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# The Integral Equation Approach for Solving the Average Run Length of EWMA Procedure for Autocorrelated Process

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**Abstract**

The main goal of statistical process control (SPC) is to improve the capacity of the process. One of quality tools is the control chart and it is usually designed under the assumption that the observations are independent and identically distributed. However, some characteristics of the production process especially processes that are continuously produced such as chemical processes, the process is autocorrelated in various time series models. In this paper, we will focus on an autoregressive integrated moving average, ARIMA(p,d,q) model. The performance of control chart is evaluated in terms of the average run length. This paper aims to solve explicit formulas and develop numerical integration for the average run length of the exponentially weighted moving average (EWMA) control chart. The accuracy of the proposed formulas is established by comparing them to the numerical integration method. A comparison of the results from explicit formula and numerical integration shows that the absolute percentage difference is less than 0.1% In terms of computational time, the explicit formula can reduce the computational time better than the numerical integration.

**Keywords**: Autoregressive integrated moving average, exponentially weighted moving average, Fredholm integral equation, explicit formula.

## 1. Introduction

In current, production process control usually uses statistical process control (SPC) to monitor and control a process. Control charts are one of the efficient tools of SPC that make the process to have more production capabilities. Therefore, the main function of control charts is to detect changes in processes. The application of control chart is not only limited to manufacturing industry, but it covers a wide range of disciplines such as engineering, health care, medicine, economics, finance, education and analytical laboratories. The control chart was first developed by Shewhart (1931). Shewhart control chart, which is an effective control chart to detect large changes in the production process. Recently, the cumulative sum (CUSUM) control chart presented by Page in 1954 and later Robert (1959) introduced the exponentially weighted moving average (EWMA) control chart, in which both control charts are effective in detecting small shifts in the process.

In general, the criteria used in the comparison of efficiency of the control chart are considered from the average run length (ARL). The ARL is divided into 2 states: when the process is in-control, the $ARL_0$ is determined and when the process is out-of-control, the $ARL_1$ is determined. The ARL can be calculated using the Monte Carlo Simulation method, which provides accurate results, but it takes a lot of processing time. There are many researchers presented the methods for evaluating the ARL of control charts. Lucas and Saccucci (1990) proposes the Markov chain method for estimating the ARL for EWMA control charts when the observations are normal distribution. After that, Borror et al (1999) studied the robustness of the EWMA control chart in cases of non-normal distribution and comparing with the Shewhart control chart. Areepong and Novikov (2009) presented the formula for the ARL of the EWMA control chart when the data were exponential distribution by Martingale Approach and comparing the efficiency of the formula for the ARL of the EWMA control chart with the CUSUM control chart. Later, Areepong and Sukparungsee (2010) proposed the ARL using the integral equation approach of the EWMA control chart and compared the results with the Monte Carlo Simulation methods. In addition, Mititelu et al. (2010) used a Fredholm's integral equation to find the explicit formula for the ARL of the CUSUM control chart when the observations have hyperexponential distribution and the one-sided EWMA control chart with Laplace distribution.

Generally, the principle of control chart is under the assumptions that the data are independent and it is effective in detecting changes in processes when the data has a normal distribution. However, there are many situations the data are autocorrelated such as daily flows of a river, wind speeds, the amount of dissolved oxygen in a river, etc. For this reason, some authors evaluate the ARL when the process is serially correlation, such as Busaba et al. (2012) presented the exact solution of ARL for the CUSUM control chart for the autoregressive of order 1 (AR(1)) process and compared the results from the exact solution with the results from the numerical integral equation method. Later, Petcharat et al (2013) proved the analytical expression for the ARL of EWMA control chart and the CUSUM control chart for the moving average of order q (MA(q)) process. Recently, Phanyaem et al. (2014) proposed the explicit formula for the ARL of the CUSUM control chart for an autoregressive and moving average (ARMA(1,1)) process. After that, Phanyaem (2017) derived the analytical formula for the ARL of CUSUM control chart for $SARMA(1,1)_L$ process and used the Gauss-Legendre quadrature rule to approximate the numerical integration for the ARL.

Therefore, the objective of paper is to prove the explicit formula of the average run length (ARL) of EWMA control chart for an autoregressive integrated moving average, ARIMA(p,d,q) process and compare the results of the explicit formula with the results of the numerical integration method. The performance evaluation criteria of the proposed explicit formula are based on absolute percentage difference and the computational time. The paper is organized as follows: in Section 1, we start with the theoretical background of control charts. The EWMA control chart based on ARIMA(p,d,q) process is described in Section 2. The proposed explicit formula and the numerical integral equation for ARL of EWMA control chart are presented in Section 3 and Section 4 respectively. In Section 5, we compared the results from the explicit formula with the results from the numerical solution of an integral equation. Finally, we provided the conclusion in Section 6.

## 2. The Exponentially Weighted Moving Average Control Chart for ARIMA(p,d,q)

In this section, we present the characteristics of the EWMA control chart for ARIMA(p,d,q) process. The EWMA control chart is a powerful tool in detecting a mean shift. Let $X_t$ be the sequence of an autoregressive integrated moving average process, ARIMA(p,d,q) with exponential white noise.

