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Weighted Inverse Nakagami Distribution

Saurav Sarma* **and Kishore K Das**
Department of Statistics, Gauhati University, Guwahati, Assam, India
*Corresponding author; e-mail: sarmasaurav@gauhati.ac.in

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Abstract

In this paper, a weighted distribution has been suggested based on inverse Nakagami distribution. A detailed study of various statistical properties, as well as ordered statistics, have been discussed. The parameters of the suggested distribution have been estimated using maximum likelihood and method of moment estimation. Comparison of the performance of the estimators has been studied. Finally, the proposed distribution has been fitted to a couple of real-life data sets.

Keywords: Inverse Nakagami distribution, Nakagami distribution, weighted distribution.

1. Introduction

With the growing dependency of human civilization on telecommunication, developing and designing a robust system of communication and understanding the uncertainty related to network signal is always a field of interest for network engineers as well as to statisticians. Nakagami after a series of experiments provided a general formula to describe how the intensity of rapid fading is distributed, see Nakagami (1960). However, the application of Nakagami-m distribution is not only limited to describing multipath signal fading in a network system (Karagiannidis et al. 2007) it is also proved to be useful in the field of image processing (Tsui et al. 2006), hydrology engineering (Sarkar et al. 2010), reliability studies (Kumar et al. 2017) and various other fields of human life. The distribution is a special case to many standard distributions in the literature such as gamma, Rayleigh, Weibull, chi-square and exponential, see Huang (2016).

Corresponding to a non-negative random variable X, the probability density function (pdf) of Nakagami-m distribution (NKG) is given by

$$f(x) = \frac{2}{\Gamma(m)}\left(\frac{m}{w}\right)^m x^{2m-1}e^{-\frac{m}{w}x^2}, \tag{1}$$

where $m > 0.5$ is the shape parameter and $w > 0$ is the spread parameter.

Extending the application of a standard distribution which is already defined in the literature through various algebraic operations is a common practice. One such operation is taking the inverse of a distribution which provides better flexibility and wider applicability to the base distribution, see Johnson et al. (1994). For example, the inverse of a standard normal or a uniform distribution becomes bimodal, see Berk et al. (1996). Inverse chi-square (Bernardo and Smith 2008), inverse gamma (Abid and Al-Hassany 2016; Revfeim 1991), inverse Weibull (Almalki and Nadarajah 2014) and inverse Rayleigh (Malik and Ahmad 2018) are some other examples of an inverse distribution defined in the literature which are proved to be more flexible in terms of application than their parent distribution.

Louzada et al. (2018) defined the inverse Nakagami (INK) distribution taking the algebraic inverse of a Nakagami-m variate. The INK distribution proved to be a better fit in a system with high failure rate. Corresponding to a non-negative random variable $Y = 1/X$ where X is distributed as (1), the pdf of INK can be written as

$$f(y) = \frac{2}{\Gamma(m)} \left(\frac{m}{w}\right)^m y^{-2m-1} e^{-\frac{m}{wy^2}} , \tag{2}$$

which is a special case of inverse Rayleigh, inverse chi-square and inverse half-normal distribution, see Louzada et al. (2018).

When observations are collected from naturally occurring phenomena, it may not always follow the original distribution unless every observation is given an equal preference for being recorded. Rao (1965) introduced weighted distribution with an arbitrary non-negative weight function $\Omega(y)$ which may exceed unity to tackle such situation. If $f(y)$ is the pdf of a non-negative random variable Y and we consider a weight function $\Omega(y)$, then the weighted density function corresponding to $f(y)$ can be written as

$$g(y) = \frac{\Omega(y) f(y)}{E[\Omega(y)]} , \tag{3}$$

where $E[\Omega(y)] = \int \Omega(y) f(y) dy$ is the normalizing factor use to make the total probability unity. Rao in his paper have given practical examples where $\Omega(y) = y$ and $\Omega(y) = y^c$ are appropriate. A weighted distribution with $\Omega(y) = y^c$ is also known as size biased distribution of which length biased (when $\Omega(y) = y$) and area biased (when $\Omega(y) = y^2$) are special cases, see Patil and Rao (1978). The study of Weighted distribution can provide a better understanding of a standard distribution by expanding further flexibility to fit data better.

In this paper, a weighted distribution has been proposed based on inverse Nakagami distribution taking the weight function as x^a. The proposed distribution is named Weighted Inverse Nakagami (WINK) distribution. The various statistical properties of the proposed WINK distribution have been discussed such as r^{th} order raw moments, corresponding central moments, mean, variance, skewness, kurtosis, mode and Shannon entropy. Various reliability properties such as hazard rate, survival functions and reverse hazard rate function are also discussed in detail. The expression for density and distribution function of the k^{th} ordered statistics are also provided.

Parameter estimation of the proposed WINK distribution has been discussed using maximum likelihood and method of moment estimation. The maximum likelihood estimator of the spread parameter w is found out to be an unbiased estimator. However, the maximum likelihood estimator for the shape parameter m is found to be positively biased and hence a bias correction has been suggested. The performance of both maximum likelihood and method of moment estimator are compared based on simulated data. Finally,we fitted the proposed WINK distribution to two real-life data sets related to the failure time of equipment. Our proposed WINK distribution performed well in describing a system with a high failure rate within a short interval with a few extreme observations.

The paper is organized as follows. Section 2 introduces the properties of the proposed distribution. Section 3 discuss the estimation of the parameters of our suggested distribution. In Section 4 we have compared the performance of the estimators based on simulated data. Section 5 explores the applicability of our proposed distribution in fitting real life data sets. Section 6 summarizes the study.

2. Weighted Inverse Nakagami

Consider a random variable X which is distributed as (2) and a weight function $\Omega(x) = x^a$, then from (3) we have the pdf of WINK distribution as

$$g(x) = \frac{2}{\Gamma\left(m - \frac{a}{2}\right)} \left(\frac{m}{w}\right)^{m-\frac{a}{2}} x^{a-2m-1} e^{-\frac{m}{wx^2}} , \tag{4}$$

for all $x > 0, m > a/2$ and $w > 0$. Some important distributions can be obtained from (4) which are discussed in Section 2.8.

Putting $a = 1$ and 2 in (4) we get respectively the density functions for the length biased inverse Nakagami (LBINK) and area biased inversed Nakagami (ABINK) distributions

$$\begin{aligned} g_1(x) &= \frac{2}{\Gamma\left(m - \frac{1}{2}\right)} \left(\frac{m}{w}\right)^{m-\frac{1}{2}} x^{-2m} e^{-\frac{m}{wx^2}} , \\ g_2(x) &= \frac{2}{\Gamma(m-1)} \left(\frac{m}{w}\right)^{m-1} x^{1-2m} e^{-\frac{m}{wx^2}} . \end{aligned}$$

The corresponding cumulative distribution function (CDF) of WINK distribution with parameter m and w is obtained as

$$\begin{aligned} G_X(x) = P\left[x \leq X\right] &= \int_0^x g(x)dx \\ &= \frac{\Gamma\left(m - \frac{a}{2}, \frac{m}{wx^2}\right)}{\Gamma\left(m - \frac{a}{2}\right)}; x > 0 , \end{aligned} \tag{5}$$

where $\Gamma(m, x) = \int_x^\infty e^{-t} t^{m-1} dt$ is the upper incomplete gamma integral. (See Appendix A for the complete derivation)

