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The Gompertz Weibull Fréchet Distribution: Properties and Applications

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Abstract

A new distribution namely the Gompertz-Weibull Fréchet (Go-WFr) distribution is proposed. It is a class of Gompertz-G family of distributions. Some statistical properties of the proposed distribution including the reliability function, hazard function, quantile function, moments, and Lorenz and Bonferroni curves are discussed. The proposed distribution has five sub-models, that is, Gompertz-exponential Fréchet, Gompertz-Weibull inverse exponential, Gompertz-Weibull inverse Rayleigh, Gompertz-exponential inverse exponential, and Gompertz-exponential inverse Rayleigh distributions. The parameters of the Go-WFr are estimated by using the maximum likelihood estimation. Simulation studies and the application of the proposed distribution are illustrated.

Keywords: Weibull Fréchet distribution, Gompertz-G family of distributions, maximum likelihood estimation, hazard function, stochastic ordering

1. Introduction

Statistical applications play an important role in our life, especially in medicine and engineering which have lifetime distribution such as Rayleigh, Fréchet, exponential, and Weibull distribution. The Weibull distribution is one of the popular standard distributions in statistics, engineering, and medicine, which invented by Waloddi Weibull in 1937. It is widely used to describe the lifetime distributions, which is suitable for modeling real-life phenomena such as the dielectric failure of multilayer ceramic capacitors (Wang et al. 1997), the strength data (Basu et al. 2009), the monotonic failure rates (Ahmad and Iqbal 2017), etc. The Weibull distribution has the Rayleigh and exponential distributions as sub-models. Moreover, the Fréchet distribution, also known as inverse Weibull distribution is a special case of the generalized extreme value distribution (Khan et al. 2008; Gusmao et al. 2011).

Various modifications of the Weibull distribution, of interest in this research, is the flexible Weibull distribution because it has many applications in applied statistics, life testing experiments, clinical studies, and reliability analysis (Bebbington et al. 2009; El-Desouky et al. 2017). The Weibull Fréchet (WFr) distribution was proposed by Afify et al. (2016), which is the Weibull family of the Fréchet distribution. The WFr distribution is applied to two real data sets such as the breaking stress of carbon fibres and the strengths of glass fibres, prove empirically its flexibility. The results show that the WFr distribution is the best model when it compares the Kumaraswamy Fréchet, exponentiated Fréchet, beta Fréchet, gamma extended Fréchet, transmuted MarshallOlkin Fréchet, transmuted Fréchet, MarshallOlkin Fréchet and Fréchet distributions (Afify et al. 2016).

Numerous existing continuous probability distributions have been extensively used for statistical data modeling in several areas, such as engineering, actuarial, environmental, and medical sciences, biological studies, demography, economics, finance, and insurance. However, in many applied areas such as lifetime analysis, finance, and insurance, there is a clear need for extended forms of these distributions Some attempts have been made to define new classes of distributions to extend well-known families and at the same time provide great flexibility in data modeling in practice. Several families are employed one or more parameters to generate new distributions have been proposed in the literature. The quests for more flexible models to model complex data has led to several new distributions that are obtained by generalizing the baseline distributions (Oguntunde et al. 2015; Alizadeh et al. 2017).

Alizadeh et al. (2017) introduced the family of distributions, which is a generator of continuous distributions with two extra parameters called the Gompertz-G (Go-G) family of distributions. They also introduced sub-models, such as the Go-normal, Go-gamma, Go-beta, Go-log-logistic, Go-Weibull, Go-Fréchet, and Go-exponentiated Weibull distributions. They investigated some probability functions, such as moments, moment generating function, incomplete moments, quantile function, etc. The model parameters are estimated by using the maximum likelihood estimation (MLE). Two real data sets were fitted by the developed distributions. The Go-G family of distributions is an interesting distribution because for many reasons, Alizadeh et al. (2017), for instance, (i) to make the kurtosis more flexible compared to the baseline model, (ii) to produce a skewness for symmetrical distributions, (iii) construct heavy-tailed distributions that are not longer tailed for modeling real data, (iv) to generate distributions with symmetric, left-skewed, right-skewed, and reversed-J shaped, (v) to define special models with all types of the hazard rate function, and (vi) to provide consistently better fits than other generated models under the same baseline distribution. Many researchers proposed Go-G distribution. Oguntunde et al. (2017) introduced Go-Lomax distribution with increasing, decreasing, and constant failure rate, and applied it apply to waiting times data. Next, Koleoso et al. (2019) proposed the three-parameter Go-Lindley distribution and it was applied to industrial data. And Oguntunde et al. (2019) introduced the Go-Fréchet distribution and applied it to real data sets about the strength of carbon fibers and civil engineering data. Recently, the Go-flexible Weibull is proposed by Khaleel et al. (2020), and it was applied to real-life data sets and it was compared with some distribution such as the Go-Weibull, Go-Burr type XII, Go-Lomax, etc.

In this study, we proposed a new flexible alternative distribution to apply the lifetime data called the Gompertz - Weibull Fréchet distribution. The preliminaries to develop a new distribution are introduced in Section 2. A new lifetime distribution development and its sub-models will be introduced in Section 3. Some properties of the proposed distribution are discussed in Section 4. The model parameter estimation is estimated by using the MLE will be illustrated in Section 5. Simulation and application studies of the proposed distribution are illustrated in Sections 6 and 7, respectively. Finally, the discussion and conclusion are presented.

2. Preliminaries

In this section, we introduced the probability function of the WFr distribution (Afify et al. 2016) and the Go-G family of distribution (Alizadeh et al. 2017) as follows.

Definition 1 If X is the WFr random variable, then its cumulative distribution function (cdf) is

$$G_{\text{WFr}}(x) = 1-\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right]-1\}^{-\gamma}\right\}, x>0, \tag{1}$$

where parameters $\alpha, \beta, \gamma, \omega > 0$. Its probability density function (pdf) corresponding is

$$\begin{aligned} g_{\text{WFr}}(x) = {} & \alpha\gamma\omega\beta^{\alpha}x^{-(\alpha+1)}\exp\left[-\gamma(\beta/x)^{\alpha}\right]\left[1-\exp\left[-(\beta/x)^{\alpha}\right]\right]^{-(\gamma+1)} \\ & \times\ \exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right]-1\}^{-\gamma}\right\}\ . \end{aligned} \tag{2}$$

Definition 2 If X be a Go-G random variable with the cdf

$$F_{\text{Go-G}}(x) = 1 - \exp\left\{\frac{\lambda}{\theta}\left\{1 - [1 - G(x;\boldsymbol{\xi})]^{-\theta}\right\}\right\}, x > 0, \tag{3}$$

where $G(x;\boldsymbol{\xi})$ is the cdf of the baseline distribution (see Alizadeh et al. (2017)) depending on a parameter vector $\boldsymbol{\xi}$ and two shape parameters $\lambda > 0$ and $\theta > 0$. Consequently, the corresponding pdf will be

$$f_{\text{Go-G}}(x) = \lambda g(x;\boldsymbol{\xi})[1 - G(x;\boldsymbol{\xi})]^{-\theta-1}\exp\left\{\frac{\lambda}{\theta}\left\{1 - [1 - G(x;\boldsymbol{\xi})]^{-\theta}\right\}\right\}. \tag{4}$$

Recently, a Go-G distribution called the Gompertz-flexible Weibull (Go-FW) distribution, which was developed recently by Khaleel et al. (2020). Its cdf and pdf are

$$F_{\text{Go-FW}}(x) = 1 - \exp\left\{\frac{\lambda}{\theta}\left[1 - \left[\exp\left(-e^{ax-b/x}\right)\right]^{-\theta}\right]\right\}, x > 0 \tag{5}$$

$$\begin{aligned} f_{\text{Go-FW}}(x) &= \lambda\left(a + \frac{b}{x^2}\right)e^{ax-b/x}\left[\exp\left(-e^{ax-b/x}\right)\right]^{-\theta} \\ &\quad \times \exp\left\{\frac{\lambda}{\theta}\left[1 - \left[\exp\left(-e^{ax-b/x}\right)\right]^{-\theta}\right]\right\}, \end{aligned} \tag{6}$$

where $\lambda, \theta, a, b > 0$

3. The Gompertz-Weibull Fréchet Distribution

In this section, a new distribution called Gompertz-Weibull Fréchet (Go-WFr) distribution, which is a Go-G family of distributions, where the WFr is a baseline distribution will be introduced.