The recursive equation of ARIMA(p,d,q) process with exponential white noise is defined as:

$$\Delta_d X_t = \mu + \varphi_1 \Delta_d X_{t-1} + \varphi_2 \Delta_d X_{t-2} + \ldots + \varphi_p \Delta_d X_{t-p} + \xi_t - \theta_1 \xi_{t-1} - \theta_2 \xi_{t-2} - \ldots - \theta_q \xi_{t-q}, \tag{1}$$

where $\xi_t$ is assumed to be a white noise process with exponential distribution. The initial value $\xi_{t-1}$ is usually to be the process mean, an autoregressive coefficient $0 \le \varphi \le 1$, a moving average coefficient $0 \le \theta \le 1$ and an initial value of ARIMA(p,d,q) process is equal to 1.

The recursive equation of EWMA statistics based on ARIMA(p,d,q) process is defined by

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t; \; t = 1,2,.... \tag{2}$$

where $X_t$ is a sequence of ARIMA(p,d,q) process, and $\lambda$ is an exponential smoothing parameter with of EWMA control chart with $0 < \lambda < 1$ and the initial value $Z_0 = u.$

The stopping time of EWMA control chart is defined as follows

$$\tau_b = \inf\{t > 0; Z_t > b\}, \quad b > u, \tag{3}$$

where $b$ is a constant parameter known as the upper control limit.

Let $\mathbb{E}_\infty(\cdot)$ denoted the expectation under density function $f(x,\beta)$ that the change-point occurs at point $\theta,$ where $\theta < \infty.$ Thus by definition, the ARL for ARIMA(p,d,q) process with an initial value $Z_0 = u$ is as follow

$$ARL = H(u) = \mathbb{E}_\infty(\tau_b) < \infty. \tag{4}$$

## 3. Explicit Formulas for Average Run Length of EWMA Control Chart for ARIMA(p,d,q)

This section is solved the explicit formulas of average run length of EWMA control chart for an autoregressive integrated moving average, ARIMA(p,d,q) process. We derive analytical explicit formulas of ARL by using the Fredhom integral equation of the second kind. Firstly, we define the function $H(u)$ is the ARL of EWMA chart for ARIMA(p,d,q) process. To assume that the lower and the upper control limits are zero and $b,$ respectively. Let $\mathbb{P}_Z$ denote the probability measure and $\mathbb{E}_Z$ denote the expectation corresponding to initial value $Z_0 = u.$ After that, we extend the function into the Fredholm integral equations of the second kind,

$$H(u) = 1 + \mathbb{E}_z\left[I\{0 < Z_1 < b\}L(Z_1)\right] + \mathbb{P}_z\{Z_1 = 0\}L(0). \tag{5}$$

The EWMA statistics $Z_1$ is an in-control state, we obtain

$$0 < (1-\lambda)Z_0 + \lambda\mu + \lambda\varphi_1\Delta_d X_{t-1} + \ldots + \lambda\varphi_p\Delta_d X_{t-p} + \lambda\xi_t - \lambda\theta_1\xi_{t-1} - \ldots - \lambda\theta_q\xi_{t-q} < b. \tag{6}$$

If $X_1$ gives an out-of-control state for $Z_1,$ thus

$$(1-\lambda)Z_0 + \lambda\mu + \lambda\varphi_1\Delta_d X_{t-1} + \ldots + \lambda\varphi_p\Delta_d X_{t-p} + \lambda\xi_t - \lambda\theta_1\xi_{t-1} - \ldots - \lambda\theta_q\xi_{t-q} > b, \tag{7}$$

or

$$(1-\lambda)Z_0 + \lambda\mu + \lambda\varphi_1\Delta_d X_{t-1} + \ldots + \lambda\varphi_p\Delta_d X_{t-p} + \lambda\xi_t - \lambda\theta_1\xi_{t-1} - \ldots - \lambda\theta_q\xi_{t-q} < 0.$$

For an initial value $Z_0 = u,$ then the Equation (6) can be rewrite as follows

$$0 < (1-\lambda)u + \lambda\mu + \lambda\varphi_1\Delta_d X_{t-1} + \ldots + \lambda\varphi_p\Delta_d X_{t-p} + \lambda\xi_t - \lambda\theta_1\xi_{t-1} - \ldots - \lambda\theta_q\xi_{t-q} < b.$$

According to the method of Champ and Rigdon (1991), we let an initial value of the EWMA statistics $Z_0 = u$ and $\xi_t \sim Exp(\beta)$ are white noise error terms, then the function $H(u)$ can be rewritten as follows

$$H(u) = 1 + \int H(Z_1) f(\xi_1) d\xi_1$$

$$=1+\int H((1-\lambda)u+\lambda\mu+\lambda\varphi_1\Delta_d X_{t-1}+\ldots+\lambda\varphi_p\Delta_d X_{t-p}+\lambda\xi_t-\lambda\theta_1\xi_{t-1}-\ldots-\lambda\theta_q\xi_{t-q})f(y)dy. \quad (8)$$

Changing the integration variable, so the function $H(u)$ is given by

$$H(u)=1+\frac{1}{\lambda}\int_0^b H(y)f\left(\frac{y-(1-\lambda)u}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-\ldots-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+\ldots+\theta_q\xi_{t-q}\right)dy \quad (9)$$

$$=1+\frac{1}{\lambda}\int_0^b H(y)\left(\frac{1}{\beta}e^{-\frac{1}{\beta}\left\{\frac{y-(1-\lambda)u}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-\ldots-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+\ldots+\theta_q\xi_{t-q}\right\}}\right)dy.$$