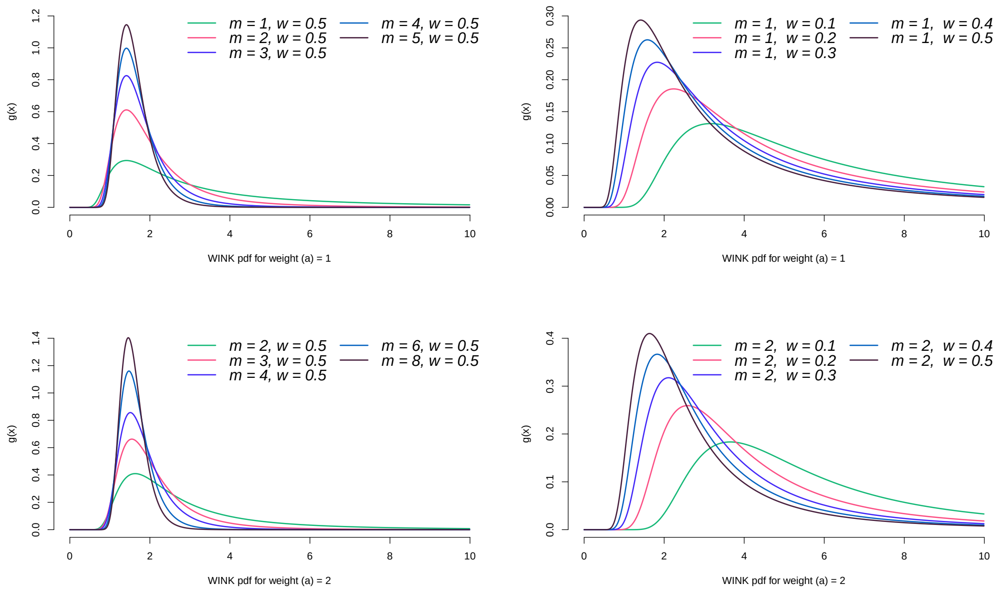

Figure 1 Density of the WINK distribution for weight a = 1 and 2

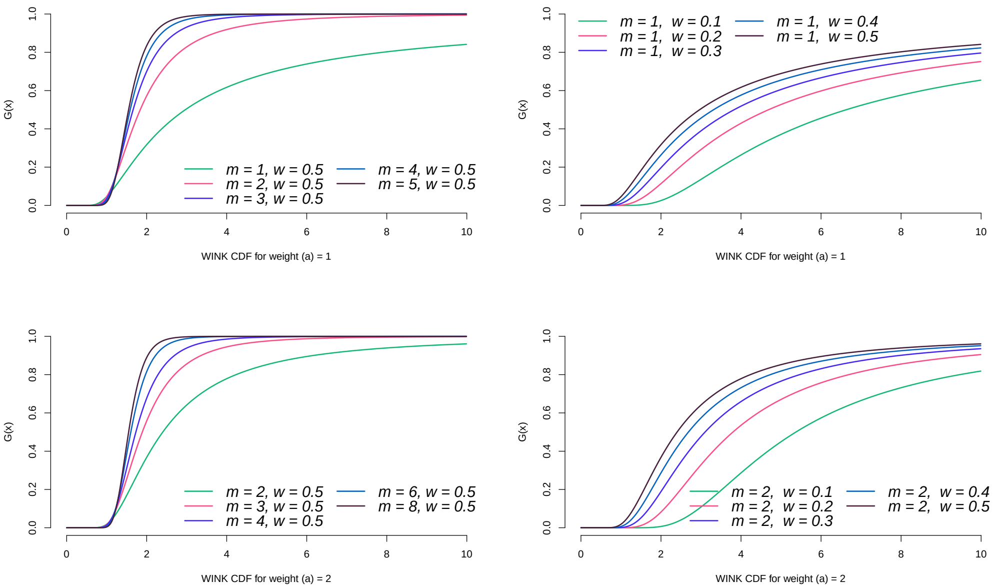

Figure 2 CDF of the WINK distribution for weight a = 1 and 2

2.1. Moments

The r^{th} order raw moment of WINK distribution is given by

$$\mu_r' = E\left[x^r\right] = \int_0^\infty x^r g(x)dx$$
$$= \left(\frac{m}{w}\right)^{\frac{r}{2}} \frac{\Gamma\left(m - \frac{a+r}{2}\right)}{\Gamma\left(m - \frac{a}{2}\right)}. \tag{6}$$

Replacing $r = 1, 2, 3$ and 4 in (6) we get the first four raw moments of WINK distributions as

$$Mean = \mu_1' = E[x] = \left(\frac{m}{w}\right)^{\frac{1}{2}} \frac{\kappa_{(1)}}{\kappa_{(0)}},$$

$$\mu_2' = E\left[x^2\right] = \left(\frac{m}{w}\right) \frac{\kappa_{(2)}}{\kappa_{(0)}}, \tag{7}$$

$$\mu_3' = E\left[x^3\right] = \left(\frac{m}{w}\right)^{\frac{3}{2}} \frac{\kappa_{(3)}}{\kappa_{(0)}},$$

$$\mu_4' = E\left[x^4\right] = \left(\frac{m}{w}\right)^2 \frac{\kappa_{(4)}}{\kappa_{(0)}}, \tag{8}$$

where, $\kappa_{(r)} = \Gamma\left(m - \frac{a+r}{2}\right); \quad r = 0, 1, 2, 3.$

Corresponding to (6), the central moments of WINK distribution are given by

$$\text{Variance} = \mu_2 = \left(\frac{m}{w}\right)\left[\frac{\kappa_{(2)}}{\kappa_{(0)}} - \frac{\kappa_{(1)}^2}{\kappa_{(0)}^2}\right],$$

$$\mu_3 = \left(\frac{m}{w}\right)^{\frac{3}{2}}\left[\frac{\kappa_{(3)}}{\kappa_{(0)}} - 3\frac{\kappa_{(1)}\kappa_{(2)}}{\kappa_{(0)}^2} + 2\frac{\kappa_{(1)}^3}{\kappa_{(0)}^3}\right],$$

$$\mu_4 = \left(\frac{m}{w}\right)^2\left[\frac{\kappa_{(4)}}{\kappa_{(0)}} - 4\frac{\kappa_{(3)}\kappa_{(1)}}{\kappa_{(0)}^2} + 6\frac{\kappa_{(2)}\kappa_{(1)}}{\kappa_{(0)}^3} - 3\frac{\kappa_{(1)}^4}{\kappa_{(0)}^4}\right].$$

2.2. Skewness and Kurtosis

The skewness and kurtosis of WINK distribution are respectively found to be

$$\beta_1 = \frac{\left[2\kappa_{(1)}^3 + \kappa_{(0)}^2\kappa_{(3)} - 3\kappa_{(0)}\kappa_{(1)}\kappa_{(2)}\right]^2}{\left[\kappa_{(0)}\kappa_{(2)} - \kappa_{(1)}^{(2)}\right]^3},$$

$$\beta_2 = \frac{\left[\kappa_{(0)}^3\kappa_{(4)} - 4\kappa_{(0)}^2\kappa_{(1)}\kappa_{(2)} + 6\kappa_{(0)}\kappa_{(1)}^2\kappa_{(2)} - 3\kappa_{(1)}^4\right]}{\left[\kappa_{(0)}\kappa_{(2)} - \kappa_{(1)}^2\right]^2}.$$

2.3. Generating functions

2.3.1 Moment generating function

To derive the moment generating function (MGF) of WINK distribution, Taylor's expansion has been used as follows

$$\begin{aligned} M_x(t) = E[e^{tx}] &= \int_0^\infty e^{tx} f(x)\,dx \\ &= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!}\right) f(x)\,dx \\ &= \int_0^\infty \sum_{r=0}^\infty \frac{t^r x^r}{r!} f(x)\,dx \\ &= \sum_{r=0}^\infty \frac{t^r}{r!}\int_0^\infty x^r f(x)dx \\ &= \sum_{r=0}^\infty \frac{t^r}{r!}\left(\frac{m}{w}\right)^{\frac{r}{2}} \frac{\Gamma\left(m - \frac{a+r}{2}\right)}{\Gamma\left(m - \frac{a}{2}\right)}. \end{aligned} \tag{9}$$

2.3.2 Characteristic function

Extending the concept of (9) we can derive the characteristic function of WINK distribution as

$$\Phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!}\left(\frac{m}{w}\right)^{\frac{r}{2}} \frac{\Gamma\left(m - \frac{a+r}{2}\right)}{\Gamma\left(m - \frac{a}{2}\right)}.$$

