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Statistical Inference for the Extended Weibull Distribution Based on Adaptive Type-II Progressive Hybrid Censored Competing Risks Data

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Abstract

This paper discussed the unknown parameters of extended Weibull distribution under adaptive type II progressive hybrid censoring scheme (AT-II PHCS) in the existence of the competing risks model. Depending on this scheme the maximum likelihood and Bayesian estimators of the distribution parameters are obtained. Bayes estimators have been developed using the standard Bayes method under square error, using gamma prior for the parameter. Also, the asymptotic confidence intervals and two bootstrap conference intervals are offered. As a final point, the maximum likelihood, bootstrap and Bayes estimates are set in a comparison via a Monte Carlo simulation study. Finally, a set of real data is used to test the hypothesis that the causes of failure follow extended Weibull distribution.

Keywords: Maximum likelihood estimation, Bayesian estimation, bootstrap confidence intervals, extended Weibull model, Markov chain Monte Carlo.

1. Introduction

The combination of type I and type II censoring schemes is known as the hybrid censoring scheme and was initially proposed by Epstein (1954) in the setting of life testing experimentations. In such a scheme, the experiment is halted at time $T^* = \min\{X_{m:n}, T\}$ where $T \in (0,\infty)$ and $1 \le m \le n$ are fixed in advance, and $X_{m:n}$ is indicated the m^{th} failure time in which n items are employed in a life test. Several authors, for example Gupta and Kundu (1998), Childs et al. (2003), Kundu (2007), Banerjee and Kundu (2008), Dube et al. (2011) and Almetwally et al. (2018), have considered this sampling scheme.

Absence of elasticity to remove the units from the experiment at any point excluding the terminal point is the foremost disadvantage of the conventional type I, type II or hybrid censoring schemes. Thus, we are driven straightforward to the range of progressive hybrid censoring. Kundu and Joarder (2006a) and Childs et al. (2008) set a general rule for the above-mentioned hybrid censoring scheme to the case in which the observed sample is progressively censored. They thought that the progressive type II hybrid censoring scheme in which n units undergo a test with censoring scheme $(R_1, R_2, ..., R_m)$ and stopping time $T^* = \min\{X_{m:m:n}, T\}$, where $X_{1:m:n} \leq X_{2:m:n} \leq ... \leq X_{m:m:n}$ be the observed ordered failure times driven from the progressively censored experiment and T is fixed ahead. Briefly, if $X_{m:m:n} < T$, the experiment terminates at time $X_{m:m:n}$ and m failures occur. As an alternative, the experiment stops at time T and only J failure happens before time T, where $X_{J:m:n} < T < X_{J+1:m:n}$, and $0 \leq J \leq m.$ The detailed description of the progressive type II hybrid censoring scheme is presented in Kundu and Joarder (2006a) and Childs et al. (2008) (see Kundu et al. (2009)). Although, in order to control the total on test, the experiment time is fixed by the experiment, so less than m failures (or even equal to zero) might be observed which delivers an advance effect on the efficiency of the inferential producer based on the progressive type II hybrid censoring scheme. Consequently, it is appropriate to have a model that takes into account an adaption process.

For the purpose of increasing the efficiency of statistical analysis as well as saving the total test time, Ng et al. (2009) presented a modification of progressive type II hybrid censoring scheme, so called adaptive type II progressive hybrid censoring scheme (AT-II PHCS), and investigated the statistics under the assumptions of experiment lifetime distribution of the experimental units. Covered by this scheme, the number of observed failures m is fixed beforehand but the experiment time is permitted to run over the (pre-fixed) threshold time $T > 0.$ If $X_{m:m:n} < T,$ the experiment stops at time $X_{m:m:n},$ and we will have a usual progressive type II censoring scheme with the pre-fixed progressive censoring scheme $(R_1, R_2, ..., R_m).$ If $X_{J:m:n} < T < X_{J+1:m:n},$ where $J+1 < m,$ we adapt the number of items progressively removed from the experiment upon failure by setting $R_{J+1} = R_{J+2} = ... = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{J} R_i.$ Accordingly, the effectively involved scheme has to be $\left(R_1, R_2, ..., R_J, 0, ..., 0, n - m - \sum_{i=1}^{J} R_i\right),$ where $J = \max\{j : X_{J:m:n} < T\},$ that is, the first observed failure time surpassing the perfect total time $T.$ To put in another way, as long as the failure happens ahead of time $T,$ the initially organized progressive scheme is applied. After passing time $T,$ we do not withdraw any items at all expect for the time of the m^{th} failure where all residual surviving items are eliminated. This resolution generates cessation of the experiment once the $(J+1)-m^{\text{th}}$ failure is higher than $T,$ and the total test time will not be too far away from time $T.$ If $T = 0,$ the scheme will lead us to the case of the conventional type II censoring scheme, and if $T \to \infty$, we will have a usual progressive type II censoring scheme. This approach illustrates how an experiment can control the experiment. The experiment can decide to change the value of T as a compromise between a shorter experiment time and a higher chance to observe extreme failures. Recently Lin et al. (2009), and Hemmati and Khorram (2013) discussed inference regarding the above two progressively censoring schemes for the Weibull and log-normal distributions, respectively. Also, extending the model of progressive type II censoring, Cramer and Iliopoulos (2010) presented an alteration procedure. It permits us to choose the following censoring number paying attention to both the preceding censoring numbers and previous failure times. In the meantime, Balakrishnan and Cramer (2014) comprehensively discuss the progressive censoring. Ashour and Abu El Azm (2016a) obtained the maximum likelihood estimates and the corresponding Fisher information matrix of the generalized Weibull distribution based on Type-I hybrid progressive censoring scheme (Type-I PHCS) in the presence of competing risks when the cause of failure of each item is known. Ashour and Abu El Azm (2016b) introduced a new scheme called progressive hybrid Type-II censoring scheme in the presence of competing risks (Type-II PHCS). Based on this scheme and assumed that the lifetimes of the failure times have an exponential distribution, the maximum likelihood and Bayes estimators of the distribution parameters are obtained, and the asymptotic confidence intervals and Bayes credible intervals are also proposed. Nassar et al. (2017) used the maximum likelihood estimation to estimate the unknown parameters and acceleration factor based on Type I progressive hybrid censoring scheme (T-I PHCS) and adaptive Type-II progressive hybrid censoring scheme (AT-II PHCS) under step-stress partially accelerated life test model. Almetwally et al. (2019) estimated the unknown parameters of the generalized Rayleigh distribution under AT-II PHCS based on maximum product spacing, MLE and Bayesian method.

Additionally, in reliability analysis, a failure is ordinarily related to one of numerous disastrous risk factors to which the test unit undergoes. As it is not commonly feasible to examine the test with a secluded risk factor, it necessitates weighing each risk factor alongside with other risk factors. For the sake of the analysis of such a competing risks model, every failure observation has to be stated in a two-way form that consists of a failure time and the root of failure. Over and above numerous causes of failure, censoring habitually takes place, in reliability experiment, for many reasons such as time restriction and cost reduction. Breaking down of different censoring schemes under competing risks gained rather wide popularity a few years ago. Kundu et al. (2004), Pareek et al. (2009) and Cramer and Schmiedt (2011) have all examined the progressive type II censoring scheme in the existence of competing risks and regarding some definite parametric lifetime distributions for every risks factor. In addition, Kundu and Joarder (2006b) made a study over the breaking down of progressively type II hybrid censored data in the current competing risks. As recognizable of failure causes is one of the established hypothesis in all the researches mentioned earlier. In definite situations, (see for example, Dinse 1982, and Miyakawa 1982) it is clear that the determination of the cause of failure may be very costly or difficult to obtain. In those conditions, one might observe the failure time, but not the corresponding cause of failure. In recent years, some of authors have investigated the competing failure models in partially accelerated life testing, see for example, Shi et al. (2013), Han and Kundu (2015), Haghighi and Bae (2015), Zhang et al. (2016), Shi et al. (2016), Lone et al. (2017), Wang (2018) and Hassan et al. (2020).