Consequently, we get the integral equation as follows

$$H(u)=1+\frac{1}{\lambda\beta}\int_0^b H(y)e^{-\frac{y}{\lambda\beta}}e^{\left\{\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+\ldots+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-\ldots-\theta_q\xi_{t-q})}{\beta}\right\}}dy. \quad (10)$$

This section will present Banach's fixed point theorem for existence and uniqueness of the results of the integral equation. On the metric space of continuous functions on a closed interval $(C(I),\|\ \|_\infty)$ where $I$ denotes the compact interval and the norm $\|H\|_\infty=Sup_{u\in I}|H(u)|$ and the operator $T$ is named on contraction, if it exists a number of $0\le q<1$ such that

$$\|T(H_1)-T(H_2)\|\le q\|H_1-H_2\| \text{ for all } H_1,H_2\in I.$$

Now, we define $C(I_1)$ as a continuous function over a range $I_1=[0,b]$ and define the operator $T$ by

$$T(H(u))=1+\frac{1}{\lambda\beta}e^{\left\{\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+\ldots+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-\ldots-\theta_q\xi_{t-q})}{\beta}\right\}}\int_0^b H(y)e^{-\frac{y}{\lambda\beta}}dy. \quad (11)$$

Therefore, the integral equation can be written as $T(H(u))=H(u)$. According to the Banach's fixed point theorem, if the operator $T$ is a contraction, then fixed point equations $T(H(u))=H(u)$ have a unique solution.

For any $u\in I$ and $H_1,H_2\in C(I)$ we have the inequality $\|T(H_1)-T(H_2)\|\le q\|H_1-H_2\|$ where $q<1$. According to (11), we get

$$\begin{aligned}\|T(H_1)-T(H_2)\| &= Sup_{u\in[0,b]}\left|H_1(y)-H_2(y)\frac{1}{\lambda\beta}e^{\frac{(1-\lambda)u}{\lambda\beta}+\frac{X_t}{\beta}}\int_0^b L(y)e^{-\frac{y}{\lambda\beta}}dy\right| \\ &\le \underset{u\in[0,b]}{Sup}\left|\|H_1-H_2\|\frac{1}{\lambda\beta}e^{\frac{(1-\lambda)u}{\lambda\beta}+\frac{X_t}{\beta}}(-\lambda\beta)(e^{-\frac{b}{\lambda\beta}}-1)\right| \\ &= \|H_1-H_2\|\underset{u\in[0,b]}{Sup}\left[\frac{1}{\lambda\beta}e^{\frac{(1-\lambda)u}{\lambda\beta}+\frac{X_t}{\beta}}(-1)(e^{-\frac{b}{\lambda\beta}}-1)\right] \\ &= \|H_1-H_2\|\underset{u\in[0,b]}{Sup}\left[e^{\frac{(1-\lambda)u}{\lambda\beta}+\frac{X_t}{\beta}}(1-e^{-\frac{b}{\lambda\beta}})\right]\end{aligned}$$

$$\leq q\left\|H_1 - H_2\right\|, \text{ where } q = \sup_{u\in[0,b]}\left[ e^{\frac{(1-\lambda)u}{\lambda\beta}+\frac{X_t}{\beta}}(1-e^{-\frac{b}{\lambda\beta}})\right] < 1.$$

We used the Fredholm integral equation to derive the explicit formula of the ARL of EWMA control chart for ARIMA(p,d,q) process.

First, to consider

$$H(u) = 1+\frac{1}{\lambda\beta}\int_0^b H(y)e^{-\frac{y}{\lambda\beta}} e^{\left\{\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right\}}dy.$$

Now, we let $C(u) = e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}$, then we have

$$H(u) = 1+\frac{C(u)}{\lambda\beta}\int_0^b H(y)e^{-\frac{y}{\lambda\beta}}dy, \quad 0\leq u\leq b.$$

Let $k = \int_0^b H(y)e^{-\frac{y}{\lambda\beta}}dy$, therefore $H(u)$ can write another form as follows

$$H(u) = 1+\frac{1}{\lambda\beta}e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}k. \tag{12}$$

To find a constant $k$ as following form

$$\begin{aligned}
k &= \int_0^b H(y)e^{-\frac{y}{\lambda\beta}}dy \\
&= \int_0^b\left(1+\frac{C(y)}{\lambda\beta}k\right)e^{-\frac{y}{\lambda\beta}}dy \\
&= \int_0^b e^{-\frac{y}{\lambda\beta}}dy + \int_0^b\frac{C(y)}{\lambda\beta}ke^{-\frac{y}{\lambda\beta}}dy \\
&= \int_0^b e^{-\frac{y}{\lambda\beta}}dy + \frac{k}{\lambda\beta}\int_0^b C(y)e^{-\frac{y}{\lambda\beta}}dy \\
&= \int_0^b e^{-\frac{y}{\lambda\beta}}dy + \frac{k}{\lambda\beta}\int_0^b e^{\left(\frac{(1-\lambda)y}{\lambda\beta}+\frac{X_t}{\beta}\right)}e^{-\frac{y}{\lambda\beta}}dy \\
&= \lambda\beta(1-e^{-\frac{b}{\lambda\beta}}) + \frac{k}{\lambda\beta}e^{\left(\frac{X_t}{\beta}\right)}\int_0^b e^{\left(\frac{(1-\lambda)y}{\lambda\beta}-\frac{y}{\lambda\beta}\right)}dy \\
&= \lambda\beta(1-e^{-\frac{b}{\lambda\beta}}) + \frac{k}{\lambda\beta}e^{\left(\frac{X_t}{\beta}\right)}\int_0^b e^{-\frac{y}{\beta}}dy
\end{aligned}$$

$$= \lambda\beta(1-e^{-\frac{b}{\lambda\beta}}) + \frac{k}{\lambda\beta}e^{\left(\frac{X_t}{\beta}\right)}\left(\beta(1-e^{-\frac{b}{\beta}})\right)$$

$$= \lambda\beta(1-e^{-\frac{b}{\lambda\beta}}) + \frac{k}{\lambda}e^{\left(\frac{X_t}{\beta}\right)}(1-e^{-\frac{b}{\beta}}).$$