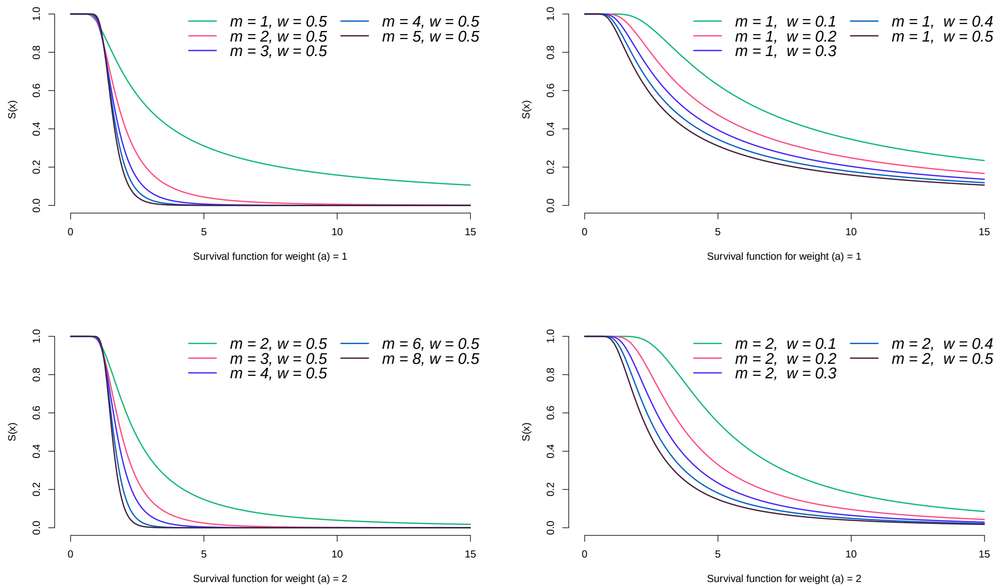

Figure 3 Survival function of the WINK distribution

2.4. Survival function

The survival function or reliability function gives the probability that an observation will not fail until x and is given by

$$S(x) = 1 - \frac{\Gamma\left(m - \frac{a}{2}, \frac{m}{wx^2}\right)}{\Gamma\left(m - \frac{a}{2}\right)} = \frac{\gamma\left(m - \frac{a}{2}, \frac{m}{wx^2}\right)}{\Gamma\left(m - \frac{a}{2}\right)},$$

where $\gamma(m,x) = \int_0^x e^{-t} t^{m-1} dt$ is the lower incomplete gamma integral.

2.5. Hazard function

The hazard function of WINK distribution is obtained as

$$h(x) = \frac{2}{\gamma\left(m - \frac{a}{2}, \frac{m}{wx^2}\right)} \left(\frac{m}{w}\right)^{m-\frac{a}{2}} x^{a-2m-1} e^{-\frac{m}{wx^2}},$$

and the corresponding reverse hazard rate function of WINK distribution is

$$\varphi(x) = \frac{2}{\Gamma\left(m - \frac{a}{2}, \frac{m}{wx^2}\right)} \left(\frac{m}{w}\right)^{m-\frac{a}{2}} x^{a-2m-1} e^{-\frac{m}{wx^2}}.$$

2.6. Mode

Consider the pdf of WINK in (4), taking logarithm we get

$$\log[g(x)] = \log(2) + \left(m - \frac{a}{2}\right) \{\log(m) - \log(w)\} + (a - 2m - 1)\log(x) - \left(\frac{m}{wx^2}\right). \quad (10)$$

Differentiating (10) w.r.t. x and equating to zero, we get the mode of WINK distribution as

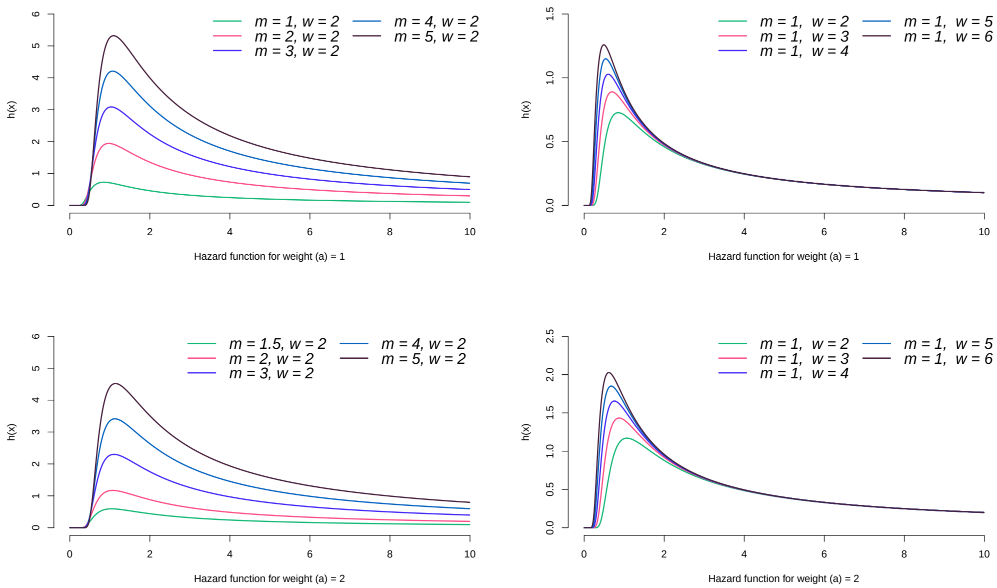

Figure 4 Hazard function of the Wink distribution

$$x_0 = \sqrt{\frac{2m}{(2m-a+1)w}}\,.$$

Thus, our proposed distribution is uni-modal.

2.7. Shannon's entropy

Entropy has an important role to play in the theory of information which gives a measure of uncertainty associated with a random variable. Shannon's entropy being the most popular among them. The Shannon's entropy of WINK distribution is obtained by solving the equation

$$\begin{aligned} H(x) &= -E[\log\{g(x)\}] \\ &= -\log C - (a-2m-1)E[\log(x)] + \left(\frac{m}{w}\right) E\left(\frac{1}{X^2}\right), \end{aligned} \tag{11}$$

where $C = \frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}}$.

Also,

$$\begin{aligned} E\left[\log(x)\right] &= \int_0^\infty \log(x) f(x) dx \\ &= \frac{1}{2}\log\left(\frac{m}{w}\right)\,. \end{aligned} \tag{12}$$

And,

$$E\left[\frac{1}{X^2}\right] = \left(\frac{w}{m}\right)\left(m-\frac{a}{2}\right)\,. \tag{13}$$

Finally, using (12) and (13) in (11) we get the Shannon's entropy as

$$H(x) = \left(m - \frac{a}{2}\right) + \log\left[\frac{\Gamma\left(m - \frac{a}{2}\right)}{2}\left(\frac{m}{w}\right)^{\frac{1}{2}}\right].$$

2.8. Related distributions

There are many standard distributions that are directly or indirectly related to our proposed WINK distribution. These relationship with different distributions are useful to generate random samples from our proposed WINK distribution and to compare efficiency of the distribution.

Theorem 1 *If X is distributed as WINK variate with parameters m, w and a=0, then $Y = \frac{1}{X}$ is distributed as Nakagami variate with scale parameter m and shape parameter w.*

Proof: Let, $Y = \frac{1}{X}$ then the Jacobian of transformation is $|J| = \frac{1}{Y^2}$. Putting $a = 0$ in (4) and using the substitution we get

$$f(y) = \frac{2}{\Gamma(m)}\left(\frac{m}{w}\right)^m y^{2m-1} e^{-\frac{m}{w}y^2},$$

which is Nakagami distribution with scale parameter m and shape parameter *w*.

Theorem 2 *If X is distributed as WINK variate with parameters m, w and a, then $Y = \frac{1}{X^2}$ is distributed as gamma variate with shape parameter $k = m - \dfrac{a}{2}$ and rate parameter $\theta = \dfrac{m}{w}$.*

Proof: Let $Y = \frac{1}{X^2}$ then $X = \frac{1}{\sqrt{Y}}$ and jacobian of transformation is $|J| = \frac{1}{2y^{3/2}}$. Using the substitution in (4) we get

$$\begin{aligned} f(y) &= \frac{1}{\Gamma\left(m - \frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} y^{m-\frac{a}{2}-1} e^{\frac{m}{w}y} \\ &= \frac{1}{\Gamma(k)}\theta^{k-1} e^{-\theta y}, \qquad \text{(where } k = m - \tfrac{a}{2} \text{ and } \theta = \tfrac{m}{w}) \end{aligned}$$

which is gamma distribution with shape parameters $k = m - \frac{a}{2}$ and rate parameter $\theta = \frac{m}{w}$.

Theorem 3 *If X is distributed as WINK variate with parameters m, w and a, then $Y = X^2$ is distributed as inverse gamma variate with shape parameter $k = m - \dfrac{a}{2}$ and scale parameter $\theta = \dfrac{m}{w}$.*

Proof: Let $Y = X^2$ then $X = \sqrt{Y}$ and Jacobian of transformation is $|J| = \frac{1}{2\sqrt{y}}$. Using the substitution in (4) we get

$$\begin{aligned} f(y) &= \frac{1}{\Gamma\left(m - \frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} y^{-m+\frac{a}{2}-1} e^{\frac{m}{wy}} \\ &= \frac{1}{\Gamma(k)}\theta^{-(k+1)} e^{-\theta/y}, \qquad \text{(where } k = m - \tfrac{a}{2} \text{ and } \theta = \tfrac{m}{w}) \end{aligned}$$

which is inverse gamma distribution with shape parameters $k = m - \frac{a}{2}$ and rate parameter $\theta = \frac{m}{w}$.

