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The Type II Topp-Leone Generalized Power Ishita Distribution with Properties and Applications

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Abstract

Motivated to obtain a flexible distribution. In this paper, we proposed a compound continuous distribution called the type II Topp-Leone power Ishita generalized (TIITLPI-G) distribution. Some of the basic statistical properties of the TIITLPI-G distribution are examined and the parameters are obtained using method of maximum likelihood estimation. The flexibility and the superiority of the proposed TIITLPI-G distribution were demonstrated by means of numerical applications on two sets of real life data. The proposed TIITLPI-G distribution was compared with other existing distributions in literature in terms of fitness and the result obtained indicates that the proposed TIITLPI-G distribution outperformed the competing distributions in terms of fitness.

Keywords: Maximum likelihood estimation, probability plots, survival plots, entropies.

1. Introduction

Over the decades, researchers have introduced in literature several extended continuous family of distributions by generalizing the parent distributions which provide many applications in different fields such as biology, engineering, physics, geology, social sciences among others. The generalization of a parent distribution to make it more flexible in describing different types of real life situations depends on how the researchers can compound one or more distributions to birth a more flexible one with simple mathematical structure.

Sangsanit and Bodhisuwan (2016) introduced a Topp-Leone generated (TLG) family of distributions using the Topp-Leone distribution (Topp and Leone 1955) as the generator. They used the TLG to obtained Topp-Leone generalized exponential (TLGE) distribution, some of its statistical properties were derived and its numerical applications were demonstrated on real life data sets. They observed that the TLGE has a better fitness on the data than the generalized exponential distribution and the exponentiated generalized exponential distribution when compared.

Haitham and Mustafa (2017) used the TLG family of distribution to derive the Topp-Leone Nadarajah_Haghighi distribution with some of its statistical properties derived. The application of the distribution was demonstrated on real life data as well as simulation.

Sirinapa (2018) used the TLG family of distribution introduced by Sangsanit and Bodhisuwan (2016) to obtain the Topp-Leone exponentiated power Lindley distribution. Some of the statistical properties were derived and its application to real life data was illustrated to determine its flexibility. All this distribution that were obtained by the extension of the parent distributions outperformed the parent distributions in all the applications. This led Elgarhy et al. (2018) to extend the Topp-Leone generated (TLG) family introduced by Sangsanit and Bodhisuwan (2016) to the type II Topp-Leone generated (TIITLG) family of distributions using half logistic distribution as the generator. They used the TIITLG family of distribution to obtained type II Topp-Leone-Weibull (TIITL-W) distribution, type II Topp-Leone-Burr XII (TIITL-Burr XII) distribution and type II Topp-Leone-Uniform (TIITL-U) distribution with the applications to real life data and the results revealed that the TIITL-W distribution provides better fits than the Topp-Leone generalized Weibull, Topp-Leone inverse Weibull and Weibull distributions when compared in terms of model adequacy.

Statistical distributions have been used to model life time data in order to study its useful properties. When there is need for more flexible distributions, many researchers use the new distributions with more generalization to provide results in sound conclusions and decisions. This makes the generalization of parent distributions more interesting to improve on its goodness of fit and determine the tail and kurtosis properties. One of such is the study of Agu and Francis (2018) where they compared goodness of fit for normal distribution using real life data set. Eghwerido et al. (2020) modeled uncensored data with a class of alpha power Gompertz distribution, Egwherido et al. (2019) proposed Kumaraswamy alpha power inverted exponential distribution with properties and applications. Agu and Eghwerido (2021) introduced the Agu-Eghwerido distribution, derived the regression model and explored the applications using real life data.

Maxwell et al. (2019) introduced a four parameter odd generalized exponentiated inverse Lomax distribution for modeling lifetime data and derived some of its basic statistical properties. Recently, Agu et al. (2019) proposed a modified Laplace distribution, derived some of its basic statistical properties and applied the proposed distribution on life data sets and simulation studied was carry out. The obtained results that revealed the outperformance of the proposed distribution to other distributions compared with in terms of fitness. Thomas et al. (2019) introduced the Kumaraswamy alpha power inverted exponential distribution and explored the applications using real life data sets of various characteristics.

The density function of the type II generalized Topp-Leone family can be symmetrical, unimodal, positively skewed, negatively skewed, reversed-J shaped, upside-down bathtub, J and reversed-J hazard rates and a decreasing and increasing function.