Thus, a constant $k$ can be found as follows

$$k = \frac{(-\lambda\beta)(e^{-\frac{b}{\lambda\beta}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{X_t}{\beta}\right)}(e^{-\frac{b}{\beta}}-1)}. \tag{13}$$

Then, on substituting a constant $k$ into (12); we obtain that

$$H(u) = 1+\frac{1}{\lambda\beta}e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}\left(\frac{(-\lambda\beta)(e^{-\frac{b}{\lambda\beta}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{X_t}{\beta}\right)}(e^{-\frac{b}{\beta}}-1)}\right).$$

$$= 1-\frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}(e^{-\frac{b}{\lambda\beta}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}(e^{-\frac{b}{\beta}}-1)}.$$

Consequently, the explicit formulas of the ARL of EWMA control chart for ARIMA(p,d,q) process for $t = 1, 2, ….$ is given by

$$H(u) = 1-\frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}(e^{-\frac{b}{\lambda\beta}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta}\right)}(e^{-\frac{b}{\beta}}-1)}. \tag{14}$$

Since the process is in-control state with exponential parameter $\beta = \beta_0$, we obtain the explicit formula for $\text{ARL}_0$ of EWMA control chart for ARIMA(p,d,q) process is as follows

$$\text{ARL}_0 = 1-\frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta_0}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta_0}\right)}(e^{-\frac{b}{\lambda\beta_0}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta_0}\right)}(e^{-\frac{b}{\beta_0}}-1)}. \tag{15}$$

On the other hand, the process is out-of-control state with exponential parameter $\beta = \beta_1$, where $\beta_1 = \beta_0(1+\delta)$, the explicit formula for $\text{ARL}_1$ of EWMA control chart for ARIMA(p,d,q) process is as follows

$$\text{ARL}_1 = 1-\frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta_1}+\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta_1}\right)}(e^{-\frac{b}{\lambda\beta_1}}-1)}{1+\frac{1}{\lambda}e^{\left(\frac{(\mu+\varphi_1\Delta_d X_{t-1}+...+\varphi_p\Delta_d X_{t-p}+\xi_t-\theta_1\xi_{t-1}-...-\theta_q\xi_{t-q})}{\beta_1}\right)}(e^{-\frac{b}{\beta_1}}-1)}, \quad (16)$$

where $\beta$ is a parameter of exponential white noise, $b$ is upper control limit, $X_{t-L},\xi_{t-L}$ are the initial values, $\varphi$ is an autoregressive coefficient; $0\le\varphi\le 1$ and $\theta$ is a moving average coefficient; $0\le\theta\le 1$.

## 4. Numerical Integration of Average Run Length of EWMA Chart for ARIMA(p,d,q)

This section presents the method of finding the numerical integration of the ARL of EWMA control chart for the ARIMA(p,d,q) model when the white noise processes are exponential distribution. We solve the integral equation of ARL by using Gauss-Legendre quadrature to approximate integrals.

Consequently, the integral equation in (9) can be rewritten as follows

$$\tilde{H}(u) = 1+\frac{1}{\lambda}\int_0^b H(y) f\left(\frac{y-(1-\lambda)u}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-...-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+...+\theta_q\xi_{t-q}\right)dy. \quad (17)$$

Numerical integration of integral equations based on quadrature rule to estimate the integrals with finite sums. The approximation for an integral has the following form

$$\int_0^b W(y)f(y)dy \approx \sum_{j=1}^m w_j f(a_j),$$

where $w_j=\frac{b}{m}$ and $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right); j = 1,2,...,m.$

The numerical approximation to integral equation is denoted by $\tilde{H}(a_i)$, the solution can be found using the method of solving systems of linear algebraic equations,

$$\tilde{H}(a_i)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-...-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+...+\theta_q\xi_{t-q}\right).$$

Thus,

$$\tilde{H}(a_1)=1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-...-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+...+\theta_q\xi_{t-q}\right)$$

$$\tilde{H}(a_2)=1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_2}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-...-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+...+\theta_q\xi_{t-q}\right)$$

$$\vdots$$

$$\tilde{H}(a_m)= 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_m}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-...-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+...+\theta_q\xi_{t-q}\right).$$

or in matrix form as

$$\mathbf{H}_{m\times 1} = \mathbf{1}_{m\times 1}+\mathbf{R}_{m\times m}\mathbf{H}_{m\times 1}, \quad (18)$$

where $\mathbf{H}_{m\times 1}=\begin{pmatrix}\tilde{H}(a_1)\\ \tilde{H}(a_2)\\ \vdots \\ \tilde{H}(a_m)\end{pmatrix}$, $\mathbf{1}_{m\times 1}=\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$,

$$[\mathbf{R}]_{ij}\approx\frac{1}{\lambda}w_j f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-\ldots-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+\ldots+\theta_q\xi_{t-q}\right)$$

and $\mathbf{I}_m=\text{diag}(1,1,\ldots,1)$. If $(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}$ there exist

$$\mathbf{H}_{m\times 1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}. \tag{19}$$

