 0.$$

Thus,
$$\int \frac{f_X'(x \mid a,b,m)}{f_X(x \mid a,b,m)} dx = -\int \frac{N(a,b,m;x)}{K(a,b,m;x)} dx .$$

With long but simple calculations proof is completed.

4. Reliability Properties

By the formulas (1) and (3), the hazard function of the $TME(a,b,m)$ is given by

$$\begin{aligned} h_X(x \mid a,b,m) &= \frac{f_X(x \mid a,b,m)}{1 - F_X(x \mid a,b,m)} \\ &= \frac{b\left(a(e^{bx}-2)-e^{bx}\right)\left(e^{2bx}(m-1)-2am+2(a-1)e^{bx}m\right)}{\left(a(e^{bx}-1)-e^{bx}\right)\left(e^{2bx}(m-1)-am+(a-1)e^{bx}m\right)}, \end{aligned}$$

for $x > 0$ and 0 otherwise. The hazard rate function of the TME distribution for some values of the parameters a,b and m, is plotted in Figures 2 and 3.

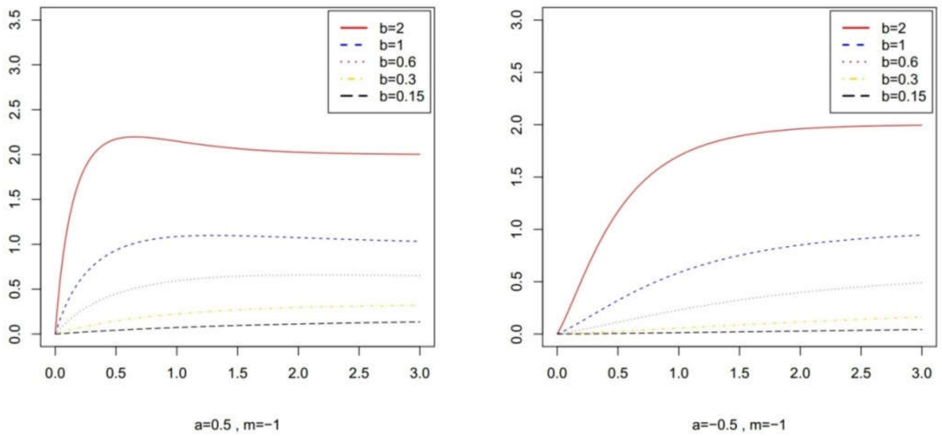

Figure 2 The hazard rate function of the TME distribution for same values of the parameters a,b,m

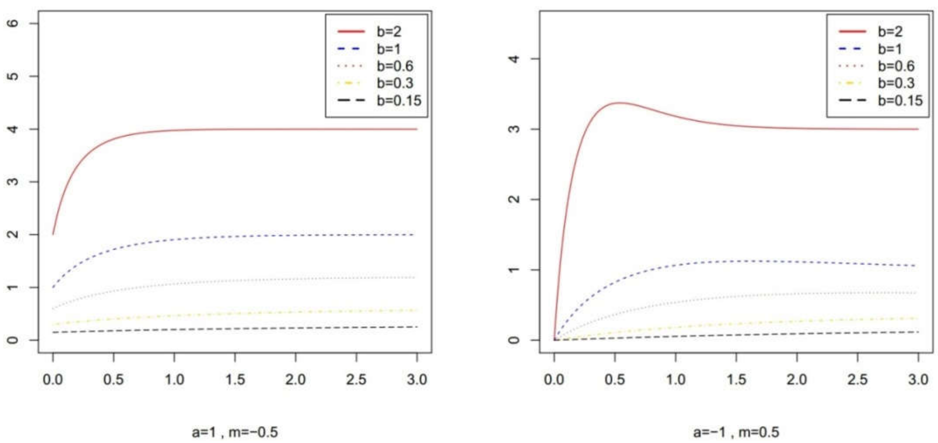

Figure 3 The hazard rate function of the TME distribution for same values of the parameters a,b,m

In the following theorem we compute the probability that one of two independently distributed TME random variables exceeds another. In the context of the reliability, this probability is in core of interest and is known as the reliability probability, $R = P(X > Y)$.