Shanker and Shukla (2017) obtained a Poisson mixture of Ishita distribution and studied its various statistical properties with application to real life data. Shukla and Shanker (2018) proposed a two parameter power Ishita (PI) distribution as a power transformation of Ishita distribution with the aim to improve on the flexibility of Ishita distribution. Some of the statistical properties were derived and the application of the distribution were demonstrated on real life data.

In this paper, we proposed a three parameter distribution called the type II Topp-Leone Power Ishita generalized (TIITLPI-G) distribution where the parent distribution is the two parameter Power Ishita (PI) distribution.

The motivations for this paper are to obtain a distribution with a flexible kurtosis, symmetric, positively skewed, negatively skewed, L-shape, J-shape and with heavy short tail for modeling real life data arising from different life situations. To give some of the statistical properties of the new distribution mathematical treatment. The maximum likelihood estimation (MLE) is used to estimate the parameters of distribution. In order to illustrate the usefulness of the proposed distribution, real life data sets are used to determine its flexibility, superiority and adequacy.

The probability density function of the Power Ishita (PI) distribution proposed by Shukla and Shanker (2018) is given by

$$g(x;\alpha,\theta)=\frac{\alpha\theta^2(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^\alpha}}{\theta^3+2},\quad \alpha>0,\theta>0,x>0, \tag{1}$$

where α is the shape parameter and θ the scale parameter.

The corresponding cumulative density function of the Power Ishita distribution is given by

$$G(x;\alpha,\theta)=1-\left[1+\frac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right]e^{-\theta x^\alpha},\quad \alpha>0,\theta>0,x>0. \tag{2}$$

Let X be a random variable with the type II Topp-Leone generalized family of distribution, that is $X\sim TIITL-G(\beta)$. The probability density function of TIITL-G family of distributions proposed by Elgarhy et al. (2018) is given by

$$f(x;\beta)=2\beta g(x)G(x)\left[1-G^2(x)\right]^{\beta-1}. \tag{3}$$

The corresponding cumulative density function is given by

$$F(x;\beta)=1-\left[1-G^2(x)\right]^{\beta}, \tag{4}$$

where $\beta>0$ is a shape parameter, $g(x)$ is the probability density function of the parent distribution and $G(x)$ is the cumulative density function of the parent distribution.

2. The New Distribution

In this section, the new distribution called the Type II Topp-Leone power Ishita generalized (TIITLPI-G) distribution is introduced. From (3) and (4), let $g(x)$ be the probability density function of Power Ishita distribution and $G(x)$ be the cumulative density function of power Ishita distribution.

Let X be a random variable of the TIITLPI-G distribution, $X\sim TIITLPI-G(\alpha,\theta,\beta)$. Put (1) and (2) into (3), we obtain the probability density function (pdf) of TIITLPI distribution as:

$$\begin{aligned} f(x;\alpha,\theta,\beta)=&\frac{2\alpha\beta\theta^3(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^\alpha}}{\theta^3+2}\left[1-\left(1+\frac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right)e^{-\theta x^\alpha}\right]\times\\ &\left[1-\left[1-\left(1+\frac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right)e^{-\theta x^\alpha}\right]^2\right]^{\beta-1}, \end{aligned} \tag{5}$$

where $\alpha>0,\theta>0,x>0.$

The pdf plots for different values of the parameters are shown in Figures 1 (a) and (b). In Figure 1 (a), the plot yielded symmetric, L-like, and a decreasing flat shapes. Similarly, by varying the parameters values in Figure 1 (b), the plot yielded a high peak with a decreasing and increasing shape from the left tail. The corresponding cumulative distribution function (cdf) of TIITLPI-G distribution is given as

$$F(x;\alpha,\theta,\beta)=1-\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)\right]^{2}e^{-2\theta x^{\alpha}}\right]^{\beta},\quad \beta>0,\alpha>0,\theta>0,x>0, \tag{6}$$

where $\beta>0,\alpha>0$ are shape parameters and $\theta>0$ is a scale parameter. Equation (6) is the cumulative density function of the proposed TIITLPI-G distribution. The cdf plots for different values of the parameters are shown in Figures 2 (a) and (b). Figures 2 (a) and (b) show that the proposed TIITLPI-G is a non-decreasing function for various parameters values.

$$\lim F(x;\alpha,\theta,\beta)=1 \text{ as } x\to\infty, \tag{7}$$

$$\lim F(x;\alpha,\theta,\beta)=0 \text{ as } x\to-\infty, \tag{8}$$

(7) and (8) shows that the new distribution (TIITLPI-G) is a proper density function.