This article could be curated in the following manner. Section 2 presents model description plus Notations. Maximum likelihood estimation and asymptotic variances and covariance matrix of the unknown parameters are to be dealt with in Section 3. Section 4 provides the two parametric bootstrap confidence intervals (CIs) and approximate confidence interval for unknown parameters. Section 5 considers the Bayesian approach that utilizes the well-known Markov chain Monte Carlo (MCMC) models. Section 6 provides an explanation of the simulation study and the theoretical results. Section 7 presents a numerical example to illustrate all methods of inference established in the article in hand. Finally, conclusions are included in Section 8. Tables are displayed in the Appendix.

2. Model Description and Notations

In reliability analysis, the failure of items could be assigned to multiple causes simultaneously. These causes are competing for the failure of the experiment unit. Let's inspect a life time experiment with $n \in N$ identical units, where its lifetimes are defined by independent and identically distributed (i.i.d.) random variables $X_1, X_2, ..., X_n$. Without loss of generality, assume that there are only two causes of failure. We have $X_i = \min\{X_{1i}, X_{2i}\}$ for $i = 1, ..., n$, where $X_{ki}, k = 1, 2,$ represents the latent failure time of the i^{th} unit under the k^{th} cause of failure. We assume that the latent failure times X_{1i} and X_{2i} are i.i.d. and the failure times follow the two parameter bathtub-shaped life time (extended Weibull) distribution. The cumulative distribution (cdf), the probability density function (pdf) and failure rate function are as follows

$$F_k(x;\alpha_k,\beta_k) = 1 - \exp\left[\alpha_k(1 - e^{x^{\beta_k}})\right];\ x > 0,\ \alpha_k, \beta_k > 0,\ k = 1, 2, \tag{1}$$

$$f_k(x;\alpha_k,\beta_k) = \alpha_k\,\beta_k\,x^{\beta_k - 1}\,e^{x^{\beta_k}}\,\exp\left[\alpha_k(1 - e^{x^{\beta_k}})\right];\ x > 0,\ \alpha_k, \beta_k > 0, k = 1, 2, \tag{2}$$

and

$$h_k(x;\alpha_k,\beta_k) = \alpha_k\beta_k x^{\beta_k - 1} e^{x^{\beta_k}};\ x > 0,$$

where $\beta_k > 0$ is the shape parameter and $\alpha_k > 0$ is the scale parameter. $h_k(x;\alpha_k,\beta_k)$ has a bathtub shape when $\beta_k < 1$ and might be increasing when $\beta_k \geq 1$. Specifically, when X is from Chen distribution with parameter β_k and α_k, let $Z = \exp(X^{\beta_k}) - 1$, the new variable Z is distributed as exponential distribution with parameter α_k.

Chen (2000) discussed exact confidence intervals and exact joint confidence regions for the parameters depending on a type II censored sample. Wu et al. (2004) explained statistical inference about the shape parameter of this distribution based on type II right censored data. Wu (2008) explored the estimation problem of progressively type II censored data from this distribution utilizing the maximum likelihood technique. Zhang and Shi (2016) introduced the maximum likelihood method to estimate the unknown parameters and speeded up factors in the general step-stress accelerated life tests depending on adaptive type II progressively hybrid censoring data.

Under AT-II PHCS competing risks data we have the following observation

$$(X_{1:m:n},\delta_1,R_1),...,(X_{D:m:n},\delta_D,R_D),(X_{D+1:m:n},\delta_{D+1},0),...,(X_{m-1:m:n},\delta_{m-1},0),(X_{m:m:n},\delta_m,R_m),$$

where $D = \max\{D : X_{D:m:n} < T\}$, $R_m = n - m - \sum_{i=1}^{D} R_i$ and $\delta_i \in (1,2)$. Here, $\delta_i = 1, 2$ means the unit i has failed at time $X_{i:m:n}$ because of the first and the second cause of failures, respectively. Let

$$I_1(\delta_i = 1) = \begin{cases} 1, & \delta_i = 1, \\ 0, & \text{elsewhere}, \end{cases} \quad I_2(\delta_i = 2) = \begin{cases} 1, & \delta_i = 2, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{and } I_3(\delta_i = *) = \begin{cases} 1, & \delta_i = *, \\ 0, & \text{elsewhere}. \end{cases}$$

Thus the random variables $m_1 = \sum_{i=1}^{m} I_1(\delta_i = 1)$ and $m_2 = \sum_{i=1}^{m} I_2(\delta_i = 2)$ describe the number of failures due to the first and the second cause of failures, respectively and $m_3 = \sum_{i=1}^{m} I_3(\delta_i = *)$ is the number of failures having failure times but corresponding causes of failure are unknown. Hemmati and Khorran (2011), wrote the likelihood function in this case as follows

$$L = C\prod_{i=1}^{m}\left[\left(f_1(x_i)\bar{F}_2(x_i)\right)^{I(\delta_i=1)}\left(f_2(x_i)\bar{F}_1(x_i)\right)^{I(\delta_i=2)}\left(f_1(x_i)\bar{F}_2(x_i)+f_2(x_i)\bar{F}_1(x_i)\right)^{I(\delta_i=*)}\right]$$
$$\times\prod_{i=1}^{D}\left[\bar{F}_1(x_i)\bar{F}_2(x_i)\right]^{R_i}\left[\bar{F}_1(x_m)\bar{F}_2(x_m)\right]^{R^*},$$

where $m = m_1 + m_2$, $f_k(x)$ is the pdf, $F_k(x)$ is the cdf, $k = 1,2$ and $\bar{F}_k(x) = 1 - F_k(x)$.

We assume that there are only two causes of failure and the case of failure is known, then under AT-II PHCS existence competing risks data, we have the following observation

$$(x_{(1)},\delta_1,R_1),\ldots,(x_{(D)},\delta_D,R_D),(x_{(D+1)},\delta_{D+1},0),\ldots,(x_{(m)},\delta_m,R^*).$$

From the above equation we can write the likelihood function in this cause as follows

$$L = C\prod_{i=1}^{m}\left[\left(f_1(x_i)\bar{F}_2(x_i)\right)^{I(\delta_i=1)}\left(f_2(x_i)\bar{F}_1(x_i)\right)^{I(\delta_i=2)}\right]\prod_{i=1}^{D}\left[\bar{F}_1(x_i)\bar{F}_2(x_i)\right]^{R_i}\left[\bar{F}_1(x_m)\bar{F}_2(x_m)\right]^{R^*}, \quad (3)$$

where C is a constant which does not dependent on parameters.