Here $\tilde{H}(u)$ denotes the numerical integration solution of $H(u)$, then the integral equation in (9) can be approximated by

$$\tilde{H}(u)\approx 1+\frac{1}{\lambda}\sum_{j=1}^{m}w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)u}{\lambda}-\mu-\varphi_1\Delta_d X_{t-1}-\ldots-\varphi_p\Delta_d X_{t-p}+\theta_1\xi_{t-1}+\ldots+\theta_q\xi_{t-q}\right). \tag{20}$$

## 5. Numerical Result

A comparison of the accuracy and precision of the $\text{ARL}_0$ and $\text{ARL}_1$ values obtained from the explicit formulas with the results of numerical integration will consider the absolute percentage difference between the exact solution and the numerical integration solution. The explicit formula solution is denoted by $H(u)$ and the numerical integration solution is denoted by $\tilde{H}(u)$.

The absolute percentage difference of ARL can be calculated by

$$\text{Diff}(\%)=\frac{\left|H(u)-\tilde{H}(u)\right|}{H(u)}\times 100. \tag{21}$$

In Table 1, we present the results obtained from the explicit formulas of ARL for EWMA control chart when observations are ARIMA(p,d,q) process with exponential white noise and compare to the ARL from numerical integral equation with $m=500$ nodes. The criterions for choosing the in-control parameter values are the smoothing parameter $(\lambda)$ and the upper control limit $(b)$ for designing the EWMA control chart with minimum of $\text{ARL}_1$ for a given $\text{ARL}_0=370$ and the value of exponential parameter $\beta_0=1$ in the case of ARIMA(p,d,q) process with parameter $(\varphi,d,\theta)=(0.10,1.0,0.10)$, (0.30,1.0,0.10) and (0.50,1.0,0.30), respectively.

The results from Table 1 present the value of parameters for EWMA control chart and show that the $\text{ARL}_0$ from analytical solution is close to the numerical integration with the absolute percentage difference less than 0.1%. In addition, the results also show that the computational time for evaluating the purposed explicit formula is much less than the computational time required for numerical integral equation method.

In Table 2 to Table 4, we present the results of $\text{ARL}_0=370$ and $\text{ARL}_1$ values obtained from the analytical formula and the results from the numerical integration method for EWMA control chart for ARIMA(p,d,q) model when an autoregressive coefficient $(\varphi)$ is equal to 0.10, the $d$ is equal to 1.00 and a moving average coefficient $(\theta)$ is equal to 0.10. In the case in-control state, the value of the in-control parameter $\beta_0$ is equal to 1. And in the case out-of-control state, we specifies that the parameter $\beta_1$ is equal to $(1+\delta)$ where $\delta$ is the magnitude of shifts are equal to 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.20, 0.30, 0.40 and 0.50, respectively.

**Table 1** Comparison of $\text{ARL}_0$ computed using explicit formulas against numerical integration for ARMA(1,1,1) process with parameter $\beta_0 = 1$ for $\text{ARL}_0 = 370$

| $\lambda$ | $u$ | $b$ | ARMA(1,1,1) Process with $\varphi = 0.10$, $d = 1.0$ and $\theta = 0.10$ | | |
|---|---|---|---|---|---|
| | | | Explicit | Numerical | Diff (%) |
| 0.01 | 0.10 | 0.001441 | 370.166 | 370.166 **(0.125)**[a] | 0.0000 |
| 0.05 | 0.40 | 0.128100 | 370.964 | 370.923 **(0.125)**[a] | 0.0111 |
| 0.10 | 1.00 | 0.038500 | 370.305 | 370.304 **(0.110)**[a] | 0.0003 |
| $\lambda$ | $u$ | $b$ | ARMA(1,1,1) Process with $\varphi = 0.30$, $d = 1.0$ and $\theta = 0.10$ | | |
| 0.01 | 0.10 | 0.000604 | 370.483 | 370.483 **(0.125)**[a] | 0.0000 |
| 0.05 | 0.40 | 0.032430 | 370.063 | 370.061 **(0.110)**[a] | 0.0005 |
| 0.10 | 1.00 | 0.015100 | 370.668 | 370.667 **(0.125)**[a] | 0.0003 |
| $\lambda$ | $u$ | $b$ | ARMA(1,1,1) Process with $\varphi = 0.50$, $d = 1.0$ and $\theta = 0.30$ | | |
| 0.01 | 0.10 | 0.000766 | 370.388 | 370.388 **(0.125)**[a] | 0.0000 |
| 0.05 | 0.40 | 0.044500 | 370.774 | 370.770 **(0.141)**[a] | 0.0011 |
| 0.10 | 1.00 | 0.019355 | 370.071 | 370.071 **(0.110)**[a] | 0.0000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

To determine the exponential smoothing parameter $(\lambda)$ is equal to 0.01 and the upper control limit $(b)$ is equal to 0.001441, showing the results as in Table 2. In addition, the exponential smoothing parameter $(\lambda)$ is equal to 0.05 and the upper control limit $(b)$ is equal to 0.1281, showing the results as in Table 3. And to determine the exponential smoothing parameter $(\lambda)$ is equal to 0.10 and the upper control limit $(b)$ is equal to 0.0385, showing the results as in Table 4.