Theorem 4 *Suppose two independent random variables X and Y are distributed according to the distributions $TME(a,b,n)$ and $TME(a,b,m)$, respectively. Then the reliability probability is given by*

$$R = P(X > Y) = \frac{1}{2}.$$

Proof: We have,

$$\begin{aligned}
F_X(x) f_Y(x) &= -4a^4bm^2e^{-8bx} + 14a^4bm^2e^{-7bx} - 18a^4bm^2e^{-6bx} + 10a^4bm^2e^{-5bx} \\
&\quad +42a^3bm^2e^{-6bx} - 45a^3bm^2e^{-5bx} + 20a^3bm^2e^{-4bx} - 3a^3bm^2e^{-3bx} \\
&\quad -6a^3bme^{-6bx} + 15a^3bme^{-5bx} - 12a^3bme^{-4bx} + 3a^3bme^{-3bx} \\
&\quad +45a^2bm^2e^{-5bx} - 38a^2bm^2e^{-4bx} + 12a^2bm^2e^{-3bx} - 18a^2bm^2e^{-6bx} \\
&\quad -a^2bm^2e^{-2bx} - 15a^2bme^{-5bx} + 32a^2bme^{-4bx} - 21a^2bme^{-3bx} \\
&\quad +4a^2bme^{-2bx} - 2a^2be^{-4bx} + 3a^2be^{-3bx} - a^2be^{-2bx} - 14a^3bm^2e^{-7bx} \\
&\quad -10abm^2e^{-5bx} + 20abm^2e^{-4bx} - 12abm^2e^{-3bx} + 2abm^2e^{-2bx} \\
&\quad +21abme^{-3bx} - 10abme^{-2bx} + abme^{-bx} - 3abe^{-3bx} - 12abme^{-4bx} \\
&\quad +4abe^{-2bx} - abe^{-bx} - 2bm^2e^{-4bx} + 3bm^2e^{-3bx} - bm^2e^{-2bx} - 3bme^{-3bx} \\
&\quad +4bme^{-2bx} - bme^{-bx} - be^{-2bx} + be^{-bx} - 2a^4bm^2e^{-4bx} \\
&= \frac{1}{2} + a\left(-3 + a\left(2a - 3\right)\right)m - \frac{am}{2}\left(a\left(-6 + a\left(4 + \left(a+4\right)m\right)\right) - 6\right) \\
&\quad + \frac{m^2}{2}\left(2a\left(-2 + a\left(6 + \left(a-2\right)a\right)\right) - 1\right) - \frac{m^2}{2}\left(a-1\right)\left(a\left(5 + \left(a-7\right)a\right) + 1\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Then,

$$R = 1 - \int_0^{+\infty} F_X(y) f_Y(y) dy = 1 - \frac{1}{2} = \frac{1}{2}.$$

The average deviation can be used as a measure of spread in a population. The average deviation of X about the mean value is given by

$$\delta_1(X) = \int_0^{\infty} |x - \mu| f(x) dx,$$

where $\mu = \mathbb{E}(X)$.

Theorem 5 *The average deviation of $X \sim TME(a,b,m)$ about the mean value is given by*

$$\begin{aligned}
\delta_1(X) &= \frac{e^{-4b\mu}}{6b}\Big[-2ae^{b\mu}(e^{b\mu}-1)^2\left(e^{bt\mu}(m-3) - 4m\right) - a^2(e^{b\mu}-1)^3(3+e^{b\mu})m \\
&\quad +6e^{2b\mu}\left(-2e^{b\mu}(m-1) + m + e^{2b\mu}(-2+m+2bt)\right)\Big],
\end{aligned}$$

where $\mu = \dfrac{\left(a(2+a)-6\right)m-6(a-2)}{12b}$.

Proof: The quantitiy $\delta_1\left(X\right)$ can be calculated as

$$\delta_1\left(X\right)=2\mu F\left(\mu\right)-2\int_0^{\mu} xf\left(x\right)dx$$

Since

$$\int_0^t xe^{-cx}dx=\frac{1-e^{-ct}(1+ct)}{c^2},\quad c\in\mathbb{R}.$$

We have,

$$\begin{aligned}\int_0^t xf(x)dx=\frac{1}{12b}\Big[&-6(m-2)+a\left(-6+(2+a)m\right)-12(a-2)e^{-bt}(m-1)(1+bt)\\&-6e^{-2bt}\left(a+m+(a-3)am\right)(1+2bt)\\&+8(a-1)ae^{-3bt}m(1+3bt)-3a^2e^{-4bt}m\left(1+4bt\right)\Big].\end{aligned}$$

Then proof is completed.