3. Maximum Likelihood Estimation

In the existence of AT-II PHCS under competing risks data (3) and from the life time distribution (1) and (2), then the likelihood function of the observed data ignoring the constant can be written as:

$$L \propto (\alpha_k\,\beta_k)^{m_k}\left(\prod_{i=1}^{m_k}x_i^{\beta_k-1}\right)\exp\left\{\sum_{i=1}^{m_k}x_i^{\beta_k}-\alpha_k\left[-\sum_{i=1}^{m_k}u_{ki}-\sum_{i=1}^{m_{3-k}}u_{ki}-\sum_{i=1}^{D}R_iu_{ki}-R^*u_{km}\right]\right\} \quad (4)$$

where $u_{ki} = u_{ki}(\beta_k) = (1-e^{x_i^{\beta_k}})$, $u_{km} = u_{km}(\beta_k) = (1-e^{x_m^{\beta_k}})$, $k = 1,2$, $L = L(\alpha_1,\alpha_2,\beta_1,\beta_2)$, $x_i = x_{(i)}$ for simplicity of notation $m_1 = \sum_{i=1}^{m_1} I(\delta_i = 1)$, and $m_2 = \sum_{i=1}^{m_2} I(\delta_i = 2)$ describe the number of failure that are attributable to the first and the second cause of failure, respectively. Taking the natural logarithm likelihood function $l = \ln L$ in (4) we obtain

$$l \propto m_1(\ln\alpha_1+\ln\beta_1)+m_2(\ln\alpha_2+\ln\beta_2)+\sum_{i=1}^{m_1}\left[(\beta_1+1)\ln x_i + x_i^{\beta_1}+\alpha_1u_{1i}+\alpha_2u_{2i}\right]$$
$$+\sum_{i=1}^{m_2}\left[(\beta_2+1)\ln x_i + x_i^{\beta_2}+\alpha_2u_{2i}+\alpha_1u_{1i}\right]+\sum_{i=1}^{D}R_i\left[\alpha_1u_{1i}+\alpha_2u_{2i}\right]+R^*\left[\alpha_1u_{1m}+\alpha_2u_{2m}\right]. \quad (5)$$

The first order derivatives of (5) with respect to α_k, β_k and $k = 1,2$ are given respectively by

$$\frac{\partial l}{\partial\beta_k} = \frac{m_k}{\beta_k}+\sum_{i=1}^{m_k}\ln x_i+\sum_{i=1}^{m_k}\left[x_i^{\beta_k}\ln x_i-\alpha_kV_{ki}\right]-\alpha_k\sum_{i=1}^{m_{3-k}}V_{ki}-\alpha_k\sum_{i=1}^{D}R_iV_{ki}-\alpha_kR^*V_{km}, \quad (6)$$

$$\frac{\partial l}{\partial\alpha_k} = \frac{m_k}{\alpha_k}+\sum_{i=1}^{m_k}u_{ki}+\sum_{i=1}^{m_{3-k}}u_{ki}+\sum_{i=1}^{D}R_iu_{ki}+R^*u_{km}, \quad (7)$$

where $V_{ki} = V_{ki}(\beta_k) = e^{x_i^{\beta_k}}x_i^{\beta_k}\ln x_i$, $V_{km} = V_{km}(\beta_k) = e^{x_m^{\beta_k}}x_m^{\beta_k}\ln x_m$ and $k = 1,2$.

Based on Equation (7) the maximum likelihood estimator (MLE) of α_1 and α_2 is expressed by

$$\hat{\alpha}_k = \frac{-m_k}{\sum_{i=1}^{m_k} u_{ki} + \sum_{i=1}^{m_{3-k}} u_{ki} + \sum_{i=1}^{D} R_i u_{ki} + R^* u_{km}}, \ k = 1, 2. \tag{8}$$

Consequently, by substituting $\hat{\alpha}_k$ into Equation (6), the system equation reduced to nonlinear equation as follows

$$\frac{m_k}{\hat{\beta}_k} + \sum_{i=1}^{m_k} \ln x_i + \sum_{i=1}^{m_k} \left[x_i^{\hat{\beta}_k} \ln x_i - \hat{\alpha}_k V_{ki} \right] - \hat{\alpha}_k \sum_{i=1}^{m_{3-k}} V_{ki} - \hat{\alpha}_k \sum_{i=1}^{D} R_i V_{ki} - \hat{\alpha}_k R^* V_{km} = 0 \ , \ k = 1, 2. \tag{9}$$

Since the closed from solution to nonlinear Equation (9) is very hard to achieve the MLE of the unidentified parameters β_1 and β_2. So a numerical method technique is required for competing the MLE of the parameters β_1 and β_2. Therefore α_1 and α_2 is calculated easily from Equation (8).

The asymptotic variance covariance matrix for $\alpha_1, \alpha_2, \beta_1$ and β_2 can be achieved by reversing the information matrix with the elements that are not positive of the expected values of the second order derivatives of logarithms of the likelihood functions. Hence, the Fisher information matrix related to $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ can be prescribed as

$$I(\theta) = \begin{bmatrix} \frac{\partial^2 l}{\partial \beta_1^2} & 0 & \frac{\partial^2 l}{\partial \beta_1 \partial \alpha_1} & 0 \\ 0 & \frac{\partial^2 l}{\partial \beta_2^2} & 0 & \frac{\partial^2 l}{\partial \beta_2 \partial \alpha_2} \\ \frac{\partial^2 l}{\partial \alpha_1 \partial \beta_1} & 0 & \frac{\partial^2 l}{\partial \alpha_1^2} & 0 \\ 0 & \frac{\partial^2 l}{\partial \alpha_2 \partial \beta_2} & 0 & \frac{\partial^2 l}{\partial \alpha_2^2} \end{bmatrix},$$

The elements of 4×4 matrix $I(\theta), I_{ij}(\theta), \ i, j = 1, 2, 3, 4$ can be obtained as follows:

$$\frac{\partial^2 l}{\partial \alpha_k^2} = -\frac{m_k}{\alpha_k^2},$$

$$\frac{\partial^2 l}{\partial \alpha_k \, \partial \beta_k} = -\left[\sum_{i=1}^{m_k} V_{ki} + \sum_{i=1}^{m_{3-k}} V_{ki} + \sum_{i=1}^{D} R_i V_{ki} + R^* V_{km} \right],$$

$$\frac{\partial^2 l}{\partial \beta_k^2} = -\frac{m_k}{\beta_k^2} + \sum_{i=1}^{m_k} x_i^{\beta_k} (\ln x_i)^2 \left[1 - \alpha_k \, e^{x_i^{\beta_k}} (1 + x_i^{\beta_k}) \right] - \alpha_k \sum_{i=1}^{m_{3-k}} V_{ki} \ln x_i (1 + x_i^{\beta_k})$$
$$- \alpha_k \sum_{i=1}^{D} R_i \, V_{ki} \ln x_i (1 + x_i^{\beta_k}) - \alpha_k \, R^* V_{km} \ln x_m (1 + x_m^{\beta_k}).$$

Now, we derive the relative risk rates, π_1 and π_2 due to case 1 and 2, respectively. The relative risk due to case 1 is defined as

$$\pi_1 = P(X_{1i} \le X_{2i}) = \int_0^\infty f_1(x) \overline{F}_2(x) \, dx = \alpha_1 \beta_1 \int_0^\infty x_i^{\beta_1 - 1} \exp\left[x_i^{\beta_1} + \alpha_1 (1 - e^{x_i^{\beta_1}}) + \alpha_2 (1 - e^{x_i^{\beta_2}}) \right] dx. \tag{10}$$

Once π_1 is computed, we determine π_2 using the relation $\pi_2 = 1 - \pi_1$,

$$\pi_2 = 1 - \alpha_1 \beta_1 \int_0^\infty x_i^{\beta_1 - 1} \exp\left[x_i^{\beta_1} + \alpha_1 (1 - e^{x_i^{\beta_1}}) + \alpha_2 (1 - e^{x_i^{\beta_2}}) \right] dx.$$

As the integral in the right side of (10) is not to be attributed to any methodical clarification, we have to use a numerical procedure to resolve the integral. As maintained by the invariance property of the MLE, the MLE of the relative risk rates π_1 and π_2, may be achieved by replacing the MLE of $\alpha_1, \alpha_2, \beta_1$ and β_2 in (10).