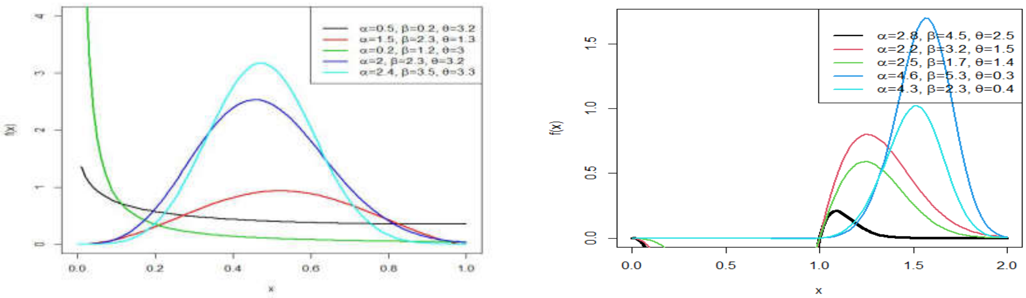
The survival function (Surv) of the TIITLPI-G distribution is defined as

$$Surv_{TIITLPI-G}=\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2}\right]^{\beta},\quad \beta>0,\alpha>0,\theta>0,x>0.$$

The survival plots for different values of the parameters are shown in Figures 3 (a) and (b). Figures 3 (a) and (b) show that the introduced TIITLPI-G model has a non-increasing survival shape.

The corresponding failure function of the TIITLPI-G distribution is defined as

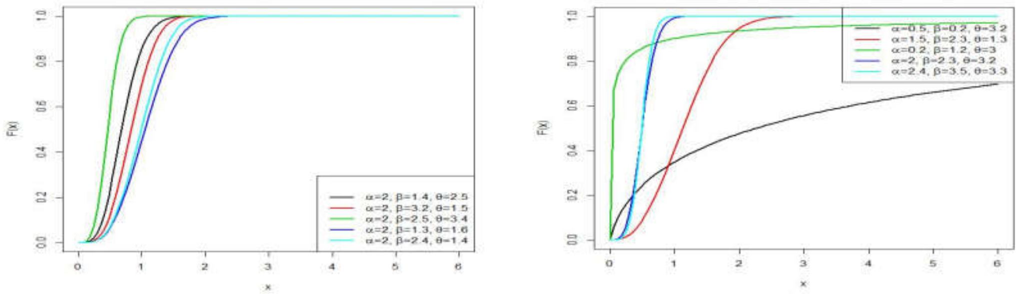
$$f_{TIITLPI-G}=2\alpha\beta\theta^{3}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}(\theta^{3}+2)^{-1}\left[1-\left(1+\theta x^{\alpha}(\theta x^{\alpha}+2)(\theta^{3}+2)^{-1}\right)e^{-\theta x^{\alpha}}\right]\times$$
$$\left[1-\left[1-\left(1+\theta x^{\alpha}(\theta x^{\alpha}+2)(\theta^{3}+2)^{-1}\right)e^{-\theta x^{\alpha}}\right]^{2}\right]^{-1}.$$



(a) The pdf plot of the TIITLPI-G yield symmetric and L-like shapes

(b) The pdf plot of the TIITLPI-G yielded high peak with increasing and decreasing shape from left tail

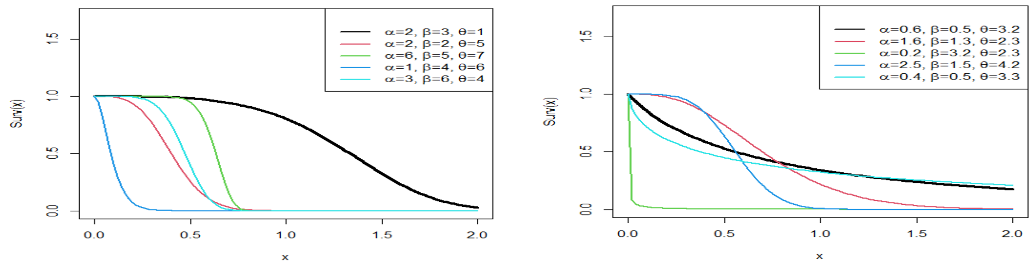
Figure 1 The pdf plot of the TIITLPI-G



(a) The cdf plot of the TIITLPI-G yield non-decreasing shape

(b) The cdf plot of the TIITLPI-G disperse from each other and yield non-decreasing shape

