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New Generalized Rank Mapped Transmuted Exponential Distribution and Some Properties

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Abstract

In this paper, the authors have introduced a new generalized rank mapped transmuted distribution and the proposed distribution has been found to generalize some existing transmuted distributions in literature. The proposed map is applied to the exponential distribution so as to obtain the generalized rank mapped transmuted exponential distribution and its various properties such as nature of the probability curve, mean, variance, skewness, kurtosis with respect to variation in the parameters and the rank of the transmutation map is studied. The hazard rate function and distributional characteristics of the largest order statistics of the generalized rank mapped transmuted exponential distribution is also studied. The generalized rank mapped transmuted exponential distribution is found to model data with higher degree of skewness and kurtosis better than the traditional exponential distribution, which is also one of the utilities of the proposed distribution. If someone is interested to locate more flexible and higher degree of skewed distribution can explore this generalized transmuted distribution for future use.

Keywords: Transmuted map, order statistics, largest order statistics, exponential distribution, beta distribution, hazard function, family of distributions.

1. Introduction

The ideas of developing new distributions are important issues in recent literatures and increased attention over the last few years. Numerous families of distributions have been proposed by several authors for modelling data in several areas such as engineering, economics, finance and actuarial science, medical and life sciences. However, in many applied areas like life time analysis, insurance analysis need extended distributions, i.e., need new distributions which are more flexible to model real life data where the parent distribution does not provide a good fit, since the data can present a high degree of skewness and kurtosis.

Lee et al. (2013) provided an overview of most of the methods used to generate family of continuous distributions earlier in 1980. For more details about these methods, refer to Pearson and Henrici (1895), Johnson (1949) and Tukey (1960). In the last few years, there has been a growing a number of literatures were discussed generalized method to generate generalized family of distributions. Details about the recent developments one may refer to Johnson et al. (1994), Eugene et al. (2002), Jones (2009), Alzaatreh et al. (2013), Bourguignon et al. (2016), Afify et al. (2016), Al-Kadim and Mohammed (2017), Granzotto et al. (2017), Jayakumar and Babu (2017), Mahdavi and Kundu (2017), Alizadeh et al. (2017), Al-Kadim (2018), Pobocikova et al. (2018), Elgarhy et al. (2018), Afify et al. (2018) and the references therein. Apart from the above, a more extended generalized n^{th} degree transmuted method suggested in this study to generate transmuted distributions. An application of the generated transmuted map is extended to apply to the Exponential distribution. Distributional characteristics of the generated transmuted distributions are also simulated to compare with ordinary exponential distribution.

This paper is organized in the following way: In Section 2, n^{th} ranked map generalized transmutation based on continuous family of distribution has been developed. Particular cases are also discussed. Some other members of generalized transmutation map are identified. In Section 3, the survival function, hazard rate function and reserved hazard rate function of newly generated generalized transmuted distribution are discussed. In Section 4, developed n^{th} degree generalized transmutation map based on exponential distribution as well as graphs for the probability density functions (pdf) are also simulated and presented in figure to compare each other. In Section 5, some distributional characteristics such as mean, variance, skewness and kurtosis are simulated and presented in tabular form to compare each other for different parametric values. In Section 6, discussion and proofs of some theorems related to order statistics of generalized transmuted exponential distribution. Simulated distributional characteristics of largest order statistics of quadratic transmuted exponential distribution are presented. Section 7 states conclusions and then the references are inserted.

2. Generalized n^{th} Rank Transmutation Map

The construction of the generalized n^{th} rank transmutation map considered here is simple and intuitive. Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with pdf $g(x),\ x \in (a,b)$ corresponding to cumulative distribution function (cdf) $G(x),\ x \in (a,b).$ Further, let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ is the corresponding order statistics obtained by arranging the preceding random sample in increasing order of magnitude.