4. Confidence Interval

Here, we introduced different confidence intervals. The first is established on the asymptotic distribution of $\alpha_k, \beta_k,\ k = 1,2$ and two different bootstrap confidence intervals.

4.1. Asymptotic confidence interval (ACI)

We may derive the approximate confidence intervals of the parameters on the asymptotic distribution of the MLEs of the elements of the vector of unknown parameters $\theta = (\alpha_k, \beta_k)$ and $k = 1,2.$ It is known that the asymptotic distribution of the MLEs of $\left(\frac{\hat{\theta} - \theta}{\sqrt{\mathrm{Var}(\hat{\theta})}}\right)$ can be approximated by a standard normal distribution, where $Var(\hat{\theta})$ is estimated as the asymptotic variance, then, the approximate $100(1-\gamma)\%$ two sided confidence interval for $\theta = (\alpha_k, \beta_k)$ and $k = 1,2$ are achieved, hence;

$$P\left[\hat{\theta} - Z_{\gamma/2}\sqrt{Var(\hat{\theta})} \le \theta \le \hat{\theta} + Z_{\gamma/2}\sqrt{Var(\hat{\theta})}\right] \cong \gamma,$$

where $Z_{\gamma/2}$ is the $100(1-\gamma/2)\%$ standard normal percentile.

4.2. Bootstrap confidence interval

In this subsection, we construct two parametric bootstrap confidence intervals for α_k, β_k and $k = 1,2$ as:

4.2.1. Percentile bootstrap confidence interval (Boot-P)

1) Compute the MLE of $\theta = (\alpha_k, \beta_k)$ and $k = 1,2$ based on AT-II PHCS under competing risks data.
2) Generated a bootstrap samples using α_k, β_k and $k = 1,2$ to obtain the bootstrap estimate of α_k say $\hat{\alpha}_k^b, \beta_k$ say $\hat{\beta}_k^b$ and $k = 1,2$ using the bootstrap sample.
3) Repeat Step 2 B times to get $\left(\alpha_k^{b(1)}, \alpha_k^{b(2)}, ..., \alpha_k^{b(B)}\right)$ and $\left(\beta_k^{b(1)}, \beta_k^{b(2)}, ..., \beta_k^{b(B)}\right)$.
4) Arrange $\left(\alpha_k^{b(1)}, \alpha_k^{b(2)}, ..., \alpha_k^{b(B)}\right)$ and $\left(\beta_k^{b(1)}, \beta_k^{b(2)}, ..., \beta_k^{b(B)}\right)$ in ascending order as $\left(\alpha_k^{b[1]}, \alpha_k^{b[2]}, ..., \alpha_k^{b[B]}\right)$ and $\left(\beta_k^{b[1]}, \beta_k^{b[2]}, ..., \beta_k^{b[B]}\right)$.
5) A two-sided $100(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters α_k, β_k and $k = 1,2$ is set by $\left\{\hat{\alpha}_k^{b[B\gamma/2]}, \hat{\alpha}_k^{b[B(1-\gamma/2)]}\right\}$ and $\left\{\hat{\beta}_k^{b[B\gamma/2]}, \hat{\beta}_k^{b[B(1-\gamma/2)]}\right\}$.

4.2.2. Bootstrap-t confidence interval (Boot-t)

1) The same steps as (1-2) in Boot-P.

2) Compute the t-statistic of $\theta=(\alpha_k,\beta_k)$ and $k=1,2$ as $T=(\hat{\theta}_k^b-\hat{\theta}_k)\Big/\sqrt{Var(\hat{\theta}_k^b)}$ where $Var(\hat{\theta}_k^b)$ is asymptotic variances of $\hat{\theta}_k^b$ and it can be obtained using the Fisher information matrix.
3) Repeat Steps 2-3 B times and obtain $T^{(1)},T^{(2)},...,T^{(B)}$.
4) Arrange $T^{(1)},T^{(2)},...,T^{(B)}$ in ascending order as $T^{[1]},T^{[2]},...,T^{[B]}$.
5) A two-sided $100(1-\gamma)\%$ percentile bootstrap-t confidence interval for the unknown parameters α_k,β_k and $k=1,2$ is given by

$$\left(\hat{\alpha}_k+T_k^{[B\gamma/2]}\sqrt{Var(\hat{\alpha}_k)},\ \ \hat{\alpha}_k+T_k^{[B(1-\gamma/2)]}\sqrt{Var(\hat{\alpha}_k)}\right),$$

and

$$\left(\hat{\beta}_k+T_k^{[B\gamma/2]}\sqrt{Var(\hat{\beta}_k)}\ \ ,\ \ \hat{\beta}_k+T_k^{[B(1-\gamma/2)]}\sqrt{Var(\hat{\beta}_k)}\right).$$

5. Bayesian Estimation

In this section, the Bayes estimate using squared error loss function under the assumption of gamma prior of the unknown parameters of the extended Weibull distribution is to be achieved depending on AT-II PHCS in the existence of competing risks data. One may consider the Bayesian estimation under the assumption that the random variables α_k,β_k and $k=1,2$ are independently distributed with gamma prior distribution with defined shape and scale parameters v_k,b_k,c_k,d_k and $k=1,2,$ with pdf as

$$\pi(\alpha_k)\propto\alpha_k^{v_k-1}\exp(-\alpha_k b_k),\ v_k,b_k>0,k=1,2$$

and

$$\pi(\beta_k)\propto\beta_k^{c_k-1}\exp(-\beta_k d_k),\ \ c_k,d_k>0,k=1,2.$$

Hence, the joint prior density of unknown parameters α_k and β_k can be written as

$$\pi\left(\alpha_k,\beta_k\right)\propto\alpha_k^{v_k-1}\beta_k^{c_k-1}\exp\left(-\alpha_k b_k-\beta_k d_k\right),\ v_k,b_k,c_k,d_k>0,k=1,2. \tag{11}$$

Combining (4) and (11) to obtain the posterior density of $\theta=(\alpha_k,\beta_k)$ and $k=1,2$ considering the next form

$$\pi^*(\theta|\,\underline{x})=\frac{L(\theta|\,\underline{x}).\pi(\alpha_k,\beta_k)}{\int_\theta L(\theta|\,\underline{x}).\pi(\alpha_k,\beta_k)d\theta}. \tag{12}$$

Therefore, the Bayes estimates of the unknown parameters $\theta=(\alpha_k,\beta_k)$ and $k=1,2$ based on AT-II PHCS in the existence of competing risks under squared error denoted by $\tilde{\theta}_{(\mathrm{BESL})}$; can be calculated through the following equations as follows

$$\tilde{\theta}_{(\mathrm{BESL})}=E(\theta|\,\underline{x})=\int_0^\infty\theta\,\pi^*(\theta|\,\underline{x})d\theta. \tag{13}$$

Normally, the ratio of four integrals given by Equation (13) are not to be obtained in a closed form. In this case, one may utilize the MCMC technique to generate samples from the posterior distributions, after that, compute the Bayes estimators of the individual parameters.