Figure 2 The cdf plot of the TIITLPI-G



(a) The survival plot of the TIITLPI-G yield non-increasing shape

(b) The survival plot of the TIITLPI-G yield slow decreasing shape

Figure 3 The survival plot of the TIITLPI-G

3. Entropies

An entropy has to do with the measure of dispersion or uncertainty of a random variable X. The Rényi entropy of a random variable Y is given by

$$I_R = \frac{1}{1-\varepsilon}\log\left(\int_0^\infty \left[f(x)\right]^\varepsilon dx\right), \quad \text{(Rényi, 1961).} \tag{9}$$

Put (5) in (9), we obtain the expression for the Rényi entropy of the TIITLPI-G distribution as

$$I_R = \frac{1}{1-\varepsilon}\log\left(\int_0^\infty \left[\begin{array}{l}\dfrac{2\alpha\beta\theta^3(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^\alpha}}{\theta^3+2}\left[1-\left(1+\dfrac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right)e^{-\theta x^\alpha}\right]\times \\ \left[1-\left[1-\left(1+\dfrac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right)e^{-\theta x^\alpha}\right]^2\right]^{\beta-1}\end{array}\right]^\varepsilon dx\right). \tag{10}$$

Let $k = 1-\left(1+\dfrac{\theta x^\alpha(\theta x^\alpha+2)}{\theta^3+2}\right)e^{-\theta x^\alpha}$ such that we can obtain $(1-k^2)^{\beta-1}$. Using series expansion (10) can be written as

$$I_R = \frac{1}{1-\varepsilon}\log\left(\int_0^{\infty}\left[\begin{array}{l}\frac{2\alpha\beta\theta^3(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{\theta^3+2}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right]\times \\ \sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}\end{array}\right]^{\varepsilon} dx\right).$$

3.1. Shannon entropy

The Shannon entropy of a random variable X is defined by

$$n_{\gamma} = -E\left\{\log\left[f(x)\right]\right\}, \text{ (Shannon, 1948).}$$

The expression for the Shannon entropy of the TIITLPI-G distribution is given as

$$n_{\gamma} = -E\left\{\log\left[\begin{array}{l}\frac{2\alpha\beta\theta^3(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{\theta^3+2}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right]\times \\ \sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}\end{array}\right]\right\}.$$

3.2. Order statistics

Let $x_1, x_2, ..., x_u$ be a random sample of size n from the TIITLPI-G distribution. Then the probability density function of the j^{th} order statistics can be obtain as follows

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}, \tag{11}$$

where $f(x)$ and $F(x)$ are the probability density function and the cumulative density function of the TIITLPI-G distribution respectively. Substituting (5) and (6) in (11), we have

$$\begin{aligned} f_{j:n}(x) = {} & \frac{n!2\alpha\beta\theta^3(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta^3+2)(j-1)!(n-j)!}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right] \\ & \times\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right]^2\right]^{\beta-1} \\ & \times\left[1-\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)\right]^2 e^{-2\theta x^{\alpha}}\right]^{\beta}\right]^{j-1} \\ & \times\left[\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)\right]^2 e^{-2\theta x^{\alpha}}\right]^{\beta}\right]^{n-j}. \end{aligned} \tag{12}$$

Using series expansion, we have the terms in (12) to be written as

$$\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2}\right]^{\beta-1}$$

$$=\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}.$$

$$\left[1-\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)\right]^{2}e^{-2\theta x^{\alpha}}\right]^{\beta}\right]^{j-1}$$

$$=\sum_{r=0}^{j-1}\binom{j-1}{r}(-1)^{j-1-r}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{j-1-r}.$$

$$\left[\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)\right]^{2}e^{-2\theta x^{\alpha}}\right]^{\beta}\right]^{n-j}$$

$$=\sum_{r=0}^{n-j}\binom{n-j}{r}(-1)^{n-1-j}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{n-1-j}.$$

Using the above series expansions, (11) can be written as

$$f_{j:n}(x)=\frac{n!2\alpha\beta\theta^{3}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta^{3}+2)(j-1)!(n-j)!}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]\times$$
$$\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}\times$$
$$\sum_{r=0}^{j-1}\binom{j-1}{r}(-1)^{j-1-r}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{j-1-r}\times$$
$$\sum_{r=0}^{n-j}\binom{n-j}{r}(-1)^{n-1-j}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{n-1-j}.$$