The results from Table 2 through Table 4 show that the results obtained from the explicit formula give the $\text{ARL}_0$ and $\text{ARL}_1$ values that are close to the results obtained by numerical integration methods. For the case of division points $m = 500$ nodes, it is found that the absolute percentage difference of ARL is less than 1.0%. In addition, when considering the time required to process the data, it is found that the successful ARL calculation of the explicit formula takes less than 0.01 seconds in comparison to the ARL calculation using the numerical integration method. The result is approximately 0.10 to 0.20 seconds. Therefore, finding the ARL value from the analytical formula successfully produces more accurate results and takes less time to process than the numerical integration method.

In Table 5 through Table 7 shows the results of the $\text{ARL}_0 = 370$ and $\text{ARL}_1$ values obtained from the analytical formula method and the results from the numerical integration method of the EWMA control chart for the ARIMA(1,1,1) model when given that an autoregressive coefficient $(\varphi)$ is equal to 0.30, the $d$ is equal to 1.0 and a moving average coefficient $(\theta)$ is equal to 0.10. In the case in-control state, the value of the in-control parameter $\beta_0$ is equal to 1. In the case out-of-control state, we specifies that the parameter $\beta_1$ is equal to $(1+\delta)$ where $\delta$ is the magnitude of shifts are equal to 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.20, 0.30, 0.40 and 0.50, respectively.

To determine the exponential smoothing parameter $(\lambda)$ is equal to 0.01 and the upper control limit $(b)$ is equal to 0.000604, showing the results as in Table 5. In addition, the exponential smoothing parameter $(\lambda)$ is equal to 0.05 and the upper control limit $(b)$ is equal to 0.03243, showing the results as in Table 6. And to determine the exponential smoothing parameter $(\lambda)$ is equal to 0.10 and the upper control limit $(b)$ is equal to 0.0151, showing the results as in Table 7.

**Table 2** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 0.10$, $\lambda = 0.01$, $b = 0.001441$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.166 | 370.166 (0.125)[a] | 0.0000 |
| 0.01 | 339.313 | 339.315 (0.125) | 0.0006 |
| 0.03 | 286.494 | 286.493 (0.125) | 0.0003 |
| 0.05 | 243.394 | 243.394 (0.140) | 0.0000 |
| 0.07 | 207.990 | 207.991 (0.125) | 0.0005 |
| 0.09 | 178.723 | 178.723 (0.141) | 0.0000 |
| 0.10 | 166.001 | 166.002 (0.156) | 0.0006 |
| 0.20 | 84.7010 | 84.7000 (0.115) | 0.0012 |
| 0.30 | 47.8277 | 47.8275 (0.140) | 0.0004 |
| 0.40 | 29.3061 | 29.3061 (0.125) | 0.0000 |
| 0.50 | 19.2095 | 19.2095 (0.109) | 0.0000 |

[a]. The values in parentheses are CPU times in numerical integration methods (seconds)

**Table 3** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 0.40$, $\lambda = 0.05$, $b = 0.1281$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.964 | 370.923 (0.125)[a] | 0.0111 |
| 0.01 | 352.040 | 352.003 (0.125) | 0.0105 |
| 0.03 | 317.988 | 317.955 (0.120) | 0.0104 |
| 0.05 | 288.317 | 288.288 (0.109) | 0.0101 |
| 0.07 | 262.343 | 262.318 (0.141) | 0.0095 |
| 0.09 | 239.508 | 239.486 (0.109) | 0.0092 |
| 0.10 | 229.119 | 229.099 (0.141) | 0.0087 |
| 0.20 | 152.844 | 152.833 (0.140) | 0.0072 |
| 0.30 | 108.062 | 108.055 (0.125) | 0.0065 |
| 0.40 | 79.9112 | 79.9068 (0.125) | 0.0055 |
| 0.50 | 61.2317 | 61.2288 (0.109) | 0.0047 |

[a]. The values in parentheses are CPU times in numerical integration methods (seconds)

**Table 4** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 1.00,\ \lambda = 0.10,\ b = 0.0385$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.305 | 370.304 (0.110) [a] | 0.0003 |
| 0.01 | 342.960 | 342.959 (0.125) | 0.0003 |
| 0.03 | 295.428 | 295.427 (0.114) | 0.0003 |
| 0.05 | 255.859 | 255.858 (0.141) | 0.0004 |
| 0.07 | 222.722 | 222.722 (0.125) | 0.0000 |
| 0.09 | 194.816 | 194.816 (0.140) | 0.0000 |
| 0.10 | 182.519 | 182.519 (0.109) | 0.0000 |
| 0.20 | 100.669 | 100.669 (0.125) | 0.0000 |
| 0.30 | 60.6216 | 60.6215 (0.120) | 0.0002 |
| 0.40 | 39.1635 | 39.1635 (0.111) | 0.0000 |
| 0.50 | 26.7963 | 26.7963 (0.140) | 0.0000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

**Table 5** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 0.10,\ \lambda = 0.01,\ b = 0.000604$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.482 | 370.483 (0.031) [a] | 0.0003 |
| 0.01 | 336.642 | 336.642 (0.047) | 0.0000 |
| 0.03 | 279.448 | 279.447 (0.031) | 0.0004 |
| 0.05 | 233.562 | 233.561 (0.031) | 0.0004 |
| 0.07 | 196.479 | 196.478 (0.109) | 0.0005 |
| 0.09 | 166.301 | 166.301 (0.047) | 0.0000 |
| 0.10 | 153.336 | 153.336 (0.031) | 0.0000 |
| 0.20 | 73.241 | 73.241 (0.110) | 0.0000 |
| 0.30 | 39.175 | 39.174 (0.062) | 0.0026 |
| 0.40 | 22.974 | 22.974 (0.062) | 0.0000 |
| 0.50 | 14.549 | 14.549 (0.046) | 0.0000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