5.1. MCMC approach

A broad diversity of MCMC schemes is accessible, and any researcher may find difficulty in selecting one of them. A vital sub-class of MCMC methods is Gibbs sampling and more general Metropolis within Gibbs samplers. The benefit of employing the MCMC method over the MLE method can be revealed as one may always attain a sound interval estimate of the parameters by constructing the probability intervals depending on empirical posterior distribution. The aforementioned may not frequently be feasible in ML estimation. Indeed, the MCMC samples may be used to completely summarize the posterior uncertainty about the parameters $\theta=(\alpha_k,\beta_k)$ and $k=1,2,$ through a kernel estimate of the posterior distribution. This is also true of any function of the parameters.

The joint posterior density functions of $\theta=(\alpha_k,\beta_k)$ and $k=1,2$ can be written as

$$\pi^*(\theta|\underline{x}) \propto \alpha_k^{m_k+v_k-1}\,\beta_k^{m_k+c_k-1}\left(\prod_{i=1}^{m_k} x_i^{\beta_k-1}\right)\exp\left(\sum_{i=1}^{m_k} x_i^{\beta_k}-\beta_k d_k-\alpha_k w_{ki}\right),$$

$$v_k,b_k,c_k,d_k,\alpha_k,\beta_k>0, k=1,2 \quad (14)$$

where $w_{ki}=w_{ki}(\beta_k)=b_k-\sum_{i=1}^{m_k}u_{ki}-\sum_{i=1}^{m_{3-k}}u_{ki}-\sum_{i=1}^{D}R_i\,u_{ki}-R^*u_{km}.$

The conditional posterior densities of $\theta=(\alpha_k,\beta_k)$ and $k=1,2$ have the following forms:

$$\pi_1^*(\alpha_1\,|\,\alpha_2,\beta_1,\beta_2,x)\sim Gamma\left(m_1+v_1,w_{1i}\right), \quad (15)$$

$$\pi_2^*(\alpha_2\,|\,\alpha_1,\beta_1,\beta_2,x)\sim Gamma\left(m_2+v_2,w_{2i}\right), \quad (16)$$

$$\pi_3^*(\beta_1\,|\,\alpha_1,\alpha_2,\beta_2,x)\propto \beta_1^{m_1+c_1-1}\left(\prod_{i=1}^{m_1} x_i^{\beta_1-1}\right)\exp\left(\sum_{i=1}^{m_1} x_i^{\beta_1}-\beta_1 d_1-\alpha_1 w_{1i}\right), \quad (17)$$

and

$$\pi_4^*\left(\beta_2\,|\,\alpha_1,\alpha_2,\beta_1,x\right)\propto \beta_2^{m_2+c_2-1}\left(\prod_{i=1}^{m_2} x_i^{\beta_2-1}\right)\exp\left(\sum_{i=1}^{m_2} x_i^{\beta_2}-\beta_2 d_2-\alpha_2 w_{2i}\right). \quad (18)$$

It is rather obvious that both (15) and (16) are gamma distributed, consequently, samples of α_1 and α_2 may be created without difficulty by employing any of the gamma generating procedures. The posterior of β_1 and β_2 in (17) and (18) are not known. Thus, to derive from this distributions, one may employ the Metropolis-Hastings method (Metropolis et al. (1953) with normal proposal distribution). For more information concerning the application of Metropolis-Hasting algorithm, readers may refer to Robert and Casella (2004).

To run the Gibbs sampler algorithm, we started with the MLEs. We then drew samples from various full conditionals, in turn, using the most recent values of all other conditioning variables unless some systematic pattern of convergence was achieved.

5.2. The algorithm Gibbs sampling

The algorithm Gibbs sampling can be described as follows.

Step 1: Start with an $\left(\alpha_1^{(0)}=\alpha_1,\alpha_2^{(0)}=\alpha_2,\beta_1^{(0)}=\beta_1,\beta_2^{(0)}=\beta_2\right)$ and set $I=1$.

Step 2: Generate α_1^I from $\pi_1^*(\alpha_1|\alpha_2,\beta_1,\beta_2,x)$.

Step 3: Generate α_2^I from $\pi_2^*(\alpha_2 \,|\alpha_1, \beta_1, \beta_2, x)$.

Step 4: Generate β_1^I from $\pi_3^*(\beta_1 \,|\alpha_1, \alpha_2, \beta_2, x)$.

Step 5: Generate β_2^I from $\pi_4^*(\beta_2 \,|\alpha_1, \alpha_2, \beta_1, x)$.

Step 6: Compute $\alpha_1^I, \alpha_2^I, \beta_1^I$ and β_2^I.

Step 7: Set $I = I + 1$.

Step 8: Repeat Steps 2-6 N times.

Step 9: We get the Bayes MCMC point estimate of $\theta_I \left(\theta_1 = \alpha_1, \theta_2 = \alpha_2, \theta_3 = \beta_1, \theta_4 = \beta_2\right)$, $q = 1, 2, 3, 4$ as

$$E(\theta_q \,|\text{data}) \propto \left(\sum_{i=M+1}^{N} \theta_q^{(i)} \right) \Big/ (N - M),$$

where M is the burn-in period (that is, some iterations beforehand the stationary distribution perform) and the posterior variance of θ becomes

$$\hat{V}(\theta_q \,|\text{data}) \propto \left[\sum_{i=M+1}^{N} \left(\theta_q^{(i)} - \hat{E}(\theta_q \,|\text{data}) \right)^2 \right] \Big/ (N - M).$$

6. Simulation Study

Here, the researcher will carry out a simulation study to assess the performance of the estimations using R package. The estimates of parameters of the extended Weibull distribution under AT-II PHCS are evaluated in terms of their Bias, mean squared errors (MSE) and length of CIs. The numerical procedure is designed as below:

For different sample size $n = 50$, 100 and 200. Choosing different effective sample sizes by using ratio of effective sample size $m / n = 0.3$.

According Balakrishnan and Sandhu (1995), we generate m random sample of size under AT-II PHCS with different schemes and in existence of competing risks data researchers may obtain the reflection to follow

$$\left(X_{1:m:n}, \delta_1, R_1\right), \ldots, \left(X_{D:m:n}, \delta_D, R_D\right), \left(X_{D+1:m:n}, \delta_{D+1}, 0\right), \ldots, \left(X_{m-1:m:n}, \delta_{m-1}, 0\right), \left(X_{m:m:n}, \delta_m, R_m\right),$$

where $D = \max\{D : X_{D:m:n} < T\}$, then $X_{i:m:n} \sim EW\left[\left(\alpha_1 + \alpha_2\right), \left(\beta_1 + \beta_2\right)\right]$.