We can construct Bonferroni and Lorenz curves, which are important in several fields such as economics, reliability, demography, insurance and medicine. They are defined as the following:

$$B\left(F(x)\right)=\frac{1}{\mu F(x)}\int_0^x t\,f(t)dt,$$

and

$$L\left(F(x)\right)=\frac{1}{\mu}\int_0^x t\,f(t)dt.$$

Also, we can compute the Bonferroni index as follows:

$$B=1-\int_0^1 \delta_1\left(X\right)dx$$

If $X\sim TME(a,b,m)$, then the mean waiting time of X for $s\in\mathbb{N}$ is given by

$$R_s(t)=\frac{1}{F(t)}\sum_{j=0}^{s}\binom{s}{j}(-1)^j t^{s-j}\int_0^t x^j f(x)dx.$$

In life testing situations, the expected additional lifetime given that a component has survived until time x is a function of x, called the mean residual life. More specifically, if the random variable X represents the life of a component, then the mean residual life is given by $m(x)=\mathbb{E}\left(X-x\mid X>x\right)$.

Theorem 6 *Suppose $X\sim TME(a,b,m)$. The mean residual life of X is given by*

$$m(x)=\frac{12e^{3bx}(a-1)(m-1)+3a^2m-8(a-1)ae^{bx}m+6e^{2bx}\left(a+m+(a-3)am\right)}{12b\left((a-1)e^{bx}-a\right)\left(e^{2bx}(m-1)-am+(a-1)e^{bx}m\right)}.$$

Proof: The mean residual life of X can be expressed as

$$m(x) = \frac{1}{\overline{F}(x)} \int_x^{\infty} \overline{F}(u)du,$$

where $\overline{F} = 1 - F$. But

$$\int_x^{\infty} \overline{F}(u)du = \frac{e^{-4bx}}{12b}\left(12(a-1)e^{3bx}(m-1) + 3a^2m - 8(a-1)ae^{bx}m + 6e^{2bx}\left(a + m + (a-3)am\right)\right).$$

Now, proof is completed.

The graphical presentation of the mean residual life time function is shown in Figure 4.

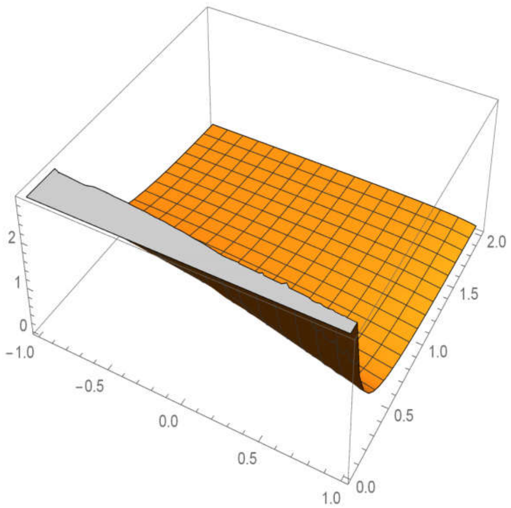

Figure 4 The mean residual life time at $m, x = 1$ for $-1 \le a \le 1$ and $0.01 \le b \le 2$

5. Order Statistics

Order statistics makes its appearance in many areas of statistical theory and practice. Moments of order statistics play an important role in quality control testing and reliability, where a professional needs to predict the failure of future items based on the times of few early failures. These predictors are often based on moments of order statistics. Suppose that $X_1, X_2, ...X_n$ is a random sample from the TME distribution. Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denote the corresponding order statistics. The PDF of the k th order statistics, say $X_{k:n}$ is given by

$$f_{X_{k:n}}(y) = k\binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}(-1)^m F^{k-1+m}(y) f(y).$$