Theorem 1 *If X be a Go-WFr random variable with a parameter vector of $\boldsymbol{\Theta} = (\lambda, \theta, \alpha, \beta, \gamma, \omega)$, denoted by $X \sim$ Go-WFr($\boldsymbol{\Theta}$), then its cdf and pdf respectively are*

$$F(x;\boldsymbol{\Theta}) = 1 - \exp\left\{\frac{\lambda}{\theta}\left\{1 - \left\{\exp\left\{-\omega[\exp\left((\beta/x)^{\alpha}\right) - 1]^{-\gamma}\right\}\right\}^{-\theta}\right\}\right\}, \tag{7}$$

$$\begin{aligned} f(x;\boldsymbol{\Theta}) &= \alpha\gamma\omega\beta^{\alpha}x^{-(\alpha+1)}\exp\left[-\gamma(\beta/x)^{\alpha}\right]\{1 - \exp\left[-(\beta/x)^{\alpha}\right]\}^{-(\gamma+1)} \\ &\quad \times \exp\left\{-\frac{\lambda}{\theta}\left\{1 - \left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right] - 1\}^{-\gamma}\right\}\right\}^{-\theta}\right\}\right\} \\ &\quad \times\left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right] - 1\}^{-\gamma}\right\}\right\}^{-\theta}, \end{aligned} \tag{8}$$

where $x > 0$, and the parameters $\lambda, \theta, \alpha, \beta, \gamma, \omega > 0$.

Proof: By replacing the WFr cdf as in Equation (1) into Equation (3), we then obtain the cdf of the Go-WFr distribution as in Equation (7). The pdf of X as in equation (8) is obtained by differentiating the cdf of the Go-WFr distribution with respect to x. In fact, $f(x;\boldsymbol{\Theta})$ as Equation (8) is defined as pdf, which is satisfied the properties, $f(x;\boldsymbol{\Theta}) > 0$ for all x and $\int_0^{\infty} f(x;\boldsymbol{\Theta})dx = 1$, that can be shown as following;

$$\begin{aligned} \text{Let } y &= \exp\left\{\frac{\lambda}{\theta}\left\{1 - \left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right] - 1\}^{-\gamma}\right\}\right\}^{-\theta}\right\}\right\} = \exp(u), \\ u &= \frac{\lambda}{\theta}\left\{1 - \left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right] - 1\}^{-\gamma}\right\}\right\}^{-\theta}\right\} = \frac{\lambda}{\theta} - \frac{\lambda}{\theta}t^{-\theta}, \\ t &= \exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right] - 1\}^{-\gamma}\right\}, \end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dt}\cdot\frac{dt}{dx} = -\lambda\alpha\beta^{\alpha}\gamma\omega x^{-\alpha-1} \\ &\quad\times\exp\left[-\gamma(\beta/x)^{\alpha}\right]\left\{1-\exp\left[-(\beta/x)^{\alpha}\right]\right\}^{-\gamma-1} t^{-\theta}\exp(u),\end{aligned}$$

consequently, we obtain

$$\begin{aligned}\int_0^{\infty} f(x;\boldsymbol{\Theta})dx &= -\exp\left\{\frac{\lambda}{\theta}\left\{1-\left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right]-1\}^{-\gamma}\right\}\right\}^{-\theta}\right\}\right\}\Bigg|_0^{\infty} \\ &= 1.\end{aligned}$$

Some plots of the Go-WFr pdf are shown in Figure 1. The behavior of Go-WFr pdf has various shapes; (i) The reversed J shape when $\gamma < 1$, and (ii) the unimodal distribution when $\gamma \geq 1$ which consisting of left-skewed, right-skewed, and approximately symmetric shapes.

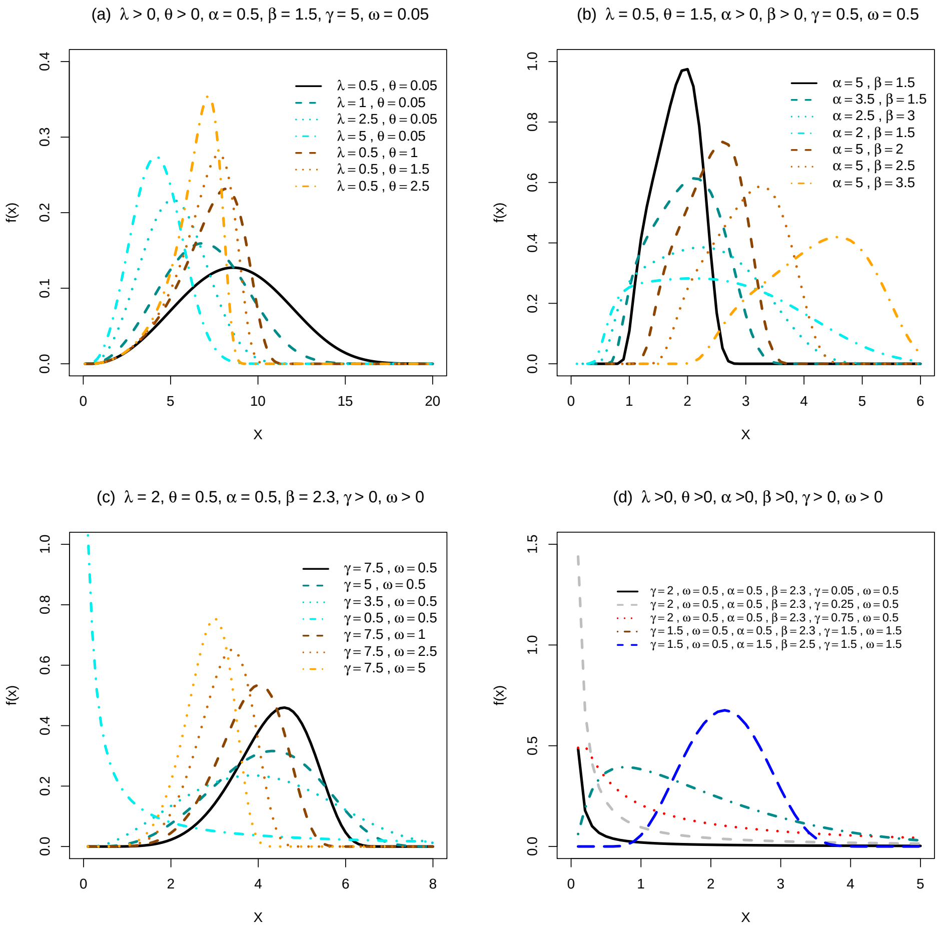

Figure 1 Plots of the Go-WFr pdf with the specified parameters $\lambda, \theta, \alpha, \beta, \gamma$ and ω

Some special case of the WFr distribution are presented by Afify et al. (2016) as follows; (i) for $\gamma = 1$, the WFr distribution reduce to the exponential Fréchet (EFr) distribution, (ii) the WFr distribution reduces to the Weibull inverse exponential (WIE) distribution when $\alpha = 1$, (iii) when $\alpha = 2$ the WFr distribution refers to Weibull inverse Rayleigh (WIR) distribution, (iv) for $\alpha = 1$ and

$\gamma = 1$, the WFr distribution reduce to the exponential inverse exponential (EIE) distribution, and (v) the WFr distribution reduces to the exponential inverse Rayleigh (EIR) distribution.