For different time selecting of trail $T = 0.5$ and 1.5.

Selecting $\alpha_1 = 1.1, \beta_1 = 0.4, \alpha_2 = 1.5, \beta_2 = 0.8$ in every cases and consider five different sampling schemes:

Scheme 1: $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - m$,

Scheme 2: $R_1 = n - m$ and $R_2 = \ldots = R_m = 0$,

Scheme 3: $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - 2m + 1$,

Scheme 4: $R_1 = n - 2m + 1$ and $R_2 = \ldots = R_m = 0$, and

Scheme 5: $R_1 = \ldots = R_{m/3} = 3$ and $R_{(m/3)+1} = \ldots = R_m = 2$.

Thus the random variables $m_1 = \sum_{i=1}^{m} I_1(\delta_i = 1)$, $m_2 = \sum_{i=1}^{m} I_2(\delta_i = 2)$ and describe the number of failures due to the first and the second cause of failures, respectively and $m_3 = \sum_{i=1}^{m} I_3(\delta_i = *)$ is the number of failures having failure times but corresponding causes of failure are unknown where

$$I_1(\delta_i = 1) = \begin{cases} 1, & \delta_i = 1, \\ 0, & \text{elsewhere}, \end{cases} \quad I_2(\delta_i = 2) = \begin{cases} 1, & \delta_i = 2, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{and } I_3(\delta_i = *) = \begin{cases} 1, & \delta_i = *, \\ 0, & \text{elsewhere}. \end{cases}$$

As for specific selections of unknown parameters and accelerated factor, we restricted the number of repeated-samples to 1,000.

The simulation methods for MLE and Bayes are set in comparison using the measures of parameters estimation, the comparison is performed by calculate the average values of Bias, MSE and the length of confidence intervals (LCIs) for each methods of estimation.

Numerical outcomes are listed in Tables 1-5 of the estimated parameters from the extended Weibull distribution under AT-II PHCS. The following observations can be detected as described below:

1. For MLE and Bayes estimations, it is clear that MSE and biases decrease as sample size increases. (see Table 1).
2. For MLE and Bayes estimations, it is clear that MSE decrease as sample removal (m) increase (see Tables 1 and 3).
3. The MSE of Bayesian estimation is better than MSE of MLE always (see Tables 1 and 3).
4. For the shape parameter (α_1, α_2), the MSE are decreasing as T increases based on maximum likelihood and Bayesian methods (see Table 3).
5. Five different samples schemes were applied on AT-II PHCS and that to get to the most effective scheme, the efficiency is the best for Scheme 2 followed by Scheme 4.
6. Schemes 1 and 2 not affected by the changes in time t, whatever the changes in time t, there is a stability in the numerical and practical results of Schemes 1 and 2. (see Tables 1 and 3), where scheme 1 is $R_1 = R_2 = ... = R_{m-1} = 0$, and $R_m = n - m$, it is type-II scheme and Scheme 2 is $R_1 = n - m$ and $R_2 = R_3 = ... = R_{m-1} = 0$ so not affected by the time.
7. In most cases, the Boot-t are smaller than the anther method as ACI, Boot-P (see Tables 2 and 4).

7. Numerical Results

To illustrate the practical benefit of the procedures proposed in this paper, the worth of the parameters through two various competing risks from the extended Weibull distribution is scrutinized under adaptive type II progressive hybrid censoring.

Lawless (2011) introduced data of life testing and Sarhan (2007) explained this data that contents of times failure or censoring times for 36 small electrical appliances submit to an automatic life test. Failures were categorized into 18 several proceeds, although between the 33 observed failures only 7 procedures were exemplified; and only procedures 6, and 9 showed more than twice. We are fundamentally centering on failure procedure 9. Therefore, the data contents of two reasons of failure: (failure procedure 9), (all other failure procedure), and (failure time is censored). The following offers the ordered failure times, and reason of failure, if available. Data Set: (11, 2), (35, 2), (49, 2), (170, 2), (329, 2), (381, 2), (708, 2), (958, 2), (1062, 2), (1167, 1), (1594, 2), (1925, 1), (1990, 1), (2223, 1), (2327, 2), (2400, 1), (2451, 2), (2471, 1), (2551, 1), (2565, 0), (2568, 1), (2694, 1), (2702, 2), (2761, 2), (2831, 2), (3034, 1), (3059, 2), (3112, 1), (3214, 1), (3478, 1), (3504, 1), (4329, 1), (6367, 0), (6976, 1), (7846, 1), (13403, 0).

This data is type-II censored competing risks data and it's a special case of AT-II PHCS model. The MLE and Bayes for the unknown parameters based on type-II censored competing risks scheme are obtained and reported in (6). Also, we calculated the Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted distributions for EW distribution is 0.15939 and p-value is 0.3353 where $\hat{\alpha}=0.00305$, $\hat{\beta}=0.22308$, and the estimations of relative risk rates π_1 and π_2 are 0.45935 and 0.54065, respectively.

Table 1 Bias and MSEs of the MLE and Bayes estimates based on the AT-II PHCS under various censoring schemes in $T=0.5$ when $\alpha_1=1.1, \beta_1=0.4, \alpha_2=1.5, \beta_2=0.8$

(n,m)	Scheme	Properties	MLE				Bayes Estimate			
			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$
(50,15)	1	Bias	2.9655	1.1810	−0.6252	0.6745	−0.3965	0.4345	−0.3003	0.3909
		MSE	2.6847	2.3712	2.5461	0.8807	0.2085	0.2071	0.3177	0.1892
	2	Bias	1.5698	1.0841	−0.4011	0.5435	−0.1984	0.5423	−0.2036	0.4173
		MSE	2.1217	1.9968	1.2624	0.5331	0.1055	0.3104	0.2313	0.2035
(50,25)	1	Bias	1.3486	0.9966	0.1275	0.5260	−0.3069	0.4840	−0.2950	0.4018
		MSE	2.5547	1.2576	2.0981	0.3984	0.1400	0.2493	0.2693	0.1913
	2	Bias	0.9180	0.9482	−0.2293	0.4812	−0.1773	0.1786	−0.2514	0.2170
		MSE	7.3748	1.1170	0.4258	0.2915	0.0866	0.0426	0.2152	0.0833
	3	Bias	1.2405	0.9326	−0.2169	0.4562	−0.2646	0.5031	−0.3394	0.3623
		MSE	1.3190	1.0868	0.5417	0.2754	0.1220	0.2681	0.2530	0.1518
(100,30)	1	Bias	1.6083	0.9641	−0.4201	0.5211	−0.3472	0.2677	−0.2334	0.3209
		MSE	2.0396	1.1350	1.6231	0.3742	0.1932	0.0906	0.2781	0.1505
	2	Bias	0.7076	0.9170	−0.2759	0.4485	−0.1474	0.3263	−0.2641	0.3237
		MSE	1.4168	0.9884	0.3599	0.2489	0.0710	0.1205	0.2227	0.1479
(100,50)	1	Bias	0.8853	0.8813	−0.1883	0.4710	−0.2951	0.2918	-0.2878	0.3341
		MSE	1.9655	0.8856	0.4007	0.2719	0.1264	0.1000	0.2393	0.1564
	2	Bias	0.5410	0.8537	−0.3155	0.4350	−0.0878	0.3442	−0.3034	0.3240
		MSE	0.5701	0.8067	0.2395	0.2145	0.0465	0.1312	0.2082	0.1427
	3	Bias	0.5766	0.8262	−0.3546	0.4072	−0.2525	0.3113	−0.2997	0.3181
		MSE	1.0445	0.7619	0.3184	0.1907	0.0993	0.1104	0.2243	0.1363
(200,60)	1	Bias	0.9582	0.8663	−0.2022	0.4461	-0.2265	0.1519	−0.2559	0.2178
		MSE	0.9548	0.8337	0.4621	0.2395	0.1214	0.0368	0.2497	0.0858
	2	Bias	0.5444	0.8647	−0.3464	0.4255	0.1015	0.1692	−0.2562	0.1906
		MSE	0.4723	0.8062	0.2389	0.2026	0.0582	0.0389	0.1836	0.0757
(200,100)	1	Bias	0.5817	0.8450	−0.3237	0.4287	−0.0756	0.1546	−0.2521	0.2214
		MSE	0.6574	0.7543	0.2138	0.2063	0.0612	0.0352	0.1817	0.0845
	2	Bias	0.4861	0.8421	−0.3674	0.4217	0.1505	0.5714	−0.3085	0.3997
		MSE	0.3290	0.7433	0.2020	0.1906	0.0571	0.3403	0.1727	0.1790
	3	Bias	0.3606	0.7729	−0.4674	0.3589	-0.0513	0.1711	−0.3655	0.2177
		MSE	0.3492	0.6370	0.3012	0.1409	0.0470	0.0405	0.1932	0.0686