In Table 8 through Table 10 shows the results of the $\text{ARL}_0 = 370$ and $\text{ARL}_1$ values obtained from the analytical formula method and the results from the numerical integration method of the EWMA control chart for the ARIMA(1,1,1) model when given that an autoregressive coefficient $(\varphi)$ is equal to 0.50, the $d$ is equal to 1.0 and a moving average coefficient $(\theta)$ is equal to 0.30. In the case in-control state, the value of the in-control parameter $\beta_0$ is equal to 1. In the case out-of-control state, we specifies that the parameter $\beta_1$ is equal to $(1+\delta)$ where $\delta$ is the magnitude of shifts are equal to 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.20, 0.30, 0.40 and 0.50, respectively.

**Table 6** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 0.40$, $\lambda = 0.05$, $b = 0.03243$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.063 | 370.061 (0.016)[a] | 0.0005 |
| 0.01 | 345.008 | 345.006 (0.062) | 0.0006 |
| 0.03 | 301.008 | 301.006 (0.032) | 0.0007 |
| 0.05 | 263.886 | 263.884 (0.031) | 0.0008 |
| 0.07 | 232.398 | 232.396 (0.031) | 0.0009 |
| 0.09 | 205.551 | 205.550 (0.062) | 0.0005 |
| 0.10 | 193.614 | 193.612 (0.031) | 0.0010 |
| 0.20 | 112.001 | 112.000 (0.047) | 0.0009 |
| 0.30 | 70.082 | 70.083 (0.015) | 0.0014 |
| 0.40 | 46.699 | 46.698 (0.031) | 0.0021 |
| 0.50 | 32.757 | 32.757 (0.063) | 0.0000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

**Table 7** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1) process with parameter $u = 1.00$, $\lambda = 0.10$, $b = 0.0151$

| Shift size $(\delta)$ | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.668 | 370.667 (0.031)[a] | 0.0003 |
| 0.01 | 339.980 | 339.979 (0.031) | 0.0003 |
| 0.03 | 287.393 | 287.392 (0.047) | 0.0003 |
| 0.05 | 244.428 | 244.428 (0.032) | 0.0000 |
| 0.07 | 209.092 | 209.092 (0.016) | 0.0000 |
| 0.09 | 179.848 | 179.848 (0.046) | 0.0000 |
| 0.10 | 167.127 | 167.127 (0.031) | 0.0000 |
| 0.20 | 85.628 | 85.628 (0.031) | 0.0000 |
| 0.30 | 48.504 | 48.504 (0.015) | 0.0000 |
| 0.40 | 29.792 | 29.792 (0.031) | 0.0000 |
| 0.50 | 19.563 | 19.563 (0.140) | 0.0000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

To determine the exponential smoothing parameter $(\lambda)$ is equal to 0.01 and the upper control limit $(b)$ is equal to 0.000766, showing the results as in Table 8. In addition, the exponential smoothing parameter $(\lambda)$ is equal to 0.05 and the upper control limit $(b)$ is equal to 0.0445, showing the results as in Table 9. And to determine the exponential smoothing parameter $(\lambda)$ is equal to 0.10 and the upper control limit $(b)$ is equal to 0.019355, showing the results as in Table 10.

**Table 8** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1)$_L$ process with parameter $u = 0.10$, $\lambda = 0.01$, $b = 0.000766$

| Shift size ($\delta$) | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.388 | 370.388 (0.094)[a] | 0.00000 |
| 0.01 | 337.363 | 337.363 (0.109) | 0.00000 |
| 0.03 | 281.349 | 281.349 (0.110) | 0.00000 |
| 0.05 | 236.202 | 236.202 (0.125) | 0.00000 |
| 0.07 | 199.553 | 199.553 (0.109) | 0.00000 |
| 0.09 | 169.601 | 169.601 (0.109) | 0.00000 |
| 0.10 | 156.692 | 156.692 (0.109) | 0.00000 |
| 0.20 | 76.202 | 76.2020 (0.110) | 0.00000 |
| 0.30 | 41.3621 | 41.3621 (0.110) | 0.00000 |
| 0.40 | 24.5445 | 24.5445 (0.125) | 0.00000 |
| 0.50 | 15.6861 | 15.6861 (0.125) | 0.00000 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

**Table 9** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1)$_L$ process with parameter $u = 0.40$, $\lambda = 0.05$, $b = 0.0445$

| Shift size ($\delta$) | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.774 | 370.770 (0.125)[a] | 0.0011 |
| 0.01 | 347.005 | 347.000 (0.140) | 0.0014 |
| 0.03 | 305.025 | 305.021 (0.125) | 0.0013 |
| 0.05 | 269.342 | 269.339 (0.125) | 0.0011 |
| 0.07 | 238.855 | 238.853 (0.094) | 0.0008 |
| 0.09 | 212.681 | 212.679 (0.109) | 0.0009 |
| 0.10 | 200.983 | 200.981 (0.125) | 0.0010 |
| 0.20 | 119.738 | 119.737 (0.125) | 0.0008 |
| 0.30 | 76.7839 | 76.7833 (0.109) | 0.0008 |
| 0.40 | 52.2185 | 52.2181 (0.125) | 0.0008 |
| 0.50 | 37.253 | 37.2528 (0.125) | 0.0005 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