The cdf of $X_{r:n} (1 \le r \le n)$ is given by

$$G_{r:n}(x) = P(X_{r:n} \le x) = \sum_{i=r}^{n} \binom{n}{i} [G(x)]^i [1 - G(x)]^{n-i} = I_{G(x)}(r, n-r+1), \tag{1}$$

where $I_x(p,q)$ is the incomplete beta function. The corresponding pdf is given by

$$\begin{aligned} g_{r:n}(x) &= \frac{\partial}{\partial x}\left[G_{r:n}(x)\right] = \frac{n!}{r!(n-r)!}\left[G(x)\right]^{r-1}\left[1-G(x)\right]^{n-r} g(x) \\ &= g(x) b[G(x); r, n-r+1];\ -\infty < x < \infty,\ 1 \le r \le n, \end{aligned} \tag{2}$$

where $b(w;\alpha,\beta) = \dfrac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1};\ 0 \le t \le 1$ and $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$

Now, consider the random variable, Y is distributed as

$$Y \xrightarrow{d} g_{1:n}(x) \text{ with probability } p_{1:n}$$

$$
\begin{aligned}
&Y \xrightarrow{d} g_{2:n}(x) \text{ with probability } p_{2:n}\\
&\quad\vdots\\
&Y \xrightarrow{d} g_{r:n}(x) \text{ with probability } p_{r:n}\\
&\quad\vdots\\
&Y \xrightarrow{d} g_{n:n}(x) \text{ with probability } p_{n:n}
\end{aligned} \tag{3}
$$

where $0 \le p_{r:n} \le 1$ with $\sum_{r=1}^{n} p_{r:n} = 1$, $n = 2,3,...$ Hence, generalized transmuted cdf of n^{th} rank mapped $n = 2,3,...$ distribution is given by

$$F_Y(x) = \sum_{r=1}^{n} p_{r:n}\, I_{G(x)}(r, n-r+1) = \sum_{r=1}^{n} m_{r:n}\,, \tag{4}$$

where $m_{r:n} = p_{r:n}\, I_{G(x)}(r, n-r+1)$. The corresponding generalized transmuted pdf of n^{th} rank mapped distribution is given from (4)

$$f_Y(x) = \frac{\partial}{\partial x}\big[F_Y(x)\big] = \sum_{r=1}^{n} \frac{\partial}{\partial x} m_{r:n} = \sum_{r=1}^{n} p_{r:n} \frac{\partial}{\partial x}\Big[I_{G(x)}(r, n-r+1)\Big] = g(x)\sum_{r=1}^{n} p_{r:n}\, b\big[G(x); r, n-r+1\big], \tag{5}$$

where $b(.;.,.)$ is same as before. Now,

$$
\begin{aligned}
b(w;\alpha,\beta) &= \frac{1}{B(\alpha,\beta)} w^{\alpha-1}(1-w)^{\beta-1} = \frac{1}{B(\alpha,\beta)}(1-w)^{\beta-1}[1-(1-w)]^{\alpha-1},\\
&= \frac{1}{B(\alpha,\beta)} \sum_{i=1}^{\alpha-1} (-1)^{\alpha-1-i} \binom{\alpha-1}{i} w^{\alpha-1}(1-w)^{\alpha+\beta-2-i}.
\end{aligned} \tag{6}
$$

Using (6) in (5), we get

$$f_Y(x) = g(x)\sum_{r=1}^{n}\sum_{i=0}^{r} \frac{(-1)^{r-1-i}}{B(r, n-r+1)} p_{r:n} \binom{r-1}{i} [1-G(x)]^{n-1-i}. \tag{7}$$

Equations (3), (4) and (7) will also be helpful for simulation study for the cdf and pdf of transmuted distribution, respectively.

2.1. Specifications

Several specifications are available from our new transmuted distribution and some of them are illustrated herewith as a particular case of our generalized transmuted distribution.

i) Put $n = 2, p_{1:2} = \pi,\ p_{2:2} = 1-\pi$ and $2\pi = \lambda$ in (4) and (7) to get the cdf and pdf, respectively of quadratic transmutation map due to Shaw and Buckley (2007) as

$$F(x) = \lambda\, G(x) + (1-\lambda)\, G^2(x) \text{ and } f(x) = g(x)[G(x) + \lambda\{1-G(x)\}].$$

ii) Put $n = 3, p_{3:3} = 1 - p_{1:3} - p_{2:3}$ and $\lambda_1 = 3p_{1:3},\ \lambda_2 = 3p_{2:3}$ in (4) and (7) to get cdf of the cubic ranked transmutation map of Granzotto et al. (2017, Equation (3), p.2761)

$$F(x) = \lambda_1\, G(x) + (\lambda_2 - \lambda_1)\, G^2(x) + (1-\lambda_2)\, G^3(x),$$