Theorem 4 *If X is a WINK variate with parameters $m = 2, w = 1$ and $a = 2$, then $Y = \dfrac{\lambda X^2}{2}$ is an inverse exponential variate with parameter λ.*

Proof: Replacing $m = 2, w = 1$ and $a = 2$ in (4) we get

$$g(x) = \frac{4}{x^3} e^{-2/x^2}.$$

Let $Y = \dfrac{\lambda X^2}{2}$ then $X = \sqrt{\dfrac{2y}{\lambda}}$ and Jacobian of transformation is $|J| = \dfrac{1}{\sqrt{2\lambda y}}$. Using the substitution in the above expression we get

$$f(y) = \frac{\lambda}{y^2} e^{-\frac{\lambda}{y}} \ ,$$

which is inverse exponential distribution with parameter λ.

Theorem 5 *If X is a WINK variate with parameters $m = p, w = 2p$ and a, then $Y = X^2$ is an inverse chi-square variate with $\eta = p - \dfrac{a}{2}$ degrees of freedom.*

Proof: Replacing $m = p, w = 2p$ in (4) we get

$$g(x) = \frac{2}{\Gamma\left(p - \frac{a}{2}\right)} \left(\frac{1}{2}\right)^{p-\frac{a}{2}} x^{a-2p-1} e^{-\frac{1}{2x^2}}.$$

Let $Y = X^2$ then $X = \sqrt{Y}$ and Jacobian of transformation is $|J| = \frac{1}{2\sqrt{y}}$. Using the substitution in the above expression we get

$$\begin{aligned} f(y) &= \frac{1}{2^{(p-\frac{a}{2})}\ \Gamma(p - \frac{a}{2})}\ y^{-(p-\frac{a}{2})-1} e^{-1/2y} \\ &= \frac{1}{2^{\eta}\ \Gamma(\eta)} y^{-\eta-1} e^{-1/2y} \ , \qquad \text{(where } \eta = p - \tfrac{a}{2}) \end{aligned}$$

which is inverse chi-square distribution with $\eta = p - \frac{a}{2}$ degrees of freedom.

Theorem 6 *If X is a WINK variate with parameters $m = 2, a = 2$ and w, then $Y = X$ is an inverse Weibull variate with shape parameters 2 and scale parameter $\lambda = \dfrac{2}{w}$.*

Proof: Replacing $m = 2$ and $a = 2$ in (4) we get

$$g(x) = \frac{4}{w\ x^3} e^{-\frac{2}{wx^2}}.$$

Assuming $Y = X$, we get

$$\begin{aligned} f(y) &= \frac{4}{w\ y^3} e^{-\frac{2}{wy^2}} \qquad (*) \\ &= \frac{2\lambda}{x^3} e^{-\frac{\lambda}{x^2}}, \qquad \text{(where } \lambda = \tfrac{2}{w}) \end{aligned}$$

which is inverse Weibull distribution with shape parameter 2 and scale parameter $\lambda = \frac{2}{w}$.

Theorem 7 If X is a WINK variate with parameters $m = 2, a = 2$ and w, then $Y = X$ is an inverse Rayleigh variate with scale parameter $\theta = \dfrac{2}{w}$.

Proof: Equation (*) can be rearranged as

$$f(y) = \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}} \ , \qquad \text{(where } \theta = \tfrac{2}{w})$$

which is inverse Rayleigh distribution with scale parameter $\theta = \frac{2}{w}$.

2.9. Order statistics

Let, $X_1, X_2, \ldots, X_n$ be a random sample from (4). Let $X_{(1)} \leq X_{(2)} \ldots \leq X_{(k)} \ldots \leq X_{(n)}$ be the corresponding ordered statistics. Then the probability density function and cumulative distribution function of the k^{th} order statistic $X_{(k)}$ (say) are given by

$$\begin{aligned} f_{(k)}(x) &= \frac{n!}{(k-1)!(n-k)!}\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k} f(x) \\ &= \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\left[F(x)\right]^{k+i-1} f(x). \end{aligned} \tag{14}$$

And

$$\begin{aligned} F_{(k)}(x) &= \sum_{j=k}^{n}\binom{n}{j}\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k} \\ &= \sum_{j=k}^{n}\sum_{i=0}^{n-k}(-1)^i\binom{n}{j}\binom{n-k}{i}\left[F(x)\right]^{i+j}. \end{aligned} \tag{15}$$

Respectively, for $k = 0, 1, 2, \ldots, n$. Corresponding to (4) and (5) and using (14) and (15), we get the density of k^{th} order statistic for WINK distribution as

$$\begin{aligned} f_{(k)}(x) = &\sum_{i=0}^{n-k}(-1)^i\frac{n!}{i!\,(k-1)!\,(n-k-i)!}\frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} x^{a-2m-1}e^{-\frac{m}{wx^2}} \\ &\left[1-e^{-\frac{m}{wx^2}}\sum_{j=0}^{\infty}\frac{\left(\frac{m}{wx^2}\right)^{m+j-\frac{a}{2}}}{\Gamma\left(m+j-\frac{a}{2}+1\right)}\right]^{k+i-1}. \end{aligned} \tag{16}$$

The corresponding distribution function is

$$F_{(k)}(x) = \sum_{j=k}^{n}\sum_{i=0}^{n-k}(-1)^i\binom{n}{j}\binom{n-k}{i}\left[1-e^{-\frac{m}{wx^2}}\sum_{l=0}^{\infty}\frac{\left(\frac{m}{wx^2}\right)^{m+l-\frac{a}{2}}}{\Gamma\left(m+l-\frac{a}{2}+1\right)}\right]^{i+j}. \tag{17}$$

3. Inferential Procedures

Our proposed distribution has three parameters, the scale parameter m, the spread parameter w and the weight parameter a. For simplicity, throughout the estimation of the parameters, we kept the weight parameter fixed at 1, 2 and 3. The shape parameter and spread parameter have been estimated using both method of moments (MME) and maximum likelihood estimation (MLE). Their performance are then compared based on a simulated dataset.

3.1. Maximum likelihood estimation

The maximum likelihood estimation procedure is preferred over all other classical method of statistical inference because of its asymptotic properties. Let, $X_1, X_2, \ldots, X_n$ be a random sample from WINK (m, w, a). In such a case, the likelihood function corresponding to (4) is given by

$$L(x; m, w, a) = \left[\frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\right]^n\left(\frac{m}{w}\right)^{\left(m-\frac{a}{2}\right)n}\prod_{i=1}^{n} x_i^{a-2m-1} e^{-\frac{m}{w}\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

The corresponding log likelihood function is

$$\begin{aligned}\log L(x;m,w,a) &= n\log(2) - n\log\Gamma\left(m-\frac{a}{2}\right) + n\left(m-\frac{a}{2}\right)\log\left(\frac{m}{w}\right) \\ &+ (a-2m-1)\sum\log x_i - \left(\frac{m}{w}\right)\sum\frac{1}{x_i^2}\,.\end{aligned} \tag{18}$$

Differentiating (18) with respect to m and w and equating to zero, we get the likelihood equations as

$$\frac{\partial}{\partial m}\log L = 0,$$

$$\log\left(m-\frac{a}{2}\right) - \psi\left(m-\frac{a}{2}\right) = \log\left[\frac{1}{n}\sum x_i^{-2}\right] - \frac{1}{n}\sum\log x_i^{-2}, \tag{19}$$

$$\frac{\partial}{\partial w}\log L = 0,$$

$$\hat{w} = \frac{2m}{n(2m-a)}\sum\frac{1}{x_i^2}\,, \tag{20}$$

where $\psi(r) = \frac{\partial}{\partial r}\log\Gamma(r) = \frac{\Gamma'(r)}{\Gamma(r)}$, is the digamma function.

Equation (20) requires knowledge of both m and a. We solve Equation (19) using numerical method for particular values of $a = 1, 2$ and 3. The value of m is then replaced in (20) to get an estimate of w. It can be shown that (20) gives an unbiased estimate of w.