According to Afify et al. (2016), we will have five sub-models of the Go-WFr distribution as Table 1

Table 1 Sub-models of the Go-WFr distribution

Parameters						Sub-models
λ	θ	α	β	γ	ω	
λ	θ	α	β	1	ω	the Gompertz-exponential Fréchet (Go-EF)
λ	θ	1	β	γ	ω	the Gompertz-Weibull inverse exponential (Go-WIE)
λ	θ	2	β	γ	ω	the Gompertz-Weibull inverse Rayleigh (Go-WIR)
λ	θ	1	β	1	ω	the Gompertz-exponential inverse exponential (Go-EIE)
λ	θ	2	β	1	ω	the Gompertz-exponential inverse Rayleigh (Go-EIR)

4. Statistical Properties

In this section, we discuss some statistical properties of the proposed distribution including the reliability function, hazard function, quantile function, moments, skewness, kurtosis, Lorenz curve, and Bonferroni curve.

4.1. Reliability and hazard functions

If $X \sim$ Go-WFr($\mathbf{\Theta}$), then its reliability and hazard functions are respectively,

$$\begin{aligned} S(x;\mathbf{\Theta}) &= \exp\left\{-\frac{\lambda}{\theta}\left\{1-\left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right]-1\}^{-\gamma}\right\}\right\}^{-\theta}\right\}\right\}, \qquad (9)\\ h(x;\mathbf{\Theta}) &= \alpha\gamma\omega\beta^{\alpha}x^{-(\alpha+1)}\exp\left[-\gamma(\beta/x)^{\alpha}\right]\{1-\exp\left[-(\beta/x)^{\alpha}\right]\}^{-(\gamma+1)}\\ &\quad \times\left\{\exp\left\{-\omega\{\exp\left[(\beta/x)^{\alpha}\right]-1\}^{-\gamma}\right\}\right\}. \qquad (10)\end{aligned}$$

Some plots of Go-WFr hazard function and its sub-model hazard function are shown in Figure 2 (a). The Go-WFr distribution has various forms of hazard function, such as an increasing, decreasing, and a unimodal hazard functions. In addition, the hazard function plots of sub-model of Go-WFr distribution are shown in Figure 2 (b)-(f).

4.2. The quantile function

From the Go-WFr cdf as in equation (1), and let $F(x;\mathbf{\Theta}) = U$ where U is a uniform random variable on $[0,1]$, its quantile function is

$$\begin{aligned} Q_F(u) &= F^{-1}(u;\mathbf{\Theta})\\ &= \beta\left\{\log\left\{1+\left\{-\frac{1}{\omega}\log\left(\left[1-\frac{\lambda}{\theta}\log(1-u)\right]^{-\frac{1}{\lambda}}\right)\right\}^{-1/\gamma}\right\}\right\}^{-1/\alpha}, \qquad (11)\end{aligned}$$

where $\lambda,\theta,\alpha,\beta,\gamma,\omega > 0$.

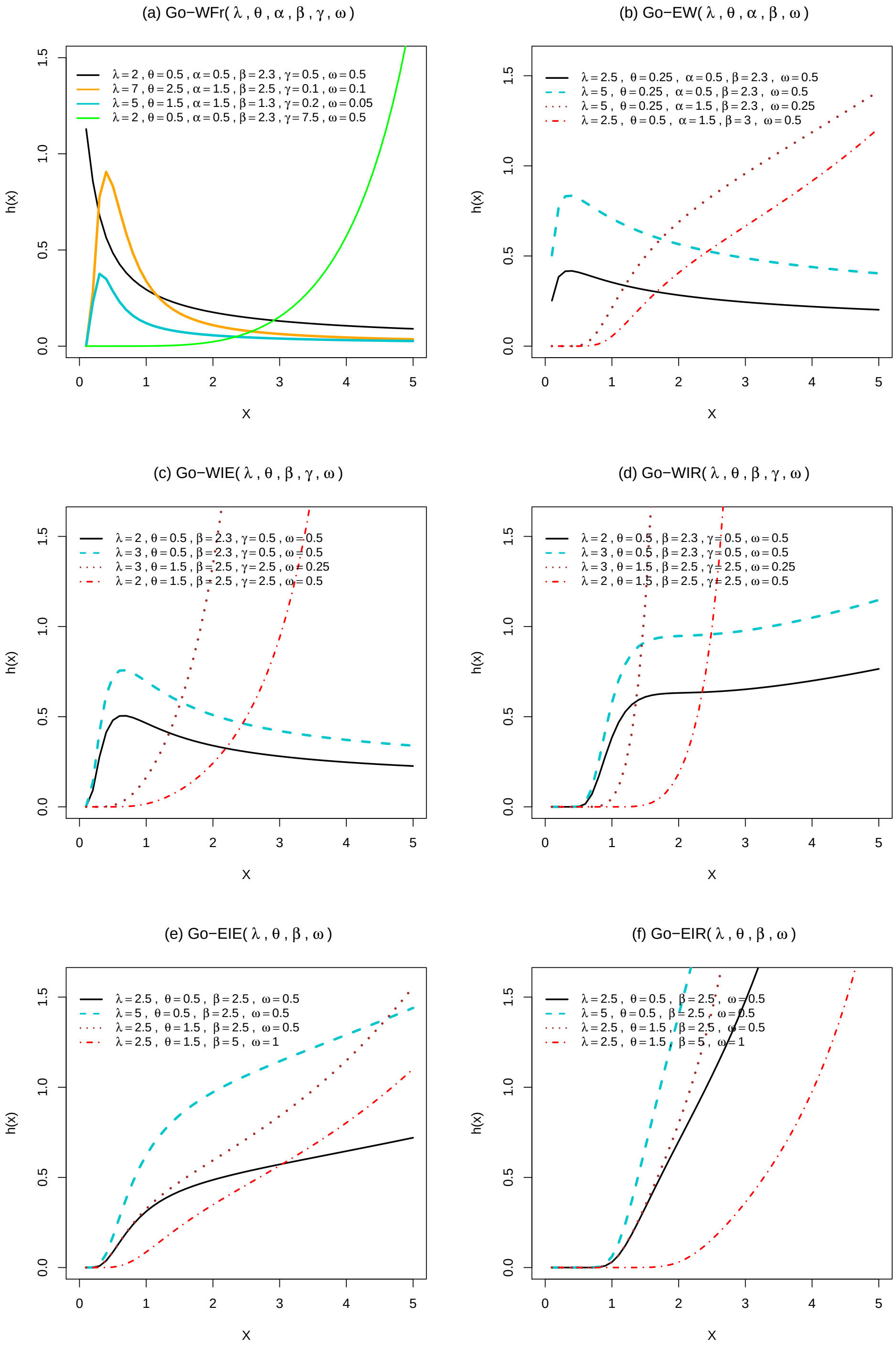

Figure 2 Plots of the hazard rate function of the Go-WFr distribution and its sub-models

4.3. Moments

From the probability weighted moment (PWM) of $\tau_{(r,k)}$, (see Alizadeh et al. 2017)

$$\tau_{(r,k)} = \int_{-\infty}^{\infty} x^r G(x)^k g(x)dx = \int_0^1 Q_G(u)^r u^k du, \tag{12}$$

we have the moments of the Go-G distribution can be expressed as an infinite linear combination of baseline PWM as follows;

$$E(X^r) = \sum_{k=0}^{\infty} (k+1) b_{k+1} \tau_{(r,k)}, \tag{13}$$

where

$$b_{k+1} = -\sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{i+k+1}}{i!} \binom{i}{j} \binom{-j\theta}{k+1} \left(\frac{\lambda}{\theta}\right)^i.$$

From the WFr quantile function

$$Q_G(u) = G^{-1}(u;\alpha,\beta,\gamma,\omega) = \beta \left\{ \log\left[1 + \left(-\frac{1}{\omega}\log(1-u)\right)^{-1/\gamma}\right]\right\}^{-1/\alpha},$$

we have the moments of the Go-WFr as,

$$\mu'_{[r]} = \sum_{k=0}^{\infty} (k+1) b_{k+1} {\tau'}_{(r,k)}, \tag{14}$$

where ${\tau'}_{(r,k)} = \int_0^1 \beta^r \left\{\log\left[1 + (-\log(1-u)/\omega)^{-1/\gamma}\right]\right\}^{-r/\alpha} u^k du$. Since it has no closed form but we can use function in R software to compute all concern integration.