The results from Table 8 through Table 10 show that the results obtained from the explicit formula give the $ARL_0$ and $ARL_1$ values that are close to the results obtained by numerical integration methods. For the case of division points $m = 500$ nodes, it is found that the absolute percentage difference of ARL is less than 1.0%. In addition, when considering the time required to process the data, it is found that the successful ARL calculation of the explicit formula takes less than 0.01 seconds in comparison to the ARL calculation using the numerical integration method. The result is approximately 0.10 to 0.20 seconds. Therefore, finding the ARL value from the analytical formula successfully produces more accurate results and takes less time to process than the numerical integration method.

**Table 10** Comparison of ARL values from the explicit formulas against numerical integration for ARIMA(1,1,1)$_L$ process with parameter $u = 1.00$, $\lambda = 0.10$, $b = 0.019355$

| Shift size ($\delta$) | Explicit | Numerical | Diff (%) |
|---|---|---|---|
| 0.00 | 370.071 | 370.071 (0.109)[a] | 0.0000 |
| 0.01 | 340.311 | 340.310 (0.109) | 0.0003 |
| 0.03 | 289.119 | 289.118 (0.110) | 0.0003 |
| 0.05 | 247.085 | 247.083 (0.125) | 0.0008 |
| 0.07 | 212.347 | 212.346 (0.110) | 0.0005 |
| 0.09 | 183.465 | 183.464 (0.109) | 0.0005 |
| 0.10 | 170.857 | 170.857 (0.125) | 0.0000 |
| 0.20 | 89.2645 | 89.2645 (0.109) | 0.0000 |
| 0.30 | 51.3855 | 51.3854 (0.110) | 0.0002 |
| 0.40 | 31.9812 | 31.9812 (0.110) | 0.0000 |
| 0.50 | 21.2244 | 21.2243 (0.140) | 0.0005 |

[a.] The values in parentheses are CPU times in numerical integration methods (seconds)

## 6. Conclusions

This research solves the explicit formula for the average run length (ARL) and proposes a method to approximate the ARL by the numerical integral method of the Exponentially Weighted Moving Average (EWMA) control chart for autoregressive integrated moving average (ARIMA (p, d, q)) process with exponential distribution white noise. We proof the analytical formula by using Fredholm's second type integral method and numerical approximation by using Gauss-Legendre quadrature rule. We used the Banach's Fixed Point theorem to verify that the results obtained from the proposed formula are existence and uniqueness of solution. In addition, we compare the accuracy of the $\text{ARL}_0$ and $\text{ARL}_1$ obtained from the analytical formula with the results obtained from numerical integration by considering the absolute percentage difference and the computational time to process the data (unit: minute).

The results show that the average run length (ARL) calculated from the explicit formula is close to the results from the numerical integral method with the absolute percentage difference of ARL is less than 1%. In addition, the computational times for computing the ARL were obtained by the numerical integral equation method was around 0.10−0.20 seconds, while the explicit formulas required a computational time of zero seconds. Therefore, the explicit formulas are efficiency high propriety of the evaluation ARL and it is easy to derive and reduce on computational times.

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## References

Areepong Y, Novikov AA. An integral equation approach for analysis of control charts. PhD [dissertation]. Australia: University of Technology; 2009.

Areepong Y, Sukparungsee S. An integral equation approach to EWMA chart for detecting a change in Lognormal distribution. Thail Stat. 2010; 8: 47-61.

Busaba J, Sukparungsee S, Areepong Y, Mititelu G. Numerical approximations of average run length for AR(1) on exponential CUSUM. IMECS 2012: Proceedings of the International Muti Conference of Engineers and Computer Scientists 2012; 2012 March 14-16; Hong Kong; 2012.

Borror CM, Montgomery D, Runger GC. Robustness of the EWMA control chart to non normality. J Qual Technol. 1999; 31: 309-316.

Champ CW, Rigdon SE. A comparison of the Markov chain and the integral equation approaches for evaluating the run length distribution of quality control charts. Commun Stat - Simul Comput. 1991; 20(1): 191-204.

Lucas JM, Saccucci MS. Exponentially weighted moving average control schemes: properties and enhancements. Technometrics. 1990; 32: 1-29.

Mititelu G, Areepong Y, Sukparungsee S, Novikov AA. Explicit analytical solutions for the average run length of CUSUM and EWMA charts. East-West J. Numer. Math. 2010; 1: 253-265.

Page ES. Continuous inspection schemes. Biometrika. 1954; 41: 100-114.

Petcharat K, Areepong Y, Sukparungsee S. Explicit formulas of average run length of EWMA chart for MA(q). Far East J Math Sci 2013; 78: 291 – 300.

Phanyaem S, Areepong Y, Sukparungsee S. Explicit formulas of average run length for ARMA(1,1) process of CUSUM control chart. Far East J Math Sci. 2014; 90: 211-224.

Phanyaem S. Average run length of cumulative sum control charts for $SARMA(1,1)_L$ models. Thail Stat. 2017; 15: 184-195.

Roberts SW. Control chart tests based on geometric moving average. Technometrics, 1959; 1: 239-250.

Shewhart WA. Economic control of quality of manufactured product. New York: D.Van Nostrand; 1931.