Theorem 8 *If X is a continuous random variable distributed as (4) with parameters m, w and a, then the MLE of w given by (20) is an unbiased estimator of w.*

Proof: Taking expectation to both sides of (20) we get

$$\begin{aligned}E(\hat{w}) &= E\left[\frac{2m}{n(2m-a)}\sum_{i=0}^{n}\frac{1}{x_i^2}\right] \\ &= \frac{2m}{n(2m-a)}\sum_{i=0}^{n}E\left[\frac{1}{x_i^2}\right] \\ &= \frac{2m}{n(2m-a)}\sum_{i=0}^{n}E\left[x_i^{-2}\right].\end{aligned} \tag{21}$$

Now,

$$\begin{aligned}E\left[x_i^{-2}\right] &= \left(\frac{m}{w}\right)^{-1}\frac{\Gamma(m-\frac{a}{2}+1)}{\Gamma(m-\frac{a}{2})} \\ &= \left(\frac{w}{m}\right)\left(m-\frac{a}{2}\right).\end{aligned} \tag{22}$$

Using (22) in (21) and simplifying we get the desired outcome.

The MLE estimates are asymptotically normally distributed with a joint bivariate normal distribution $(\hat{m},\hat{w}) \sim N_2\left[(m,w), \frac{1}{I(m,w)}\right]$ for $n \to \infty$ where $I(m,w)$ is the Fisher information matrix given by

$$I(m,w) = n\begin{bmatrix}\psi_1\left(m-\frac{a}{2}\right) - \frac{2m+a}{2m^2} & \frac{a}{2mw} \\ \frac{a}{2mw} & \frac{2m-a}{2w^2}\end{bmatrix}, \tag{23}$$

where $\psi_1(r) = \frac{\partial}{\partial r}\psi(r)$ is the trigamma function.

3.2. Bias corrected MLE

Let $logL$ be the log Likelihood function corresponding to a random sample of size n from a distribution with p-dimensional parameter vector $\Theta = [\theta_1, \theta_2, \cdots, \theta_p]$. When the sample is independent but not identically distributed, then the bias in estimating the parameters is given by, see Cox and Snell (1968).

$$\text{Bias}\left(\hat{\theta}_s\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} t^{si}t^{jk}\left[t_{ij,k} + 0.5t_{ijk}\right] + O\left(n^{-2}\right); \quad s = 1, 2, \ldots, p \tag{24}$$

where

$$\begin{aligned}
t_{ij} &= E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log L\right], \\
t_{ijk} &= E\left[\frac{\partial^3}{\partial\theta_i\partial\theta_j\partial\theta_k}\log L\right], \\
t_{ij,k} &= E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log L \cdot \frac{\partial}{\partial\theta_k}\log L\right], \\
t_{ij}^{k} &= \frac{\partial}{\partial\theta_k} t_{ij}; \quad i, j, k = 1, 2, \ldots, p,
\end{aligned}$$

where t_{ij} is the $(i,j)^{\text{th}}$ element of the Fisher Information matrix (23) and t^{ij} is the $(i,j)^{\text{th}}$ element of the variance-covariance matrix T.

In case the sample is not independent, than (24) can be rewritten as (Cordeiro and Klein 1994),

$$\text{Bias}\left(\hat{\theta}_s\right) = \sum_{i=1}^{p} t^{si}\sum_{j=1}^{p}\sum_{k=1}^{p} t^{jk}\left[t_{ij,k} - 0.5t_{ijk}\right] + O\left(n^{-2}\right); \quad s = 1, 2, \ldots, p. \tag{25}$$

Let $a_{ij}^{(k)} = t_{ij}^{k} - 0.5t_{ijk}$ and $A = \left[A^{(1)} \left|A^{(2)}\right| \ldots |A^{(p)}\right]$
where $A^{(k)} = \left\{a_{\text{ij}}^{k}\right\}; \; i, j, k = 1, 2, \ \ldots, p.$

Then (25) can be rewritten as

$$\text{Bias}\left(\hat{\theta}\,\right) = T^{-1}\text{A vec}\left(T^{-1}\right).$$

And the bias corrected MLE is given by

$$\begin{aligned}
\theta^c &= \hat{\theta} - \text{Bias}\left(\hat{\theta}\,\right) \\
&= \hat{\theta} - T^{-1}\text{A vec}\left(T^{-1}\right).
\end{aligned}$$

Since the MLE of *w* is an unbiased estimate, therefore the correction is applied only on the MLE of *m.* After some simplification, the bias corrected MLE (CMLE) for shape parameter *m* is obtained as

$$m^c = \hat{m} - \frac{2\hat{m}\left(\hat{m}^3\psi_2\left(\hat{m} - \frac{a}{2}\right) + \hat{m} + a\right)}{n\left(2\hat{m}\psi_1\left(\hat{m} - \frac{a}{2}\right) - 2\hat{m} - a\right)^2}. \tag{26}$$

3.3. Method of moments

Using (7) and (8) and assuming the weight parameter $a = 1$, we have

$$\mu_2^{'} = \left(\frac{m}{w}\right)\frac{1}{\left(m-\frac{3}{2}\right)}, \tag{27}$$

$$\mu_4^{'} = \left(\frac{m}{w}\right)^2\frac{1}{\left(m-\frac{3}{2}\right)\left(m-\frac{5}{2}\right)}. \tag{28}$$

Solving (27) and (28) we get

$$\hat{m} = \frac{\mu_4^{'}}{\mu_4^{'}-\mu_2^{2}} + \frac{3}{2}, \tag{29}$$

and

$$\hat{w} = \frac{1}{\mu_2^{'}} + \frac{3}{2}\frac{\mu_4^{'}-\mu_2^{'2}}{\mu_2^{'}\mu_4^{'}}. \tag{30}$$

Substituting the population moments $\mu_2^{'}$ and $\mu_4^{'}$ by corresponding sample moments m_2 and m_4 in (29) and (30) we get the moment estimator of m and w for $a = 1$.

Similarly, for weight parameter $a = 2$ we have

$$\hat{m} = \frac{\mu_4^{'}}{\mu_4^{'}-\mu_2^{2}} + 2,$$

$$\hat{w} = \frac{1}{\mu_2^{'}} + 2\frac{\mu_4^{'}-\mu_2^{'2}}{\mu_2^{'}\mu_4^{'}}.$$

And for weight parameter $a = 3$ we have

$$\hat{m} = \frac{\mu_4^{'}}{\mu_4^{'}-\mu_2^{2}} + \frac{5}{2},$$

$$\hat{w} = \frac{1}{\mu_2^{'}} + \frac{5}{2}\frac{\mu_4^{'}-\mu_2^{'2}}{\mu_2^{'}\mu_4^{'}}.$$

In general, for any integer value of the weight parameter $a = r(r = 1, 2, 3, \ldots)$ we have

$$\hat{m} = \frac{\mu_4^{'}}{\mu_4^{'}-\mu_2^{2}} + \frac{r+2}{2}, \tag{31}$$

$$\hat{w} = \frac{1}{\mu_2^{'}} + \frac{(r+2)}{2}\frac{\mu_4^{'}-\mu_2^{'2}}{\mu_2^{'}\mu_4^{'}}. \tag{32}$$

Substituting the sample moments in (31) and (32) we get

$$\hat{m} = \frac{m_4}{m_4-m_2^{2}} + \frac{r+2}{2},$$

$$\hat{w} = \frac{1}{m_2} + \frac{(r+2)}{2}\frac{m_4-m_2^{2}}{m_2 m_4},$$

where $m_2 = \frac{1}{n}\sum x^2$ and $m_4 = \frac{1}{n}\sum x^4$.

4. Comparison of Estimators

In this section the performance of maximum likelihood estimator (MLE), corrected MLE (CMLE) for m and moment estimator (MME) have been compared based on randomly generated 10000 samples of size $n = 20, 50, 100$ and 1,000 for different values of the parameters.

The performance of the estimators have been studied by computing their mean relative error (MRE) and mean square error (MSE) as follows

$$MRE = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{\theta}_{r,i}}{\theta}; \;\; MSE = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_{r,j} - \theta\right)^2 ,$$

where $r = 1, 2$ and N is the number of iterations.

Figures 5 to 16 present the comparison of MLE and MME estimators for particular values of the parameters. Figures 17 and 18 present the MRE and MSE for MLE, CMLE and MME. MRE value close to 1 indicates a good estimation whereas in case of MSE, a value close to zero is desirable.