4.4. Skewness and Kurtosis

In this section, we consider the effects of the shape parameters on the skewness and kurtosis. It can be defined based on quantile measures, the Bowley skewness (Alizadeh et al. 2017; Kenney and Keeping 1962).

From quantile function in Equation (11), $Q_F(u;\mathbf{\Theta})$, we have the Bowley skewness of the Go-WFr distriubiton as

$$\mathrm{B} = \frac{Q_F\left(\frac{3}{4};\mathbf{\Theta}\right) + Q_F\left(\frac{1}{4};\mathbf{\Theta}\right) - 2Q_F\left(\frac{1}{2};\mathbf{\Theta}\right)}{Q_F\left(\frac{3}{4};\mathbf{\Theta}\right) - Q_F\left(\frac{1}{4};\mathbf{\Theta}\right)}. \tag{15}$$

In addition, the Moors kurtosis (Alizadeh et al. 2017; Moors 1998) is considered as the kurtosis of the Go-WFr family, i.e.,

$$\mathrm{M} = \frac{Q_F\left(\frac{3}{8};\mathbf{\Theta}\right) - Q_F\left(\frac{1}{8};\mathbf{\Theta}\right) + Q_F\left(\frac{7}{8};\mathbf{\Theta}\right) - Q_F\left(\frac{5}{8};\mathbf{\Theta}\right)}{Q_F\left(\frac{6}{8};\mathbf{\Theta}\right) - Q_F\left(\frac{2}{8};\mathbf{\Theta}\right)}. \tag{16}$$

Figure 3 displays some plots of the measures B and M for the Go-WFr distribution with the different of the parameters.

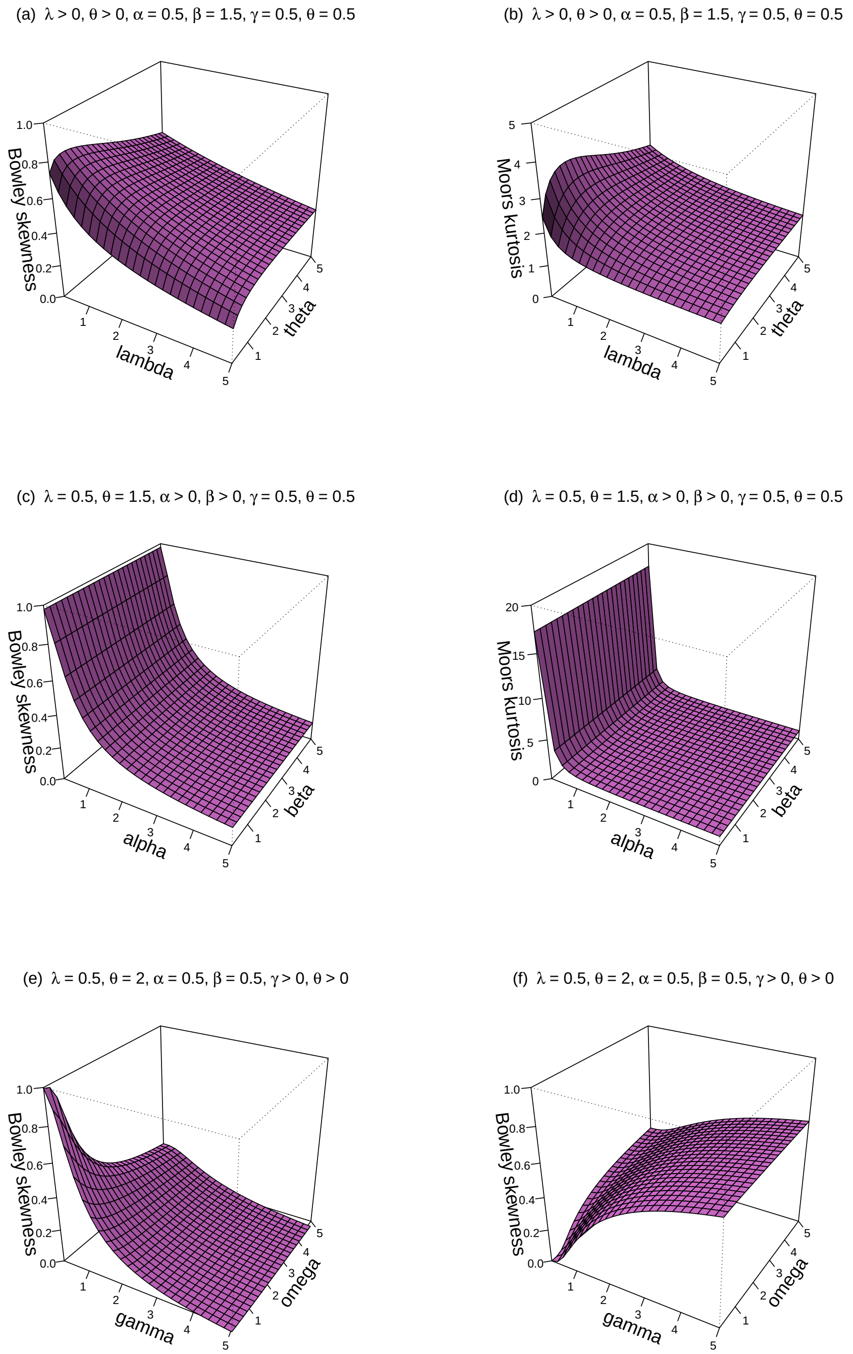

Figure 3 Plots of the measures B and M for the Go-WFr distribution with the different values of the parameters

4.5. Lorenz and Bonferroni curves

The Lorenz curve is a way of showing the distribution of income (or wealth) within an economy. It was developed by Lorenz in 1905 for describing wealth distribution. The Lorenz curve for a cumulative income distribution of $F(x)$ with mean μ is defined by Aaberge (1993)

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)dt, 0 \leq p \leq 1,$$

where $F^{-1}(t) = Q_F(t)$ is the inverse cdf or the quantile function of t. The Lorenz curve of the Go-WFr distribution is

$$L(p) = \frac{\beta}{\mu}\int_0^p \left\{\log\left\{1 + \left\{-\frac{1}{\omega}\log\left[\left(1 - \frac{\lambda}{\theta}\log(1-p)\right)^{-1/\lambda}\right]\right\}^{-1/\gamma}\right\}\right\}^{-1/\alpha}, \quad (17)$$

where $\mu = \sum_{k=0}^{\infty}(k+1)b_{k+1}\tau_{(1,k)}$ is the mean of the Go-WFr distribution. The corresponding Bonferroni curve in (17) is

$$B(p) = \frac{L(p)}{p}, 0 \leq p \leq 1. \quad (18)$$

Figure 4 shows plots of the Lorenz and Bonferroni curves of $X \sim$ Go-WFr $(\lambda, \theta, \alpha, \beta, \gamma, \omega)$.