and the corresponding pdf is

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)G^2(x)]\,.$$

iii) Put $p_{1:n} = 1$ and $p_{i:n} = 0,\ 2 \le i \le n$ in (4) and (7) to develop the simple transmutation map of Eugene et al. (2002) on using beta distribution as a generator. The cdf of simple transmutation map due to Eugene et al. (2002) is given as

$$F(x)=I_{G(x)}\left(\alpha,\beta\right),$$

and the corresponding pdf is

$$f(x)=g(x)b[G(x);\alpha,\beta].$$

iv) Put $n=3,\ \lambda=\lambda_1=3p_{1:3}=-\lambda_2,\ \lambda_2=3p_{2:3}$ in (4) and (7) to develop the cubic ranked transmutation map as given by Al-Kadim and Mohammed (2017). Thus, the cdf is given as

$$F(x)=(1+\lambda)G(x)-2\lambda G^2(x)-\lambda G^3(x),$$

and the corresponding pdf is

$$f(x)=g(x)[(1+\lambda)-4\lambda G(x)-3\lambda G^2(x)].$$

v) Put $n=4,\ \lambda_i=2p_{i:4};\ 1\le i\le 4$ in (3) and (7) to develop newly proposed distribution, called the generalized quartic ranking transmutation with cdf

$$F(x)=2\lambda_1G(x)+3(\lambda_2-\lambda_1)G^2(x)+2(\lambda_1-2\lambda_2+\lambda_3)G^3(x)+(1-\lambda_1+\lambda_2-2\lambda_3)G^4(x), \quad (8)$$

and the corresponding pdf is

$$f(x)=g(x)[2\lambda_1+6(\lambda_2-\lambda_1)G(x)+6(\lambda_1-2\lambda_2+\lambda_3)G^2(x)+4(1-\lambda_1+\lambda_2-2\lambda_3)G^3(x)]. \quad (9)$$

vi) Put $n=5,\ \lambda_i=5p_{i:5};\ 1\le i\le 5$ in (4) and (7) to develop the generalized quintic ranked transmutation map with new cdf as suggested is

$$F(x)=\lambda_1G(x)+2(\lambda_2-\lambda_1)G^2(x)+2(\lambda_1-2\lambda_2+\lambda_3)G^3(x)+(3\lambda_2-\lambda_1-3\lambda_3+\lambda_4)G^4(x)$$
$$+(1-\lambda_2-\lambda_3-\lambda_4)G^5(x) \quad (10)$$

and the corresponding generalized quintic ranked transmutation map which has a new pdf as suggested is

$$f(x)=g(x)[\lambda_1+4(\lambda_2-\lambda_1)G(x)+6(\lambda_1-2\lambda_2+\lambda_3)G^2(x)+4(3\lambda_2-\lambda_1-3\lambda_3+\lambda_4)G^3(x)$$
$$+5(1-\lambda_2-\lambda_3-\lambda_4)G^4(x)] \quad (11)$$

vii) Similarly, put $\lambda_i=np_{i:n};\ 1\le i\le n$ and $n=6,7,8,...$ in (4) and (7) to develop the cdf of suggested generalized desired n^{th} $(n\ge 6)$ ranked transmutation map as well as the corresponding pdf of generalized desired n^{th} ranked transmutation map.

viii) If one put, $p_{i:n}=\dfrac{1}{n};\ 1\le i\le n,$ then

$$F(x)=G(x).$$

ix) If we put, $p_{i:n}=\binom{n}{i}/\left(2^n-1\right),$ for all $1\le i\le n$ and $n=2,3,4,...,$ then from (4) and (7), another new generalized n^{th} ranked transmuted map of cdf for generating another new families of distributions are obtained as

$$F_{TDn}(x)=\sum_{r=1}^{n}p_{r:n}\,I_{G(x)}(r,n-r+1)=\sum_{r=1}^{n}p_{r:n}\,F_{r:n}(x)=\sum_{r=1}^{n}\sum_{i=r}^{n}p_{r:n}\binom{n}{r}[G(x)]^i[1-G(x)]^{n-i},$$
$$=\frac{1}{(2^n-1)}\sum_{r=1}^{n}\sum_{i=r}^{n}\binom{n}{r}\binom{n}{i}[G(x)]^i[1-G(x)]^{n-i}=\sum_{r=1}^{n}\sum_{i=r}^{n}m_{ri:n}(x), \quad (12)$$