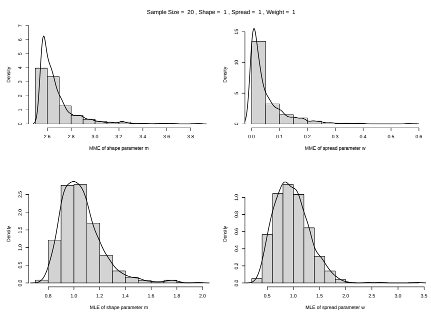

Figure 5 Comparison of MLE and MME for sample size 20 and weight = 1

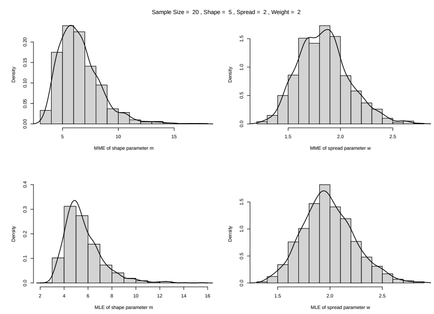

Figure 6 Comparison of MLE and MME for sample size 20 and weight = 2

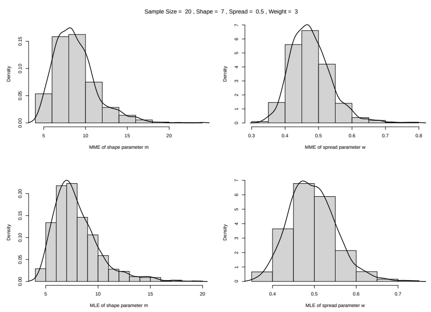

Figure 7 Comparison of MLE and MME for sample size 20 and weight = 3

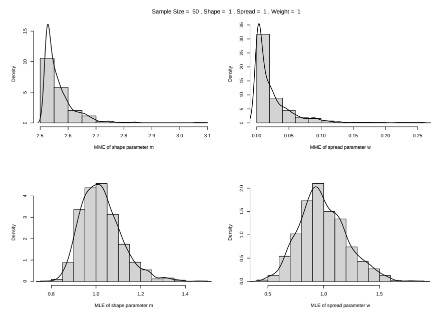

Figure 8 Comparison of MLE and MME for sample size 50 and weight = 1

Examining Figure 5 to 16 it can be inferred that the MLE performs consistently better than MME. For weight $a = 1$ (LBINK) MME is highly bias compared to weight $a = 2$ and 3. Further, MLE for the spread parameter w approaches normality much faster compare to MLE of shape parameter m.

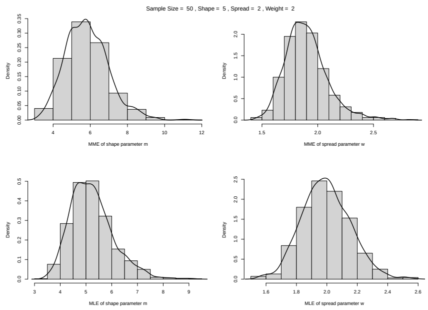

Figure 9 Comparison of MLE and MME for sample size 50 and weight = 2

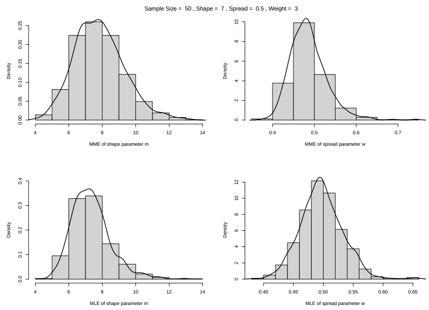

Figure 10 Comparison of MLE and MME for sample size 50 and weight =3

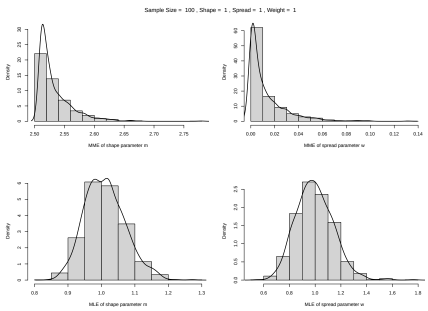

Figure 11 Comparison of MLE and MME for sample size 100 and weight = 1

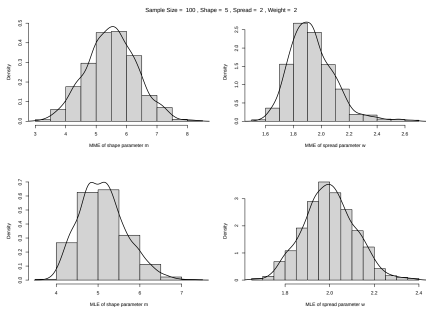

Figure 12 Comparison of MLE and MME for sample size 100 and weight = 2

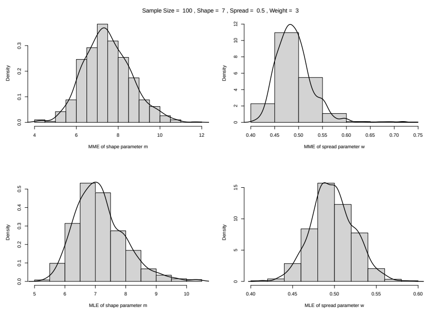

Figure 13 Comparison of MLE and MME for sample size 100 and weight = 3

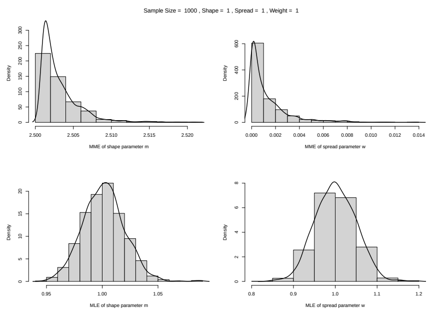

Figure 14 Comparison of MLE and MME for sample size 1000 and weight = 1

From Figures 17 and 18, we can infer that MLE of w is an unbiased estimator. Whereas, MLE of m is positively biased and approaches the true value for higher sample sizes. In all different combinations of parameter values, MLE is found out to be performing better than MME. However, except for LBINK, the MME of w is also found to be unbiased.

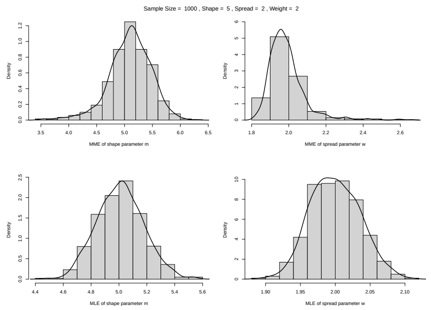

Figure 15 Comparison of MLE and MME for sample size 1000 and weight = 2

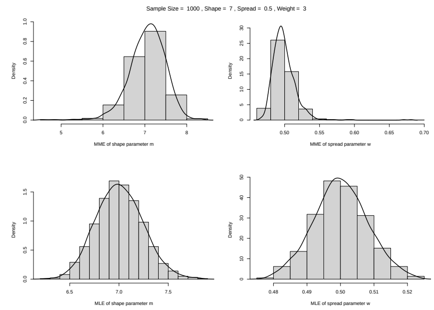

Figure 16 Comparison of MLE and MME for sample size 1000 and weight = 3

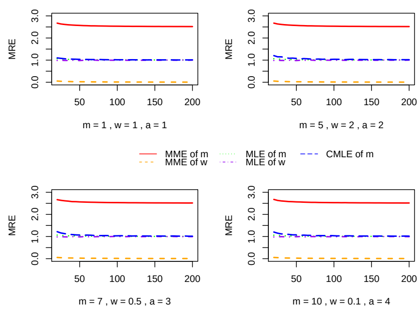

Figure 17 MRE related to the estimates of m and w for different values of a and n.

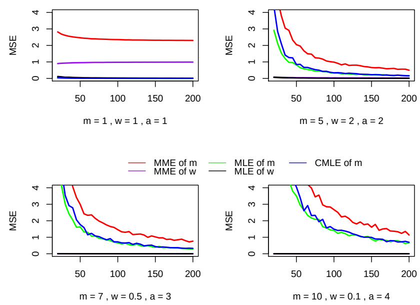

Figure 18 MSE related to the estimates of m and w for different values of a and n.

5. Applications

In this section the proposed distribution has been fitted to two real life data sets taken from Louzada et al. (2018). The first data set is related to failure of Harvester's elevator and the second data set is related to failure rate time of Harvester's motor.

The outcomes were compared with inverse Nakagami (INK), Nakagami (NKG), weighted Nakagami (WNKG), Rayleigh (RAY), weighted gamma (WGAM) and weighted Rayleigh (WRAY) distribution. To compare the performance of WINK distribution with these distributions, different goodness-of-fit test such as Kolmogorov-Smirnov statistic (KS), Cramer-von Mises statistic (CVM) and Anderson-Darling statistic (AD) as well as different discrimination criterion have been constructed under the log likelihood function. The discrimination criterion are

Akaike information criterion, $AIC = -2l(\hat{\theta}, x) + 2k$
Bayesian information criterion, $BIC = -2l(\hat{\theta}, x) + log(n)k$
Corrected Akaike information criterion, $AICC = AIC + \dfrac{2k(k+1)}{n-k-1}$
Hannan-Quinn information criterion, $HQIC = -2l(\hat{\theta}, x) + 2k\ log\ (log(n))$
Consistent Akaike information criterion, $CAIC = AIC + k\ log(n) - k$,
where n = sample size, k = number of parameters to be estimated, $l(\hat{\theta}, x)$ = maximized log likelihood function.