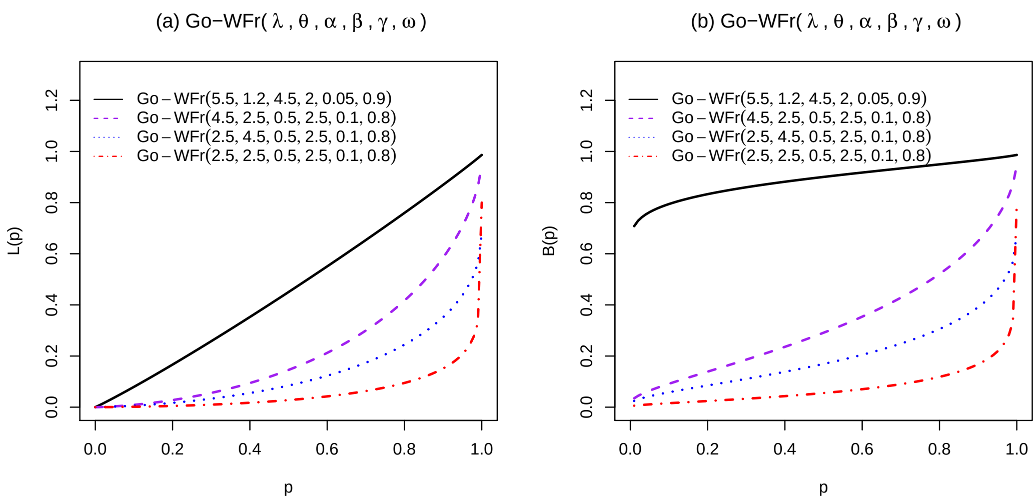

Figure 4 The Lorenz and Bonferroni curves of the Go-WFr distribution

Since it has no closed form but we can use function in R software to compute all concern integration.

5. Parameter Estimation

In this section, the proposed model parameters will be estimated by using the MLE. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be independent and identically distributed the Go-WFr distribution with a parameter vector of $\boldsymbol{\Theta} = (\lambda, \theta, \alpha, \beta, \gamma, \omega)$. If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is a random sample then its log-likelihood function is, $\ell(x|\mathbf{\Theta}) = \log L_n(\mathbf{\Theta}; \boldsymbol{x})$,

$$\begin{aligned} \ell(x|\mathbf{\Theta}) &= \log \prod_{i=1}^{n} f(x_i;\mathbf{\Theta}) = n\log\alpha + n\log\gamma + n\log\omega - (\alpha+1)\log\sum_{i=1}^{n} x_i \\ &+ \log\sum_{i=1}^{n}\{\exp[-\gamma(\beta/x)^{\alpha}]\} - (\gamma+1)\log\sum_{i=1}^{n}\{1-\exp[-(\beta/x_i)^{\alpha}]\} \\ &+ n\alpha\log\beta + \log\sum_{i=1}^{n}\left\{\exp\left\{-\frac{\lambda}{\theta}\{1-\{\exp\{-\omega\{\exp[-(\beta/x_i)^{\alpha}]\right.\right.\right. \\ &\left.\left.\left. -1\}^{-\gamma}\}\}^{-\theta}\right\}\right\}\right\} - \theta\log\sum_{i=1}^{n}\left\{\exp\left\{-\omega\{\exp[(\beta/x_i)^{\alpha}]-1\}^{-\gamma}\right\}\right\}, \end{aligned}$$

To estimate the model parameters, we take the partial derivatives of the log-likelihood function with respect to $\lambda, \theta, \alpha, \beta, \gamma$, and ω, and equate them to zero, i.e.,

$$\frac{\partial\ell(x|\mathbf{\Theta})}{\partial\lambda} = 0, \frac{\partial\ell(x|\mathbf{\Theta})}{\partial\theta} = 0, \frac{\partial\ell(x|\mathbf{\Theta})}{\partial\alpha} = 0,$$
$$\frac{\partial\ell(x|\mathbf{\Theta})}{\partial\beta} = 0, \frac{\partial\ell(x|\mathbf{\Theta})}{\partial\gamma} = 0, \frac{\partial\ell(x|\mathbf{\Theta})}{\partial\omega} = 0.$$

We obtain the MLE of $\lambda, \theta, \alpha, \beta, \gamma$, and ω from the above equations of the partial derivatives of $\log L$ with respect to each parameter by using the numerical method of Newton - Raphson type procedure. Since of the difficulty and complexity of a system of nonlinear equations, we therefore solve the system of equations by using the *nlm* function in R language R Core Team (2020).

6. Simulation study

In this section, we carry out simulation study for parameter estimation of the Go-WFr parameters. The simulations are described as below:

(i) The sample sizes are taken as $n = 30, 50, 100, 200, 300, 400$, and 500 .

(ii) The data are generated from

$$X_i = \beta\left\{\log\left\{1+\left\{-\frac{1}{\omega}\log\left(\left[1-\frac{\lambda}{\theta}\log(1-u_i)\right]^{-\frac{1}{\lambda}}\right)\right\}^{-1/\gamma}\right\}\right\}^{-1/\alpha},$$

where u_i is the value of a uniform random variable on interval (0,1). (iii) The parameter values are set as two cases, i.e., (a) $\lambda = 0.10$, $\theta = 0.55$, $\alpha = 1.3$, $\beta = 2.6$, $\gamma = 2.4$, and $\omega = 0.02$; and (b) $\lambda = 0.06$, $\theta = 0.41$, $\alpha = 1.45$, $\beta = 2.09$, $\gamma = 2.18$, and $\omega = 0.01$.

(iv) Each sample size is replicated 1,000 times.

(v) Formulas used for calculating MSE, mean, and bias of $\hat{\lambda}$ is given by

$$\text{MSE}(\hat{\lambda}) = \frac{1}{1000}\sum_{t=1}^{1000}\left(\hat{\lambda}_t - \lambda\right)^2, \hat{\lambda}_{\text{mean}} = \frac{1}{1000}\sum_{t=1}^{1000}\hat{\lambda}_t, \text{and Bias} = \hat{\lambda}_{\text{mean}} - \lambda.$$

(vi). Step (v) is also repeated for the other parameters, i.e., $\theta, \alpha, \beta, \gamma, \omega$.

The results are provided in Tables 2 and 3. It shows that the estimated value of each maximum likelihood estimators is close to the true value of the parameters of the Go-WFr distribution. The MSE values of each maximum likelihood estimators are decreasing when n increasing. The simulation study that was conducted shows that the parameters of the Go-WFr distribution are stable; though values for biasedness were generated, these values are small, indicating that the maximum likelihood estimates of the Go-WFr distribution are not too far from the true parameter values; the absolute bias and the mean square values also decreases as the sample size increases.

Table 2 The mean and MSE values of the estimated values of parameters of the Go-WFr distribution; (a) $\lambda = 0.10$, $\theta = 0.55$, $\alpha = 1.3$, $\beta = 2.6$, $\gamma = 2.4$, and $\omega = 0.02$

Sample sizes		Parameters					
(n)		λ	θ	α	β	γ	ω
30	Mean	0.2096	0.8372	1.5670	2.4882	2.2860	0.0227
	MSE	0.0964	1.5602	0.1746	2.7699	0.5383	0.0058
50	Mean	0.1669	1.0885	1.4310	2.4445	2.0381	0.0529
	MSE	0.0634	1.4924	0.1482	2.4807	0.5283	0.0145
100	Mean	0.0905	0.8399	1.4017	2.3651	2.0482	0.0651
	MSE	0.0189	0.9049	0.1051	1.8103	0.4378	0.0130
200	Mean	0.0779	0.7060	1.3876	2.3238	2.1100	0.0517
	MSE	0.0070	0.4782	0.0755	1.1596	0.2970	0.0063
300	Mean	0.0754	0.6308	1.3799	2.3169	2.1542	0.0442
	MSE	0.0047	0.2760	0.0575	0.7536	0.2155	0.0035
400	Mean	0.0729	0.5815	1.3808	2.3138	2.1857	0.0384
	MSE	0.0031	0.1869	0.0462	0.4992	0.1583	0.0026
500	Mean	0.0717	0.5659	1.3766	2.3136	2.1885	0.0345
	MSE	0.0027	0.1386	0.0415	0.4254	0.1180	0.0014