We have

$$F^v(x) = \left(1 - \left(1 - F^v(x)\right)\right)^v = \sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{v}{i}\binom{i}{k}F^k(x) = \sum_{k=0}^{\infty} s_k(v) F^k(x),$$

where

$$s_k(v) = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{v}{i}\binom{i}{k}.$$

Then,

$$f_{X_{k:n}}(y)=k\binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}(-1)^m\sum_{k=0}^{\infty}s_k\left(k-1+m\right)F^k(x)f(y).$$

For example,

$$\begin{aligned}f_{X_{n:n}}(y)&=nF_X^{n-1}(y)f_X(y)\\&=nbe^{-4nbx}\left((e^{bx}-1)(a+e^{bx})\left(e^{2bx}+am-(a-1)e^{bx}m\right)\right)^{n-1}\\&\quad\times\left(a\left(e^{bx}-2\right)-e^{bx}\right)\left(e^{2bx}(m-1)-2am+2(a-1)e^{bx}m\right).\end{aligned}$$

6. Maximum Likelihood Estimation

In this section we consider the maximum likelihood (ML) estimation of parameters of the TME distribution. Consider a random sample $X_1,\ldots,X_n$ from the $TME(a,b,m)$ distribution. Then, the likelihood function is given by

$$\begin{aligned}L(a,b,m\mid x_1,\ldots,x_n)=b^n\prod_{i=1}^{n}\left(a\left(e^{bx_i}-2\right)-e^{bx_i}\right)e^{-4bx_i}\\\times\left(e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m\right).\end{aligned}\tag{5}$$

Now, we can use the likelihood of the parameters (a,b,m) to find their maximum likelihood estimators. According to (5), the log-likelihood function of a, b and m is obtained to be

$$\begin{aligned}\ell\left(a,b,m\right)=n\log b+\sum_{i=1}^{n}(-4bx_i+\log\left(a\left(e^{bx_i}-2\right)-e^{bx_i}\right)\\+\log\left(e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m\right)).\end{aligned}\tag{6}$$

Therefore equating the first order derivations of (6) with respect to a,b and m to zero leads to the following system of equations:

$$\begin{aligned}\frac{\partial\ell\left(a,b,m\right)}{\partial a}&=\sum_{i=1}^{n}\left(\frac{e^{bx_i}-2}{a\left(-2+e^{bx_i}\right)-e^{bx_i}}+\frac{2e^{bx_i}m-2m}{e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m}\right)=0.\\\frac{\partial\ell\left(a,b,m\right)}{\partial b}&=\frac{n}{b}+\sum_{i=1}^{n}\left(-4x_i+\frac{ae^{bx_i}x_i-e^{bx_i}x_i}{a\left(e^{bx_i}-2\right)-e^{bx_i}}+\frac{2e^{2bx_i}\left(m-1\right)x_i+2\left(a-1\right)e^{bx_i}mx_i}{e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m}\right)=0,\\\frac{\partial\ell\left(a,b,m\right)}{\partial m}&=\sum_{i=1}^{n}\frac{2\left(a-1\right)e^{bx_i}+e^{2bx_i}-2a}{e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m}=0.\end{aligned}\tag{7}$$

One can use an iterative maximization procedure to find the solutions of equations (7). Since the MLE of the vector of unknown parameters $\theta=(a,b,m)$ cannot be derived in closed forms, therefore it is not easy to derive the exact distributions of the MLEs and hence we cannot get the exact bounds of the parameters. The idea is to use the large sample approximation. It suffices to compute the Fisher information matrix for obtaining the asymptotic variance of the estimators.

Consider the log-likelihood function of a,b and m in (6). The observed Fisher information matrix is obtain to be

$$I\left(a,b,m\right)=\begin{pmatrix}I_{11}\left(a,b,m\right)&I_{12}\left(a,b,m\right)&I_{13}\left(a,b,m\right)\\I_{21}\left(a,b,m\right)&I_{22}\left(a,b,m\right)&I_{23}\left(a,b,m\right)\\I_{31}\left(a,b,m\right)&I_{32}\left(a,b,m\right)&I_{33}\left(a,b,m\right)\end{pmatrix},$$

where

$$I_{11}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial a^2}=\sum_{i=1}^{n}\left(\frac{(e^{bx_i}-2)^2}{\left(-e^{bx_i}+a(-2+e^{bx_i})\right)^2}\right)+\frac{2(e^{bx_i}-1)m\left(2e^{bx_i}m-2m\right)}{\left(e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}\right)m)^2},$$