Table 1 presents the first data set with high failure after a short interval related to Harvester's motor. Table 2 gives the estimated values of the parameters m and w for fixed values of weight parameter a, their standard deviation and 95% confidence intervals. Tables 3 and 4 respectively give the goodness of fit criteria and goodness of fit statistics under different probability distributions for the data set.

Similarly, Table 5 presents the second data set with high failure related to Harvester's elevator. Table 6 gives the estimated values of the parameters m and w for fixed values of weight parameter, their standard deviation and 95% confidence intervals. Tables 7 and 8 respectively give the goodness of fit criteria and goodness of fit statistics under different probability distributions for the data set.

Table 1 Harvester's motor data

1	1	1	1	1	2	3	4	5	5	9	11	13	18	24	33
1	1	1	1	1	2	3	4	5	7	9	11	16	18	29	41
1	1	1	1	1	2	3	4	5	8	11	12	17	19	32	41
1	1	1	1	2	2	4	5	5	8	11	12	17	22	33	121

Table 2 MLE, standard deviation and 95% confidence interval for m and w for $a = 1, 2, 3$

a	θ	MLE	SD	95% CI	
1	$m_{Corrected}$	0.8454	0.0478	0.7517	0.9392
	w	0.8209	0.1892	0.4499	1.1918
2	$m_{Corrected}$	1.3448	0.0478	1.2510	1.4385
	w	1.3094	0.3129	0.6961	1.9228
3	$m_{Corrected}$	1.8445	0.0478	1.7507	1.9382
	w	1.7983	0.4378	0.9402	2.6563

Table 3 Results of AIC, BIC, AICC, HQIC & CAIC for different probability distributions considering the Harvester's motor data

Test	**WINK**	INK	NKG	WNKG	RAY	WGAM	WRAY
AIC	**414.10**	421.92	448.38	450.38	614.51	432.99	448.38
BIC	**418.41**	426.23	452.70	456.86	616.67	439.46	452.70
AICC	**414.29**	422.11	448.58	450.78	614.57	433.39	448.58
HQIC	**415.80**	423.62	450.08	452.93	615.36	435.54	450.08
CAIC	**420.41**	428.23	454.70	459.86	617.67	442.46	454.70

Table 4 Goodness of fit statistics for different probability distribution considering Harvester's motor data. (Figures inside parentheses represent p-value)

Statistic	**WINK**	INK	NKG	WNKG	RAY	WGAM	WRAY
KS	**0.2011 (0.02)**	0.2328 (0.02)	0.1854 (<0.01)	0.1851 (0.02)	0.5195 (<0.01)	0.1500 (0.10)	0.1853 (0.02)
CVM	**0.4525 (0.05)**	0.9505 (0.04)	0.6774 (<0.01)	0.6762 (0.01)	6.5123 (<0.01)	0.3182 (0.10)	0.6777 (0.01)
AD	**3.1474 (0.02)**	5.3098 (0.02)	3.9200 (<0.01)	3.9155 (0.01)	65.0167 (<0.01)	2.1162 (0.07)	3.9220 (0.01)

Table 5 Harvester's elevator data

1	1	1	1	1	2	3	4	7	9	12	21	25	61
1	1	1	1	2	2	3	5	7	11	17	23	31	122
1	1	1	1	2	2	4	6	7	11	17	23	56	
1	1	1	1	2	3	4	7	9	11	17	24	61	

Table 6 MLE, standard deviation and 95% confidence interval for m and w for $a = 1, 2, 3$

a	θ	MLE	SD	95% CI	
1	$m_{Corrected}$	0.8471	0.0521	0.7449	0.9493
	w	0.8811	0.2209	0.4481	1.3142
2	$m_{Corrected}$	1.3575	0.0521	1.2553	1.4597
	w	1.4060	0.3658	0.6891	2.1230
3	$m_{Corrected}$	1.8563	0.0521	1.7541	1.9586
	w	1.9317	0.5121	0.9279	2.9355

Table 7 Results of AIC, BIC, AICC, HQIC & CAIC for different probability distributions considering the Harvester's elevator data

Test	**WINK**	INK	NKG	WNKG	RAY	WGAM	WRAY
AIC	**345.86**	352.46	384.90	386.90	558.21	373.31	384.90
BIC	**349.84**	354.45	388.88	392.87	560.20	379.28	388.88
AICC	**346.10**	352.54	385.14	387.38	558.29	373.79	385.14
HQIC	**347.39**	353.23	386.44	389.20	558.98	375.61	386.44
CAIC	**351.84**	355.45	390.88	395.87	561.20	382.28	390.88

Table 8 Goodness of fit statistics for different probability distribution considering Harvester's elevator data. (Figures inside parentheses represent p-value)

Statistic	**WINK**	INK	NKG	WNKG	RAY	WGAM	WRAY
KS	**0.2100 (0.03)**	0.2338 (0.02)	0.1966 (<0.01)	0.1968 (0.02)	0.5664 (<0.01)	0.1710 (0.03)	0.1970 (0.03)
CVM	**0.3233 (0.11)**	0.6116 (0.10)	0.7770 (<0.01)	0.7758 (0.01)	6.7331 (<0.01)	0.4244 (0.06)	0.7765 (0.01)
AD	**2.3741 (0.05)**	3.6324 (0.05)	4.2266 (<0.01)	4.2216 (0.01)	72.7564 (<0.01)	2.5943 (0.03)	4.2244 (0.01)

6. Conclusion

In this article, the weighted inverse Nakagami (WINK) distribution has been proposed as a generalization of inverse Nakagami distribution. The suggested distribution is positively skewed with a reverse bathtub shape hazard function. The distribution can be used to explore data with a high amount of recurrent observations within a small interval of duration with some observations far left from the mode of the distribution. The various statistical properties of the proposed distribution such as moments, mode, entropy and reliability properties have been discussed. Expressions for the density and distribution function of the k^{th} ordered statistic are also provided.

The parameters of the distribution have been estimated using both MLE and MME procedure and it was found from a simulation study that under different combination of parameter values, the maximum likelihood estimators performed better than moment estimators. Bias correction has been applied to the MLE of shape parameter m to reduce the positive bias of the estimator. Finally, the distribution has been fitted to two real-life data sets with a high failure rate. A system with high initial failure is very common in practice therefore the proposed distribution can be extended and applied in a large number of practical situations.

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Appendix A: CDF of WINK Distribution

The cumulative distribution function (CDF) corresponding to (4) can be obtained as

$$
\begin{aligned}
G_X(x) = P\left[x \le X\right] &= \int_0^x g(t)dt \\
&= \int_0^x \frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} t^{a-2m-1} e^{-\frac{m}{wt^2}}\, dt \\
&= \frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} \int_0^x t^{a-2m-1} e^{-\frac{m}{wt^2}}\, dt\ .
\end{aligned}
$$

Let, $\frac{m}{wt^2} = y$, thus the Jacobian of transformation is

$$
|J| = -\frac{\sqrt{m}}{2\sqrt{w}\, y^{3/2}}\ ,
$$

when $t \to 0$; $y \to \infty$ and when $t \to x$; $y \to \frac{m}{wx^2}$.
Thus we have

$$
\begin{aligned}
G_X(x) &= \frac{2}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} \int_{\infty}^{\frac{m}{wx^2}} \left(\sqrt{\frac{m}{w\,y}}\right)^{a-2m-1} e^{-y}\left[-\frac{\sqrt{m}}{2\sqrt{w}\,y^{3/2}}\right] dy \\
&= \frac{1}{\Gamma\left(m-\frac{a}{2}\right)}\left(\frac{m}{w}\right)^{m-\frac{a}{2}} \int_{\frac{m}{wx^2}}^{\infty} \left(\frac{m}{w}\right)^{\frac{a}{2}-m} e^{-y}\, y^{m-\frac{a}{2}-1} dy \\
&= \frac{1}{\Gamma\left(m-\frac{a}{2}\right)} \int_{\frac{m}{wx^2}}^{\infty} e^{-y}\, y^{m-\frac{a}{2}-1} dy \\
&= \frac{\Gamma\left(m-\frac{a}{2}, \frac{m}{wx^2}\right)}{\Gamma\left(m-\frac{a}{2}\right)}; x > 0,
\end{aligned}
$$

where, $\Gamma(m, x) = \int_x^\infty e^{-t} t^{m-1} dt$ is the upper incomplete gamma integral.