Table 3 The mean and MSE values of the estimated values of parameters of the Go-WFr distribution; (b) $\lambda = 0.06$, $\theta = 0.41$, $\alpha = 1.45$, $\beta = 2.09$, $\gamma = 2.18$, and $\omega = 0.01$

Sample sizes		Parameters					
(n)		λ	θ	α	β	γ	ω
30	Mean	0.2000	1.1055	1.5670	2.5404	2.3153	0.0237
	MSE	0.0899	1.4914	0.1451	2.3855	0.5806	0.0038
50	Mean	0.1916	0.9190	1.4216	2.4907	2.0566	0.0548
	MSE	0.0883	1.1922	0.1414	2.6545	0.4565	0.0134
100	Mean	0.0809	0.8254	1.3526	2.3613	2.0197	0.0825
	MSE	0.0143	0.8717	0.1215	1.7071	0.4043	0.0157
200	Mean	0.0758	0.8199	1.3387	2.4278	2.1179	0.0688
	MSE	0.0067	0.7708	0.0981	1.1040	0.2965	0.0104
300	Mean	0.0754	0.7685	1.3056	2.4272	2.1511	0.0682
	MSE	0.0063	0.6082	0.1011	1.0050	0.2416	0.0102
400	Mean	0.0721	0.7357	1.2924	2.4835	2.2153	0.0642
	MSE	0.0035	0.6188	0.0955	0.8055	0.1898	0.0088
500	Mean	0.0787	0.6886	1.3046	2.5711	2.2915	0.0537
	MSE	0.0036	0.4413	0.0840	0.7443	0.1632	0.0052

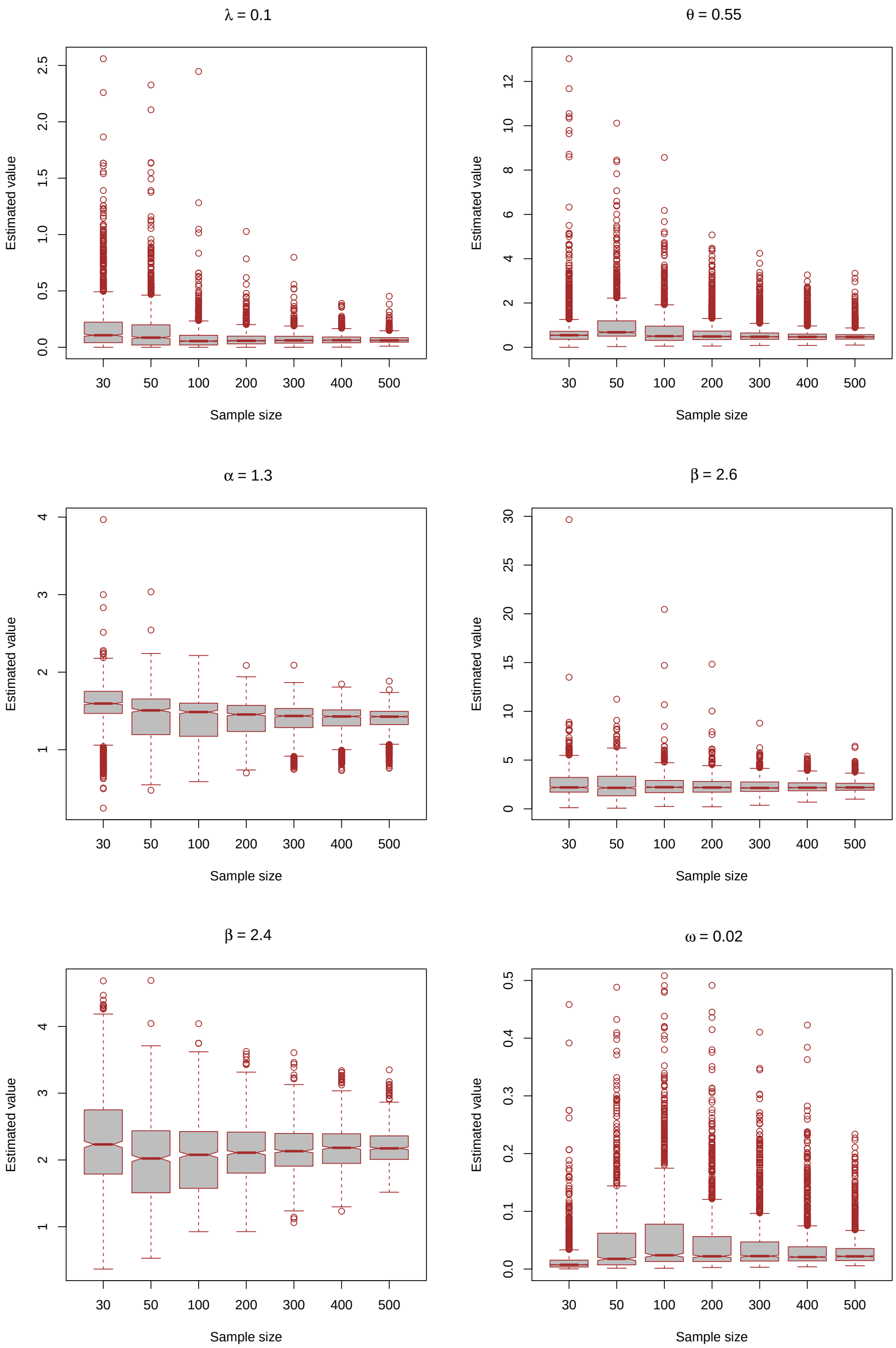

Figure 5 Box plots of the estimated values (1,000 times) of the Go-WFr parameters; (a) $\lambda = 0.10$, $\theta = 0.55$, $\alpha = 1.3$, $\beta = 2.6$, $\gamma = 2.4$, and $\omega = 0.02$