where $m_{ri:n}(x) = \frac{1}{(2^n - 1)}\binom{n}{r}\binom{n}{i}[G(x)]^i[1-G(x)]^{n-i}$, and the corresponding generalized n^{th} ranked transmuted map of pdf for generating families of distribution are given by

$$f_{TDn}(x) = \frac{\partial}{\partial x}[F_Y(x)] = \sum_{r=1}^{n}\frac{\partial}{\partial x}[m_{r:n}(x)] = \sum_{r=1}^{n} p_{r:n}\frac{\partial}{\partial x}[I_{G(x)}(r, n-r+1)],$$

$$= g(x)\sum_{r=1}^{n} p_{r:n}\, b[G(x); r, n-r+1] = \sum_{r=1}^{n} p_{r:n}\, g_{r:n}(x) = \sum_{r=1}^{n} m_{r:n}\ , \tag{13}$$

where $m_{r:n} = p_{r:n}\, g_{r:n}(x)$ with $g_{r:n}(x) = g(x)b[G(x); r, n-r+1]$.

2.2. Some specific cases of (12) and (13)

1) Quadratic rank transmuted distribution (TD2)

For $n = 2$ in (12) and (13), the new form quadratic ranked transmuted cdf is given by

$$F_{TD2}(x) = \frac{1}{3}[4G(x) - G^2(x)] = \frac{1}{3}G(x)[3 + \{1 - G(x)\}], \tag{14}$$

and the corresponding new form quadratic ranked transmuted pdf is given by

$$f_{TD2}(x) = \frac{g(x)}{3}[4 - 2G(x)] = \frac{2g(x)}{3}[1 + \{1 - G(x)\}]. \tag{15}$$

2) Cubic rank transmuted distribution (TD3)

For $n = 3$ in (12) and (13), the new form cubic ranked transmuted cdf is given by

$$F_{TD3}(x) = \frac{1}{7}[7 - 3\{1 - G(x)\} - 6\{1 - G(x)\}^2 + 2\{1 - G(x)\}^3], \tag{16}$$

and the corresponding new form cubic ranked transmuted pdf is given by

$$f_{TD3}(x) = \frac{g(x)}{7}[3 + 12\{1 - G(x)\} - 6\{1 - G(x)\}^2]. \tag{17}$$

3) Quartic rank transmuted distribution (TD4)

For $n = 4$ in (12) and (13), the new form quartic ranked transmuted cdf is given by

$$F_{TD4}(x) = \frac{1}{15}[15 - 4\{1 - G(x)\} - 18\{1 - G(x)\}^2 + 4\{1 - G(x)\}^3] + 3\{1 - G(x)\}^4, \tag{18}$$

and the corresponding new form quartic ranked transmuted pdf is given by

$$f_{TD4}(x) = \frac{g(x)}{15}[19 + 36\{1 - G(x)\} - 12\{1 - G(x)\}^2 - 12\{1 - G(x)\}^3]. \tag{19}$$

4) Quintic rank transmuted distribution (TD5)

For $n = 5$ in (12) and (13), the new form of quintic ranked transmuted cdf is given by

$$F_{TD5}(x) = \frac{1}{31}[31 - 5\{1 - G(x)\} - 40\{1 - G(x)\}^2 - 10\{1 - G(x)\}^3] + 30\{1 - G(x)\}^4 - 6\{1 - G(x)\}^5, \tag{20}$$

and the corresponding new form of quintic ranked transmuted pdf is given by

$$f_{TD5}(x) = \frac{g(x)}{31}[36 + 80\{1 - G(x)\} + 30\{1 - G(x)\}^2 - 120\{1 - G(x)\}^3] + 30\{1 - G(x)\}^4. \tag{21}$$

In a similar manner, one can generate any desired higher order $(n \geq 6)$ rank transmuted map from (12) and (13).