Table 2 The length of the difference intervals for the AT-II PHCS under various censoring schemes at $T = 0.5$

(n,m)	Scheme	$\hat{\alpha}_1$			$\hat{\beta}_1$			$\hat{\alpha}_2$			$\hat{\beta}_2$		
		LCI	Boot-P	Boot-t	LCI	Boot-P	Boot-t	LCI	Boot-P	Boot-t	LCI	Boot-P	Boot-t
(50, 15)	1	4.2656	0.3516	0.3456	3.8756	0.1248	0.1279	4.3647	0.3265	0.3546	2.5592	0.0823	0.0798
	2	4.1570	0.6266	0.6276	3.5548	0.1143	0.1123	4.4038	0.1354	0.1450	1.9122	0.0611	0.0595
(50, 25)	1	3.5465	0.4869	0.4863	2.0169	0.0605	0.0667	3.6589	0.1855	0.1819	1.3686	0.0421	0.0429
	2	3.0237	0.3313	0.3201	1.8306	0.0574	0.0575	2.3961	0.0761	0.0763	0.9599	0.0308	0.0305
	3	3.9456	0.5190	0.5254	1.8272	0.0593	0.0567	2.7583	0.0879	0.0869	1.0179	0.0322	0.0303
(100, 30)	1	3.9465	0.3156	0.3195	1.7782	0.0580	0.0568	3.8042	0.1951	0.1867	1.2566	0.0396	0.0401
	2	3.7537	0.1198	0.1202	1.5065	0.0457	0.0472	2.0891	0.0647	0.0679	0.8569	0.0266	0.0262
(100, 50)	1	3.0557	0.3222	0.3101	1.2944	0.0406	0.0422	2.3703	0.0790	0.0755	0.8777	0.0277	0.0283
	2	2.0657	0.0618	0.0657	1.0939	0.0362	0.0344	1.4671	0.0472	0.0463	0.6236	0.0196	0.0189
	3	3.3093	0.1042	0.1066	1.1048	0.0334	0.0338	1.7213	0.0552	0.0538	0.6187	0.0200	0.0192
(200, 60)	1	2.5466	0.2396	0.2388	1.1314	0.0360	0.0365	2.5455	0.0831	0.0805	0.7897	0.0250	0.0247
	2	1.6451	0.0525	0.0539	0.9483	0.0298	0.0303	1.3524	0.0413	0.0422	0.5762	0.0177	0.0175
(200, 100)	1	2.2153	0.0724	0.0717	0.7866	0.0252	0.0245	1.2952	0.0412	0.0392	0.5884	0.0182	0.0185
	2	1.1937	0.0368	0.0374	0.7247	0.0224	0.0229	1.0151	0.0326	0.0336	0.4426	0.0145	0.0139
	3	1.8361	0.0599	0.0583	0.7812	0.0239	0.0249	1.1286	0.0356	0.0348	0.4314	0.0143	0.0134

Table 3 Bias and MSEs of the MLE and Bayes estimates based on the AT-II PHCS under various censoring schemes at $\alpha_1 = 1.1, \beta_1 = 0.4, \alpha_2 = 1.5, \beta_2 = 0.8$

T	(n,m)	Scheme	Properties	MLE				Bayes Estimate			
				$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$
0.5	(50,15)	3	Bias	2.1651	1.2046	−0.5652	0.5821	−0.3869	0.4435	−0.2852	0.3979
			MSE	2.4953	2.6165	1.4596	0.5746	0.1995	0.2151	0.3007	0.1914
		4	Bias	1.7498	1.0716	0.1598	0.5463	−0.2789	0.4894	−0.2965	0.3918
			MSE	2.4565	1.7996	2.1079	0.4554	0.1494	0.2566	0.2581	0.1827
		5	Bias	1.8456	1.1012	0.3585	0.5629	−0.3510	0.4801	−0.2994	0.4002
			MSE	2.5495	1.8128	1.8947	0.6562	0.1815	0.2457	0.2742	0.1895
	(100,30)	3	Bias	1.1652	0.9456	−0.3101	0.5095	−0.3375	0.2674	−0.2596	0.3262
			MSE	1.8655	1.1465	0.6599	0.3496	0.1743	0.0893	0.2511	0.1525
		4	Bias	0.8428	0.8766	−0.2311	0.4382	−0.2696	0.3043	−0.2411	0.3290
			MSE	1.4655	0.8942	0.4868	0.2406	0.1212	0.1082	0.2476	0.1538
		5	Bias	0.9565	0.9099	−0.1772	0.4482	−0.3327	0.2913	−0.2354	0.3159
			MSE	1.2955	0.9752	0.4885	0.2495	0.1635	0.1019	0.2501	0.1432
	(200, 60)	3	Bias	0.9456	0.8585	−0.1792	0.4496	−0.2108	0.1553	−0.2183	0.2131
			MSE	0.8905	0.8179	0.3995	0.2389	0.1111	0.0365	0.2327	0.0822
		4	Bias	0.5197	0.8205	−0.3748	0.3932	−0.0678	0.1568	−0.2805	0.2121
			MSE	0.7709	0.7353	0.3019	0.1760	0.0631	0.0359	0.2255	0.0812
		5	Bias	0.6189	0.8332	−0.3039	0.4277	−0.1281	0.1583	−0.2412	0.2094
			MSE	0.9692	0.7457	0.2715	0.2036	0.0749	0.0370	0.2250	0.0799
1.5	(50,15)	3	Bias	1.6946	1.2299	1.7466	0.5912	−0.3806	0.4595	-0.2957	0.3901
			MSE	2.2456	2.6282	2.2315	0.5808	0.1892	0.2296	0.3117	0.1900
		4	Bias	1.7346	1.0701	0.1153	0.5390	−0.2662	0.4973	−0.2796	0.4059
			MSE	2.3546	1.7408	1.7782	0.4333	0.1397	0.2626	0.2586	0.1987
		5	Bias	1.7095	1.0872	0.2547	0.5658	−0.3113	0.4747	−0.2948	0.4018
			MSE	2.2565	1.7612	1.8850	0.6690	0.1627	0.2417	0.2799	0.1910
	(100,30)	3	Bias	0.8546	0.9584	0.1005	0.5119	−0.3492	0.2800	−0.2615	0.3187
			MSE	1.5056	1.1252	1.5472	0.3518	0.1781	0.0905	0.2537	0.1487
		4	Bias	0.8997	0.9079	−0.1934	0.4616	−0.2675	0.3014	−0.2452	0.3271
			MSE	1.5288	0.9720	0.4249	0.2598	0.1228	0.1066	0.2390	0.1495
		5	Bias	0.8516	0.9035	−0.1574	0.4564	−0.3375	0.2869	−0.2354	0.3179
			MSE	1.4236	0.9636	0.4652	0.2565	0.1465	0.0989	0.2487	0.1448
	(200, 60)	3	Bias	0.7938	0.8610	−0.1828	0.4593	−0.2195	0.1602	−0.2206	0.2286
			MSE	0.6959	0.8202	0.4065	0.2477	0.1294	0.0419	0.2264	0.0837
		4	Bias	0.6069	0.8509	−0.2988	0.4295	−0.0463	0.1605	−0.2217	0.2148
			MSE	0.7864	0.7794	0.2353	0.2076	0.0613	0.0377	0.2281	0.0857
		5	Bias	0.6390	0.8392	−0.2708	0.4345	−0.1345	0.1589	−0.2225	0.2110
			MSE	1.0597	0.7585	0.2736	0.2120	0.0831	0.0372	0.2392	0.0830