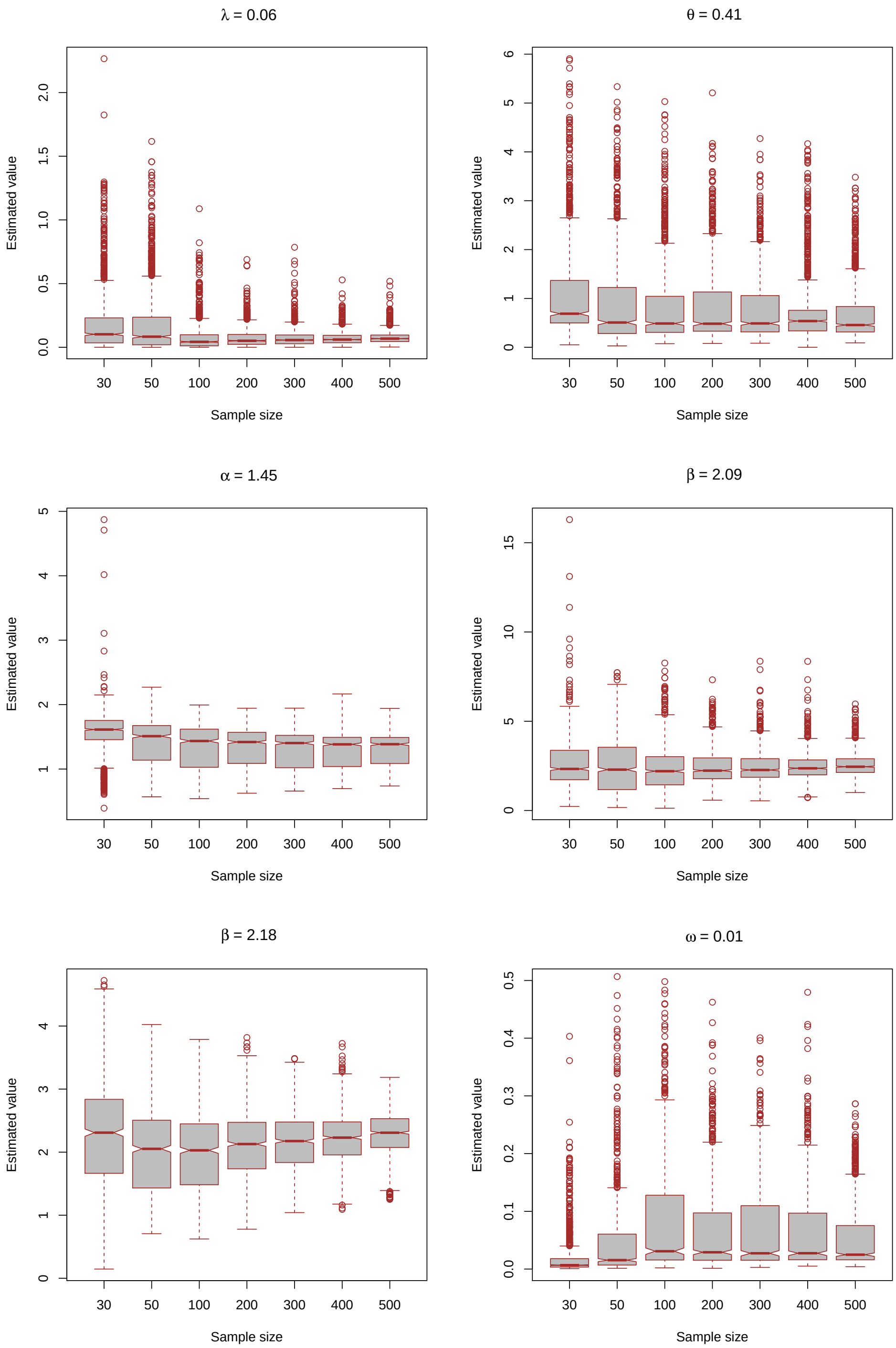

Figure 6 Box plots of the estimated values (1,000 times) of the Go-WFr parameters; (b) $\lambda = 0.06$, $\theta = 0.41$, $\alpha = 1.45$, $\beta = 2.09$, $\gamma = 2.18$, and $\omega = 0.01$

7. Applications

In this section, we present the flexibility of the proposed distribution by means of four real data sets. We compare the fits of its sub-models, such as the Go-EF, Go-WIE, Go-WIR, Go-EIE, and Go-EIR distributions. In addition, they are compared with the Go-FW distribution, which is a new recently Go-G distribution.

Data I: This data set consists of the waiting times (in seconds), between 65 successive eruptions of the Kiama Blowhole. These values were recorded with the aid of digital watch on July 12, 1998 by Jim Irish and recently has been referenced by Aryal et al. (2017). The data are as follows: 83, 51, 87, 60, 28, 95, 8, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27.

Data II: This dat set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975), see Shanker et al. (2015). The data are as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

Data III: This data set relates to the strength of carbon fibers tested under tension at gauge lengths of 10 mm. The data has been recently reported and analyzed by Bi and Gui (2017) among others. The observations are as follows: there are: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Data IV: The data consists of 100 observations of breaking stress of carbon fibres (inGba) given by Nichols and Padgett (2006). The data are as follows: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 4.42, 3.11, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.93, 3.22, 2.67, 2.38, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 1.57, 3.65, 3.56, 3.15, 2.35, 2.55, 2.59, 2.81, 2.77, 3.19, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 1.25, 3.68, 1.84, 1.59, 0.81, 5.56, 1.73, 1.59, 2.00, 2.82, 1.89, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 2.05, 3.51, 2.17, 1.69, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 2.88.

For the estimating the parameters of each distribution, we compare the Go-WFr model to the other models, such as the Go-EFr, Go-WIE, Go-WIR, Go-EIE, Go-EIR, and Go-FW distributions by using the minimum values of the criterion such as the AIC (Akaike information criterion) and the BIC (Bayesian Information Criterion). A good model is the one that has minimum AIC or BIC among the competed models. For goodness of fit tests, the Kolmogorov-Smirnov (K-S), Anderson-Darling (AD) and Cramer-von Mises (CVM) tests are used. The smallest values of these statistics give the best fit for the data. All analyses in this study were performed using R Software. The results are shown in Tables 4 and 5 respectively. The histogram of the data set and the estimated pdfs for the competing models, and the probability plot (P-P) of the best model are presented in Figures 7 to 10 .

The results of MLE of selected model parameters and some statistics of model fitting to these real data sets are shown in Tables 4 and 5 respectively. The Go-WFr distribution gives the smallest value of AIC, BIC, K-S, AD, and CVM among its sub-models and the Go-FWr model for all data sets, i.e., these results indicate that the Go-WFr distribution an appropriate to fit these data sets(see Figures 7 to 10). However, the third data set

From the application results, the Go-WFr distribution provides the best model when it compared among its sub-model and the Go-FW distribution. However, the Go-WFr distribution has maybe the efficiency that less than the other lifetime distributions. In this study, we show the illustrated to apply the Go-WFr distribution with some real data set and compares it among five sub-model and the recently Go-G distribution (Go-FW).

Table 4 The maximum likelihood estimators of selected model parameters for the real data sets

Data		Maximum likelihood estimators							
		$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\omega}$	$\hat{a}$	$\hat{b}$
I	Go-WFr	0.1493	0.0206	0.1813	0.1689	5.5020	0.0703	-	-
	Go-EFr	0.0637	0.0708	0.7486	1.2670	-	0.7556	-	-
	Go-WIE	0.0437	0.0652	-	1.0541	0.7116	0.9688	-	-
	Go-WIR	0.0433	0.0929	-	1.0127	0.3310	0.9904	-	-
	Go-EIE	1.0000	0.0100	-	0.0500	-	0.0012	-	-
	Go-EIR	2.0000	0.0100	-	1.0000	-	0.0002	-	-
	Go-FW	0.3002	0.0012	-	-	-	-	0.0209	0.1001
II	Go-WFr	0.5743	3.8527	4.2982	2.5422	0.0880	0.7277	-	-
	Go-EFr	0.2668	0.0010	1.8000	2.0000	-	6.1000	-	-
	Go-WIE	30.451	255.25	-	55.674	0.0402	0.0259	-	-
	Go-WIR	0.1191	0.0044	-	3.5646	0.4612	36.596	-	-
	Go-EIE	0.8000	0.0023	-	7.4000	-	57.556	-	-
	Go-EIR	1.0000	0.1000	-	0.0500	-	0.0006	-	-
	Go-FW	0.1677	0.0108	-	-	-	-	0.8619	0.5933
III	Go-WFr	1.8432	1.9934	4.0577	3.7965	0.3530	0.7076	-	-
	Go-EFr	10.000	0.1348	4.9589	0.9237	-	0.0002	-	-
	Go-WIE	9.9850	0.0443	-	0.7664	4.4826	0.0002	-	-
	Go-WIR	3.4054	0.0949	-	6.0563	0.6889	3.3104	-	-
	Go-EIE	8.9999	0.0008	-	0.2053	-	0.0077	-	-
	Go-EIR	9.0000	0.0009	-	0.5012	-	0.0029	-	-
	Go-FW	0.0872	0.0605	-	-	-	-	0.7239	0.8558
IV	Go-WFr	5.6057	0.0572	0.4072	0.2765	5.6135	0.0024	-	-
	Go-EFr	8.6502	5.9261	0.5180	39.538	-	4.0628	-	-
	Go-WIE	1.9995	0.3414	-	0.3676	2.3366	0.0041	-	-
	Go-WIR	14.092	0.7958	-	0.4571	1.3192	0.0005	-	-
	Go-EIE	2.0040	3.1430	-	3.2505	-	0.6026	-	-
	Go-EIR	0.5274	0.2373	-	0.7682	-	0.1093	-	-
	Go-FW	0.1546	0.0277	-	-	-	-	0.6227	0.4557

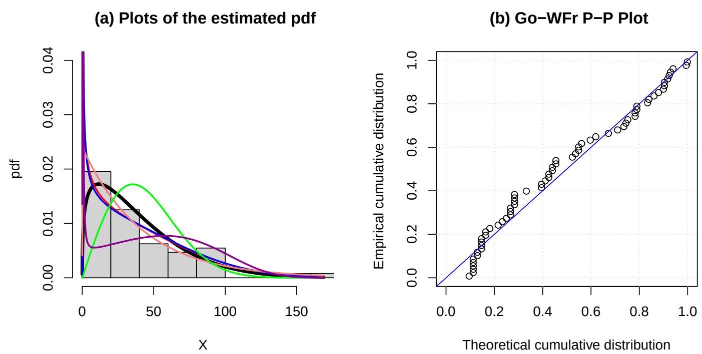

Figure 7 Plots of estimated pdf of the distributions, and the Go-WFr P-P plot of the real data sets

Table 5 Some statistics of model fitting to the real data sets.