$$I_{22}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial b^2}=\frac{n}{b^2}+\sum_{i=1}^{n}\frac{(-e^{bx_i}x_i+ae^{bx_i}x_i)^2}{(-e^{bx_i}+a(-2+e^{bx_i}))^2}$$
$$+\frac{(2e^{2bx_i}(m-1)x_i+2(a-1)e^{bx_i}mx_i)^2}{(e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m)^2}-\frac{-e^{bx_i}x_i^2+ae^{bx_i}x_i^2}{-e^{bx_i}+a(-2+e^{bx_i})}$$
$$-\frac{4e^{2bx_i}(m-1)x_i^2+2(a-1)e^{bx_i}mx_i^2}{e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m},$$

$$I_{33}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial m^2}=\sum_{i=1}^{n}\frac{(-2a+2(a-1)e^{bx_i}+e^{2bx_i})^2}{(e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m)^2},$$

$$I_{12}(a,b,m)=I_{21}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial b\partial a}=-\sum_{i=1}^{n}(\frac{e^{bx_i}x_i}{-e^{bx_i}+a(-2+e^{bx_i})}$$
$$+\frac{2e^{bx_i}mx_i}{e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m}-\frac{(-2+e^{bx_i})\left(-e^{bx_i}x_i+ae^{bx_i}x_i\right)}{\left(-e^{bx_i}+a\left(-2+e^{bx_i}\right)\right)^2}$$
$$-\frac{2(-1+e^{bx_i})m\left(2e^{2bx_i}(m-1)x_i+2(a-1)e^{bx_i}mx_i\right)}{\left(e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m\right)^2},$$

$$I_{13}(a,b,m)=I_{31}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial m\partial a}$$
$$=-\sum_{i=1}^{n}\frac{-2+e^{bx_i}}{-e^{bx_i}+a\left(-2+e^{bx_i}\right)}+\frac{2\left(-1+e^{bx_i}\right)m}{e^{2bx_i}\left(m-1\right)-2am+2\left(a-1\right)e^{bx_i}m},$$

$$I_{23}(a,b,m)=I_{32}(a,b,m)=-\frac{\partial^2\ell(a,b,m)}{\partial m\partial b}$$
$$=-\sum_{i=1}^{n}\frac{2(a-1)e^{bx_i}x_i+2e^{2bx_i}x_i}{e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m}$$
$$-\frac{\left(-2a+2(a-1)e^{bx_i}+e^{2bx_i}\right)\left(2e^{2bx_i}(m-1)x_i+2(a-1)e^{bx_i}mx_i\right)}{(e^{2bx_i}(m-1)-2am+2(a-1)e^{bx_i}m)^2}.$$

Now, based on normality and optimality of the maximum likelihood estimators, we have

$$\hat{\theta}=(\hat{a},\hat{b},\hat{m})\sim AN_3\left((a,b,m),I(a,b,m)^{-1}\right).$$

This asymptotic distribution can be used to construct confidence interval and to test hypotheses concerning a,b and m. The $100(1-\alpha)\%$ asymptotic confidence interval of the parameters a,b and m are obtained, respectively, to be $\hat{a}\pm z_{\alpha/2}\sqrt{I_{11}(\hat{a},\hat{b},\hat{m})^{-1}}$, $\hat{b}\pm z_{\alpha/2}\sqrt{I_{22}(\hat{a},\hat{b},\hat{m})^{-1}}$ and $\hat{m}\pm z_{\alpha/2}\sqrt{I_{33}(\hat{a},\hat{b},\hat{m})^{-1}}$, where z_α is the upper α^{th} quantile of the standard normal distribution.