Table 4 The length of the difference intervals for the AT-II PHCS under various censoring schemes

(n,m)	Scheme	Method	$T=0.5$				$T=1.5$			
			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$
(50, 15)	3	LCI	4.0946	4.1424	4.1956	1.8860	5.4655	4.4424	5.0465	1.9396
		Boot-P	0.5946	0.1280	0.5165	0.0598	0.6456	0.1459	0.5562	0.0604
		Boot-t	0.5981	0.1321	0.5395	0.0598	0.6456	0.1425	0.5771	0.0604
	4	LCI	4.1895	3.1650	4.6595	1.5541	4.2365	3.0269	4.2103	1.4819
		Boot-P	0.6189	0.0968	0.1850	0.0517	0.6965	0.0938	0.1652	0.0473
		Boot-t	0.6165	0.0983	0.1737	0.0485	0.6814	0.0975	0.1612	0.0472
	5	LCI	3.9546	3.0381	3.2565	2.2849	4.3565	2.9849	4.2945	2.3164
		Boot-P	0.4565	0.0994	0.3495	0.0766	0.7002	0.0947	0.2916	0.0790
		Boot-t	0.4345	0.0946	0.3246	0.0718	0.7155	0.0937	0.2985	0.0739
(100, 30)	3	LCI	3.5470	1.7832	3.8624	1.1749	4.4565	1.6298	4.6707	1.1674
		Boot-P	0.4192	0.0595	0.1574	0.0381	0.4649	0.0522	0.1544	0.0401
		Boot-t	0.4198	0.0573	0.1567	0.0373	0.4513	0.0495	0.1486	0.0365
	4	LCI	5.2806	1.3904	2.5819	0.8648	3.8155	1.5077	2.4413	0.8473
		Boot-P	0.1721	0.0421	0.0828	0.0281	0.2598	0.0444	0.0785	0.0266
		Boot-t	0.1747	0.0424	0.0837	0.0266	0.2624	0.0487	0.0759	0.0262
	5	LCI	3.5414	1.5052	2.6515	0.8646	3.3855	1.5049	2.6029	0.8611
		Boot-P	0.5116	0.0476	0.0798	0.0276	0.5110	0.0473	0.0838	0.0283
		Boot-t	0.4680	0.0483	0.0856	0.0271	0.4972	0.0477	0.0824	0.0273
(200, 60)	3	LCI	2.6563	1.1017	2.3956	0.7516	3.1565	1.0332	2.2050	0.7346
		Boot-P	0.3312	0.0336	0.0785	0.0231	0.3195	0.0321	0.0688	0.0234
		Boot-t	0.3779	0.0360	0.0781	0.0246	0.3185	0.0324	0.0678	0.0233
	4	LCI	2.7756	0.9778	1.5758	0.5738	2.5362	0.9230	1.4987	0.5968
		Boot-P	0.0838	0.0315	0.0529	0.0183	0.0807	0.0307	0.0480	0.0194
		Boot-t	0.0883	0.0314	0.0489	0.0176	0.0810	0.0290	0.0455	0.0183
	5	LCI	3.0026	0.8898	1.6602	0.5637	3.1651	0.9132	1.7554	0.5969
		Boot-P	0.0965	0.0284	0.0507	0.0173	0.1005	0.0284	0.0556	0.0193
		Boot-t	0.0945	0.0283	0.0498	0.0182	0.1002	0.0291	0.0542	0.0188

8. Conclusions

This paper explained a competing risks model under adaptive type II progressive hybrid censoring scheme when the fixed number of causes of failure in known. Assuming that the lifetime distributions are extended Weibull distribution. We have derived the MLEs, propose different confidence intervals using asymptotic distributions and bootstrap confidence intervals for the parameters of extended Weibull distribution. Also, the Bayes estimates obtained based on squared error loss function under the assumption of independent gamma priors. A simulation study has been conducted to examine and compare the performance of the proposed methods for different sample sizes and different censoring schemes. Finally, a numerical example is provided to illustrate the inference methods described in the paper.

Table 5 The MLE and Bayes with different sample based on AT-II PHCS under competing risks

T	Scheme	Properties	MLE				Bays Estimate			
			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$
0.5	1	Mean	0.5670	1.3182	5.1907	3.4798	0.9860	1.5758	0.6955	1.6252
		SE	0.4876	0.6504	7.9401	1.5117	0.5443	0.4219	0.4159	0.6432
	2	Mean	0.7348	1.3369	1.2727	2.8163	0.8802	1.4685	0.7520	1.3162
		SE	0.3887	0.4571	0.6358	0.9884	0.4363	0.3521	0.4805	0.4563
0.5	3	Mean	0.5778	1.3182	3.7272	3.2497	0.9436	1.5228	1.0464	1.9042
		SE	0.4705	0.6225	4.6394	1.3136	0.6788	0.5343	0.6396	0.5108
	4	Mean	0.5745	1.2533	1.4104	2.6005	0.6465	1.6995	0.7891	1.5503
		SE	0.3561	0.4983	0.9516	0.9465	0.3626	0.7097	0.3260	0.3816
0.25	3	Mean	0.5501	1.2105	2.3288	2.9186	0.6218	1.3781	0.6339	1.4727
		SE	0.3590	0.5867	2.7203	1.2520	0.3537	0.5873	0.3415	0.4205
	4	Mean	0.3061	0.9284	0.4982	1.8310	0.4090	1.0214	0.7579	1.7208
		SE	0.1610	0.3995	0.2848	0.8056	0.2153	0.3501	0.2412	0.4324

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