Data	Distributions	$-\log L$	AIC	BIC	AD	CVM	K-S
I	Go-WFr	297.09	606.18	619.13	0.9212	0.1213	0.1073
	Go-EFr	300.37	610.74	621.53	1.3646	0.1581	0.1532
	Go-WIE	302.71	615.42	626.21	1.5175	0.1813	0.1612
	Go-WIR	302.89	615.78	626.57	1.6269	0.1983	0.1675
	Go-EIE	302.68	613.36	622.00	1.4343	0.1602	0.1587
	Go-EIR	311.69	631.38	640.02	9.2274	1.3377	0.2692
	Go-FW	333.76	675.52	684.16	5.4890	0.9233	0.2902
II	Go-WFr	15.13	42.26	48.23	0.1394	0.0224	0.0941
	Go-EFr	19.76	49.52	54.50	0.5531	0.0923	0.1498
	Go-WIE	19.86	49.72	54.70	0.9151	0.1479	0.1548
	Go-WIR	19.64	49.28	54.26	0.2410	0.0460	0.1317
	Go-EIE	19.28	46.56	50.54	0.2381	0.0431	0.1139
	Go-EIR	22.30	52.60	56.58	1.7489	0.3131	0.2550
	Go-FW	25.1	59.02	63.00	2.0876	0.3854	0.2393
III	Go-WFr	55.94	123.88	136.74	0.2540	0.0457	0.0711
	Go-EFr	62.12	134.24	144.96	1.0064	0.1030	0.1059
	Go-WIE	62.33	134.66	145.38	2.2074	0.4213	0.1424
	Go-WIR	58.38	126.76	137.48	0.3166	0.0563	0.0821
	Go-EIE	131.4	270.80	279.37	18.519	3.9199	0.4808
	Go-EIR	92.71	193.42	201.99	10.829	2.1481	0.3568
	Go-FW	78.04	164.08	172.65	4.6276	0.7689	0.2442
IV	Go-WFr	141.42	294.84	310.47	0.4257	0.0676	0.0642
	Go-EFr	143.98	297.96	310.99	0.5181	0.0715	0.0672
	Go-WIE	144.08	298.16	311.19	0.5298	0.0779	0.0673
	Go-WIR	144.69	299.38	312.41	0.5500	0.0759	0.0713
	Go-EIE	147.15	302.30	312.72	0.6645	0.1049	0.0734
	Go-EIR	147.95	303.90	314.32	0.9686	0.1480	0.0783
	Go-FW	156.56	321.12	331.54	3.9990	0.6141	0.1431

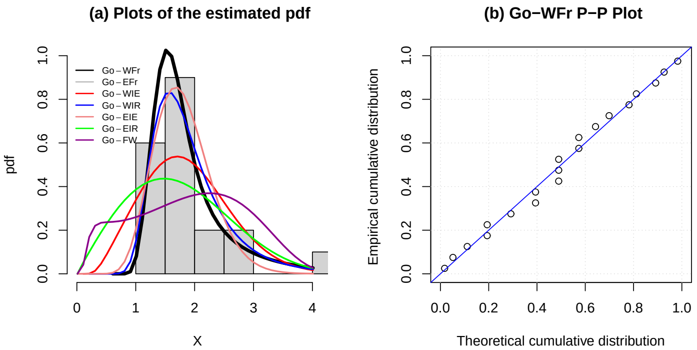

Figure 8 Plots of estimated pdf of the distributions, and the Go-WFr P-P plot of Data II

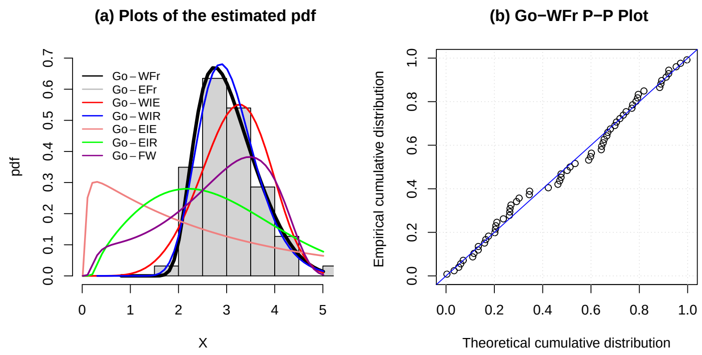

Figure 9 Plots of estimated pdf of the distributions, and the Go-WFr P-P plot of Data III

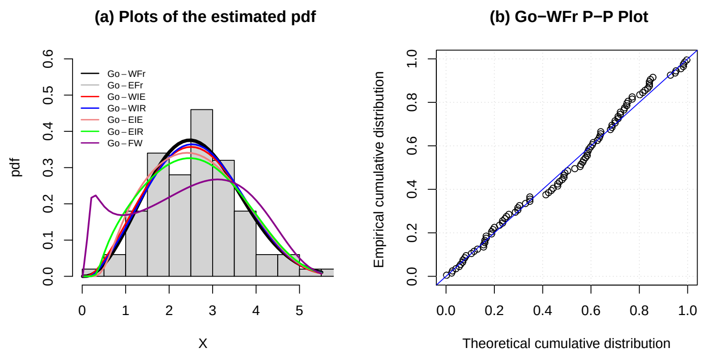

Figure 10 Plots of estimated pdf of the distributions, and the Go-WFr P-P plot of Data IV

8. Discussion and conclusions

The new distributions for modeling lifetimes data, the Gompertz-Weibull Fréchet (Go-WFr) distribution, is proposed. It has five sub-models, i.e., the Gompertz-exponential Fréchet, the Gompertz-Weibull inverse exponential, the Gompertz-Weibull inverse Rayleigh, the Gompertz-exponential inverse exponential and the Gompertz-exponential inverse Rayleigh. Some statistical properties of the proposed distribution are introduced including the reliability function, hazard function, Lorenz and Bonferroni curves, quantile function, moments, skewness, and kurtosis. We estimate the model parameters by maximum likelihood. We present a simulation study to illustrate the performance of the estimates. The simulation results indicate that the estimates are quite stable and, more important, are close to the true values for the these sample sizes. Furthermore, as the sample size increases, the MSE decreases as expected. The new distribution applied to four real data sets.

From the application results, the Go-WFr distribution provides the best model when it compared among its sub-model and the Go-FW distribution. However, in other situations (any real lifetime data), the Go-WFr distribution has maybe the efficiency that less than the other lifetime distributions. Thus, in practice, we will analyze the goodness of fit test of data by using the distribution that more than one model to compare the efficiency of the model or distribution. We hope that the proposed model will attract wider applications in areas such as engineering, survival and lifetime data, meteorology, hydrology, economics (income inequality) and others.

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