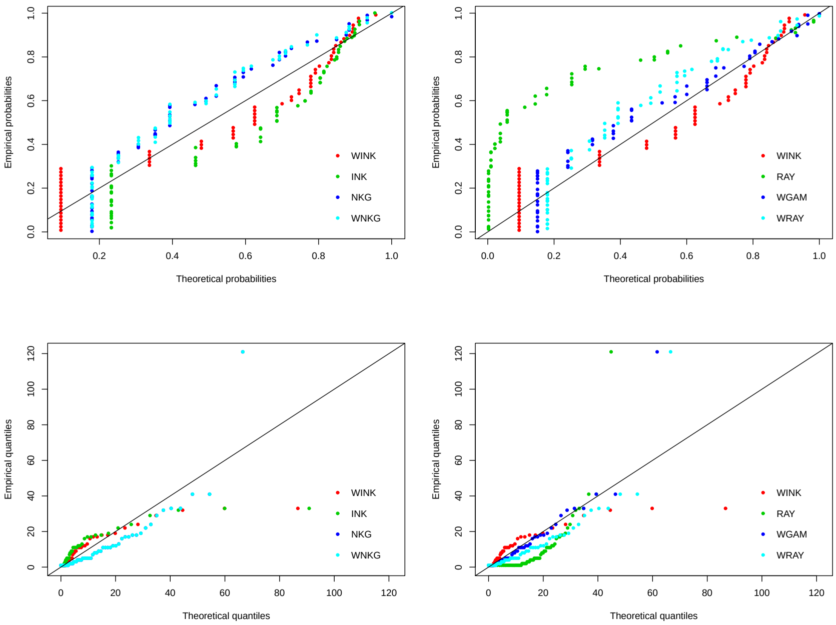


Figure 19 PP and QQ plots under different probability distributions for Harvester's motor data

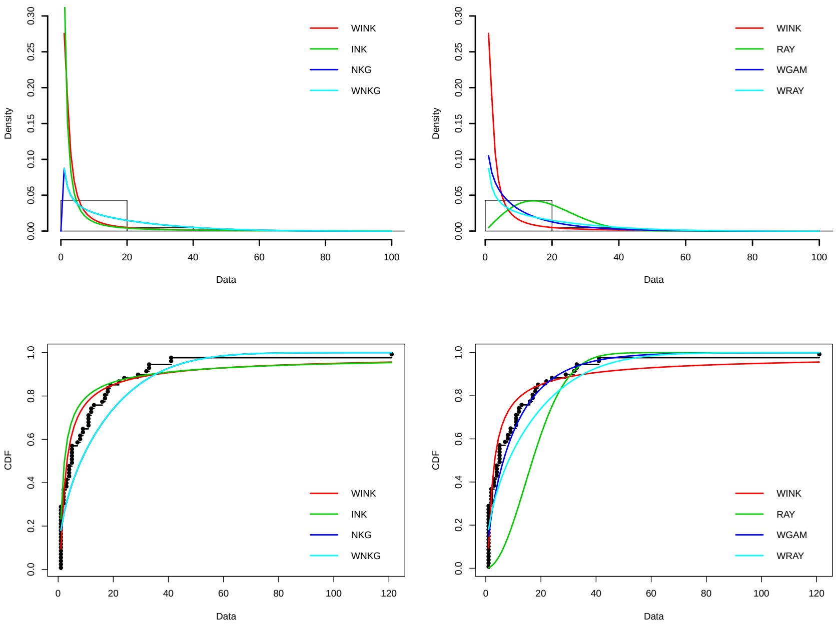


Figure 20 CDF and density plots under different probability distributions for Harvester's motor data

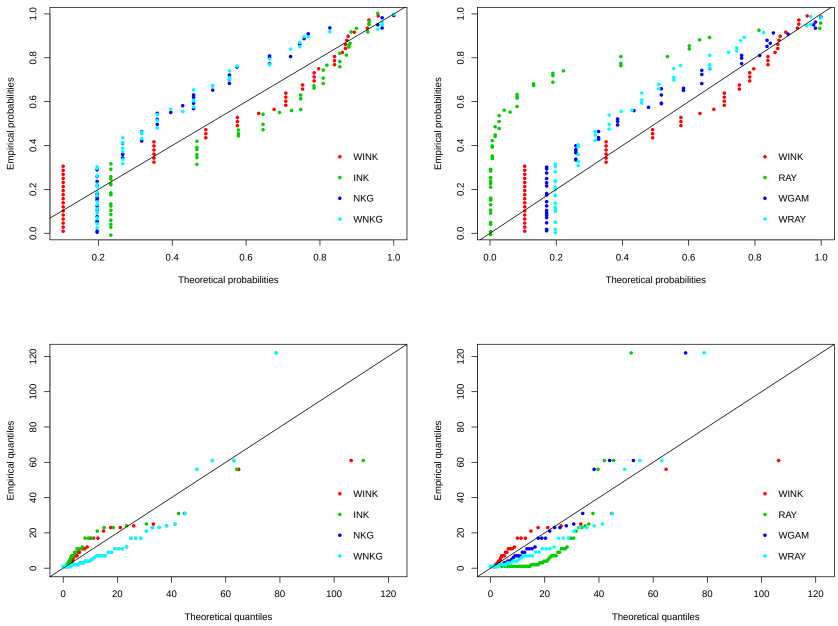


Figure 21 PP and QQ plots under different probability distributions for Harvester's elevator data

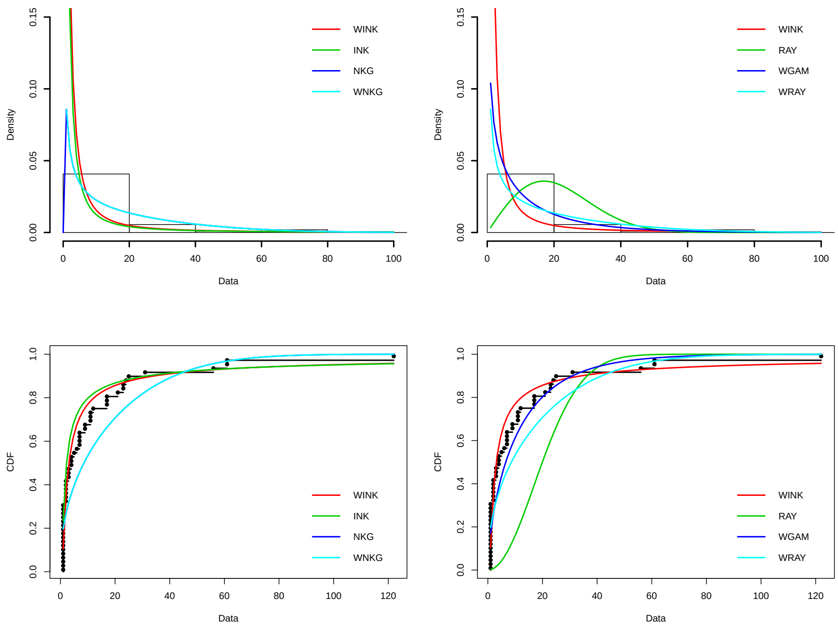
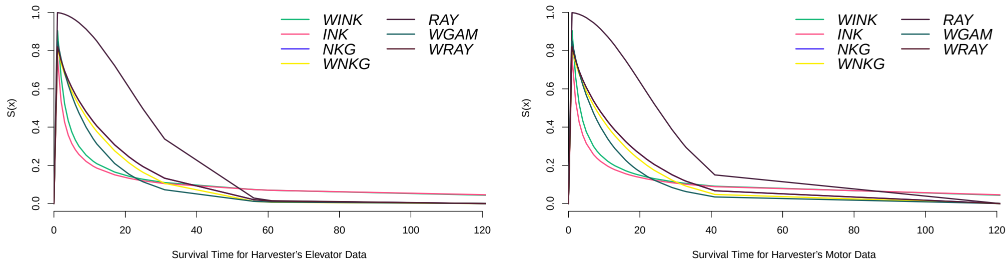


Figure 22 CDF and density plots under different probability distributions for Harvester's elevator data



(a) Harvester's elevator data

(b) Harvester's motor data

Figure 23 Survival function under different probability distributions