For $j=1$, we obtain the minimum order statistic for the TIITLPI-G distribution as

$$f_{1:n}(x)=\frac{n!2\alpha\beta\theta^{3}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta^{3}+2)(n-1)!}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]\times$$
$$\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}\times$$
$$\sum_{r=0}^{n-1}\binom{n-1}{r}(-1)^{n}(-1)^{-r}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]\times$$

$$\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{n-r},$$

and for $j=n$ we obtain the maximum order statistic for the TIITLPI distribution as

$$f_{j:n}(x)=\frac{n!2\alpha\beta\theta^{3}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta^{3}+2)(n-1)!}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]\times$$

$$\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}(-1)^{\beta-1-r}\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2(\beta-1-r)}\times$$

$$\sum_{r=0}^{n-1}\binom{n-1}{r}(-1)^{n-1-r}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{n-1-r}\times$$

$$(-1)^{-1}\left[\left(1-\left(\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}e^{-\theta x^{\alpha}}\right)^{2}\right)^{\beta}\right]^{-1}.$$

Table 1 presents the probability density function of the three parameter generalized Lindley distribution (TPGLD), exponentiated transmuted exponential distribution (ETED), Power Ishita distribution (PID) and the proposed TIITLPI-G distribution.

Table 1 Existing distributions with their corresponding PDF and the proposed distribution

Model	PDF	Authors
TPGLD	$\frac{\alpha\lambda^{2}(\beta+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(1+\lambda\beta)}$	Ekhosuehi and Opone (2018)
ETED	$\alpha\lambda e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\times\left((1-e^{-\lambda x})(1+\beta e^{-\lambda x})\right)^{\alpha-1}$	Al-Kadim and Mahdi (2018)
PID	$\frac{\alpha\theta^{3}}{\theta^{3}+2}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}$	Shukla and Shanker (2018)
TIITLPI-G	$f(x;\alpha,\theta,\beta)=\frac{2\alpha\beta\theta^{3}(\theta+x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{\theta^{3}+2}\times\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]\times\left[1-\left[1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^{3}+2}\right)e^{-\theta x^{\alpha}}\right]^{2}\right]^{\beta-1}.$	This paper

4. Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_u$ be a random sample of size u from the TIITLPI-G distribution. Then, the log-likelihood function is defined by

$$\text{n}\,L=\sum_{i=1}^{u}\ln f(x_{i};\alpha,\theta,\beta)$$

$$\begin{aligned}\ln L &= n(\log 2 + \log\alpha + \log\beta + 3\log\theta) + \\ &(\alpha-1)\sum\nolimits_{r=1}^{u}\log(x_r) - \theta\sum\nolimits_{r=0}^{u}x_r^{\alpha} - n\log(\theta^3+2) - \\ &\sum\nolimits_{r=1}^{u}\log\left[\left(1+\theta x_r^{\alpha}(\theta x_r^{\alpha}+2)(\theta^3+2)^{-1}\right)e^{-\theta x_r^{\alpha}}\right] - \\ &(\beta-1)\sum\nolimits_{r=1}^{u}\log\left[1-\left(1+\theta x_r^{\alpha}(\theta x_r^{\alpha}+2)(\theta^3+2)^{-1}\right)e^{-\theta x_r^{\alpha}}\right]^2.\end{aligned} \tag{13}$$

In this paper, the estimates of the parameters in (13) are obtained iteratively using R package (R Core Team, 2018) on the considered real life datasets since it solution may not be easily obtain in a close form.

4.1. Numerical analysis

This section present numerical analysis of the proposed TIITLPI-G distribution on two sets of survival data. The TIITLPI-G distribution is compared with other distributions and to verify which distribution fits better with the survival data sets Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan Quinn information criterion (HQIC) are used.

Data set I: This dataset represent the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee, 1992). This data set has recently been studied by Al-Kadim and Mahdi (2018). Table 2 presents the data set.

Table 2 Data set I

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0

Table 3 Summary of the survival times of 121 patients with breast cancer (Data set I)

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
0.30	17.50	40.00	46.33	60.00	154.00	1.04	3.40

Table 4 Parameter estimates and the criteria values for the survival times of 121 patients with breast cancer data

Model	Estimation			LL	AIC	BIC	HQIC	K-S	p-value
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$						
TPGLD	0.042	4.990	0.986	−578.783	1163.57	1171.95	1166.98	0.215	0.160
ETED	0.002	1.000	0.009	−900.548	1807.00	1815.48	1810.50	0.351	0.061
PID	0.334	-	0.749	−213.786	31.57	439.96	434.98	0.207	0.231
TIITLPID*	0.621	0.313	0.014	3979.625	−7953.30	−7944.70	−7949.80	0.121	0.304

*Proposed model

Data Set II: The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, Bader and Priest (1982). This data set has recently been studied by Shukla and Shanker (2018). Table 5 presents the data set.