7. Application

In this section, we use two real data sets to show that the transmuted mixture exponential distribution can be a better model than the mixture exponential distribution. The first set of data involves the intervals in days between successive coal-mining disasters in Great Britan for the period 1875-1951. A disaster is defined as involving the death of 10 or more men. They were originally discussed by Maguire et al. (1952) and analyzed by Cox and Lewis (1978), and Adamidisa et al. (2005). The second set of data, are survival times of 43 patients suffering from chronic granulocytic leukemia. This data set reported by Bryson and Siddiqui (1969) and reanalyzed in Hollander and Proschan (1975). For these data sets, Mirhossaini and Dolati (2008) showed that the mixture exponential distribution can be a better model than the exponential distribution. The maximum likelihood estimates, the log-likelihood (LL) and the Kolmogorov-Smirnov (K-S) statistic presented in Table 1. It is observed that, the TME distribution fits marginally better than ME model in both cases. We used from R software for our data analysis.

For the data set 1, the variance covariance matrix of the MLEs under the transmuted mixture exponential distribution is computed as

$$I(\hat{a},\hat{b},\hat{m})^{-1} = \begin{pmatrix} 0.241\times10^{-4} & 0.325\times10^{-4} & 0.461\times10^{-4} \\ 0.325\times10^{-4} & 0.140\times10^{-7} & 0.295\times10^{-4} \\ 0.461\times10^{-4} & 0.295\times10^{-4} & 0.215\times10^{-3} \end{pmatrix}.$$

Thus, 95% confidence intervals for data set 1 are

$$a\in\left[0.7105, 0.7298\right],\quad b\in\left[0.0019, 0.0024\right],\quad m\in\left[-0.5973, -0.5398\right].$$

For the data set 2, the variance covariance matrix of the MLEs under the transmuted mixture exponential distribution is computed as:

$$I(\hat{a},\hat{b},\hat{m})^{-1} = \begin{pmatrix} 0.335\times10^{-4} & 0.541\times10^{-4} & 0.462\times10^{-4} \\ 0.541\times10^{-4} & 0.120\times10^{-7} & 0.398\times10^{-4} \\ 0.462\times10^{-4} & 0.398\times10^{-4} & 0.276\times10^{-3} \end{pmatrix}.$$

Thus, the 95% confidence intervals for data set 2 are:

$$a\in\left[-0.6375, -0.6148\right],\quad b\in\left[0.0018, 0.0023\right],\quad m\in\left[-0.5659, -0.5008\right].$$

We can use the LR test statistic to check whether the transmuted mixture exponential distribution for a given data set is statistically superior to the mixture exponential distribution. The hypothesis test of the type $H_0: m = m_0$ versus $H_1: m \neq m_0$ can be performed using a LR test. In this case, the LR test statistics for testing H_0 versus H_1 is $W = 2\left(\ell(\hat{m};x) - \ell(\hat{m}_0;x)\right)$, where $\hat{m}$ and $\hat{m}_0$ are the MLEs under H_1 and H_0, respectively. The statistic W is asymptotically (as $n\to\infty$) distributed as χ_1^2. The LR test rejects H_0 if $W > \chi_{1,\gamma}^2$, where $\chi_{1,\gamma}^2$ denotes the upper $100\gamma\%$ quantile of the χ_1^2 distribution. For data set 1, the LR test statistic to test the hypotheses $H_0: m = 0$ versus $H_1: m \neq 0$ is $W = 61.1415 > 3.841$ so we reject the null hypothesis. Also, for data set 2, $W = 58.3345$.

Table 1 Estimates, log-likelihood and Kolmogorov-Smirnov statistic

Data Set	Distribution	Parameter Estimate	LL	K-S
($n=109$)	ME	$\hat{a}=0.7126$	−700.9823	0.0667
		$\hat{b}=0.0028$		
	TME	$\hat{a}=0.7201$	−698.3432	0.0521
		$\hat{b}=0.0022$		
		$\hat{m}=-0.5685$		
($n=43$)	ME	$\hat{a}=-0.6103$	−335.2714	0.0678
		$\hat{b}=0.0014$		
	TME	$\hat{a}=-0.6261$	−332.8879	0.0598
		$\hat{b}=0.0021$		
		$\hat{m}=-0.5333$		

8. Conclusions

Here we propose a new model, the so-called the transmuted mixture exponential distribution which extends the mixture exponential distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the transmuted form provides larger flexibility in modeling real data. We derive expansions for the mean, variance, moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the transmuted mixture exponential distribution to two real data sets show that the new distribution can be used quite effectively to provide better fits than the mixture exponential distribution.

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