Table 5 Data set II

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

Table 6 Summary of the tensile strength of 69 carbon fibers (Data set II)

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
1.312	2.098	2.478	2.451	2.773	3.585	−0.028	2.941

Table 7 Parameter estimates and the criteria values for the tensile strength of 69 carbon fibers data

Model	Estimation			LL	AIC	BIC	HQIC	K-S	p-value
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$						
TPGLD	0.838	1.181	1.054	161.788	−317.58	−310.87	−314.92	2.483	4.028
ETED	1.684	9.105	0.181	−184.747	375.40	382.10	378.15	1.352	0.493
PID	1.295	-	0.783	−105.458	214.92	221.62	217.58	0.762	0.024
TIITLPID*	0.098	0.093	0.017	10501.610	−20997.20	−20990.00	-20994.50	0.330	6.802

*Proposed model

5. Discussion

Table 1 presents three existing distributions to be used in comparison with the proposed TIITLPI-G distribution with their corresponding authors. Table 2 presents data set I for the numerical application of the proposed distribution (TIITLPI-G). Table 3 shows the summary of data set 1 presented in Table 2. Table 4 shows the parameter estimates and the criteria values for measure of fitness of the models with their corresponding ranking based on the best model with the least criteria value. Table 5 presents data set II for the numerical application of the proposed distribution (TIITLPI-G). Table 6 shows the summary of the data set II. Table 7 shows the parameter estimates and the criteria values obtained using data set II. The model is ranked based on the model with least criteria values and such model is taken as the best model.

6. Conclusions

The paper has successfully introduced the TIITLPI-G distribution and some of its basic statistical properties were given mathematical treatment. The probability density function of the TIITLPI-G distribution is unimodal, L-shape, J-shape and with heavy short tail and the survival rate has an L-shape, S-shape and has stick tinny tail, this means that the proposed TIITLPI-G distribution would be very useful to model real life situations with unimodal, L, J, S shapes and stick tinny tail survival rate. The proposed TIITLPI-G distribution has a tractable mathematical structure and shows high flexibility property in modelling real life situations as it outperformed the Power Ishita distribution, exponentiated transmuted exponential distribution and three parameter generalized Lindley distribution. This judgement was based on the obtained AIC, BIC and HQIC values of these distributions. The parameters of these distributions were obtained using method of maximum likelihood estimation with the aid of R-statistical package. The proposed TIITLPI-G distribution been the competitive model among others would be of use in fields like biology, engineering, physics, geology, social sciences among others. Further studies can be carried out on some of the statistical properties not derived in this paper as well as simulation studies.

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Appendix

Derivation of (6). Recall that the corresponding probability density function of TIITL-G family of distributions proposed by Elgarhy et al. (2018) is given in (4) as

$$F(x;\beta) = 1 - \left[1 - G^2(x)\right]^{\beta},$$

where $G(x)$ is the cumulative density function of a baseline distribution and $\beta > 0,$ the shape parameter. In this study we consider the two parameter Power Ishita distribution as the baseline distribution with a cumulative density function given in (2) as

$$G(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta x^{\alpha}(\theta x^{\alpha} + 2)}{\theta^3 + 2}\right] e^{-\theta x^{\alpha}},$$

where $\alpha > 0, \theta > 0, x > 0.$

To derive the cumulative density function of the proposed TIITLPI-G distribution, we put the cumulative density function of the baseline distribution into the cumulative density function of the TIITL-G family of distributions proposed by Elgarhy et al. (2018) above to have

$$F(x;\alpha,\theta,\beta) = 1 - \left[1 - \left[1 - \left(1 + \frac{\theta x^{\alpha}(\theta x^{\alpha} + 2)}{\theta^3 + 2}\right) e^{-\theta x^{\alpha}}\right]^2\right]^{\beta}$$

$$F(x;\alpha,\theta,\beta) = 1 - \left[1 - \left[1 - \left(1 + \frac{\theta x^{\alpha}(\theta x^{\alpha} + 2)}{\theta^3 + 2}\right)\right]^2 e^{-2\theta x^{\alpha}}\right]^{\beta}.$$

where $\alpha > 0, \theta > 0, x > 0.$