3. Hazard Function

The survival function $S(x)$, hazard rate function $h(x)$ and reserved hazard rate function $r(x)$ of newly generated generalized transmuted cdf $F_{TDn}(x)$ (12) corresponding to pdf $f_{TDn}(x)$ (13) are respectively given by

$$S_{TDn}(x)=1-F_{TDn}(x), \tag{22}$$

$$h_{TDn}(x)=f_{TDn}(x)/S_{TDn}(x)=f_{TDn}(x)/\{1-F_{TDn}(x)\}, \tag{23}$$

$$r_{TDn}(x)=f_{TDn}(x)/F_{TDn}(x), \tag{24}$$

where $f_{TD_n}(x)$ and $F_{TD_n}(x)$ are as before.

Using (4) in (22) to get survival function for newly derived n^{th} ranked transmuted distribution as,

$$S_{TDn}(x)=1-\sum_{r=1}^{n}\sum_{i=r}^{n}m_{ri:n}=1-\tfrac{1}{(2^n-1)}\sum_{r=1}^{n}\sum_{i=r}^{n}\binom{n}{r}\binom{n}{i}[G(x)]^i[1-G(x)]^{n-i},$$

$$=1-\frac{1}{(2^n-1)}\sum_{r=1}^{n}\sum_{i=r}^{n}\sum_{j=0}^{i}(-1)^{i-j}\binom{n}{r}\binom{n}{i}\binom{i}{j}[1-G(x)]^{n-j}, \tag{25}$$

where $m_{ri:n}=\dfrac{1}{(2^n-1)}\dbinom{n}{r}\dbinom{n}{i}[G(x)]^i[1-G(x)]^{n-i}.$

Using (13) and (25) in (23) to get hazard rate function for newly derived n^{th} ranked transmuted distribution as,

$$h_{TDn}(x)=\left[\sum_{r=1}^{n}m_{r:n}(x)\right]\Big/\left[1-\sum_{r=1}^{n}\sum_{i=r}^{n}m_{ri:n}(x)\right]. \tag{26}$$

Using (3) and (7) in (24) to get reserved hazard rate function for newly derived n^{th} ranked transmuted distribution as,

$$r_{TDn}(x)=\sum_{r=1}^{n}\left[m_{r:n}(x)\Big/\sum_{i=r}^{n}m_{ri:n}(x)\right]. \tag{27}$$

For more about the hazard function one can refer to Zubair et al. (2018) and references therein.

Theorem 1 *The quadratic generalized transmuted hazard rate function is given by*

$$h_{TD2}(x)=\frac{2g(x)[1+\{1-G(x)]}{\{1-G(x)\}^2-3\{1-G(x)\}}.$$

Proof: From (23) and for $n=2$, we get the generalized transmuted hazard rate function as,

$$h_{TD2}(x)=f_{TD2}(x)/[1-F_{TD2}(x)]. \tag{28}$$

Now using (14) and (15) in (28) and on algebraic simplification gives the proof of the required theorem.

Simulated hazard function (ED, TED2, TED3, TED4 and TED5) for some specific parametric (θ) values are presented in Figures 1 to 4 to compare among themselves. It is observed from figures that the shape of hazard function changes with the change of transmutations degree, i.e., hazard rate goes too skewed as transmutation degree increases. The hazard function is one of the most important quantities to character life phenomenon. Compare with many other modified exponential distributions, the shape of the hazard function is easy to make a decision. It can be derived from (23) and it is flexible. As we know, it is very common for a bathtub-shaped hazard function of a system or component to have a long useful lifetime with low constant rate portion in the middle and sharp change in the initial and wear-out of phase, so a distribution which can fit this kind of hazard rate would be very useful in reliability studies.

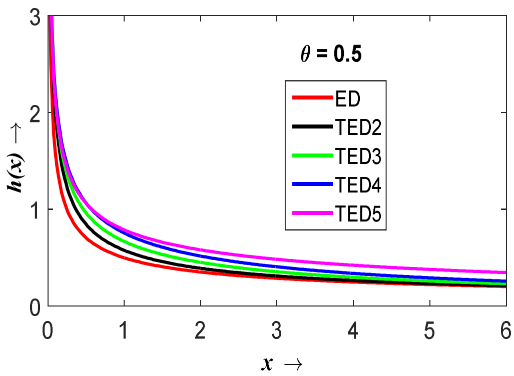

Figure 1 Hazard rate curve $h_{TED_n}\,(n=1,2,3,4,5)$ for $\theta=0.5$

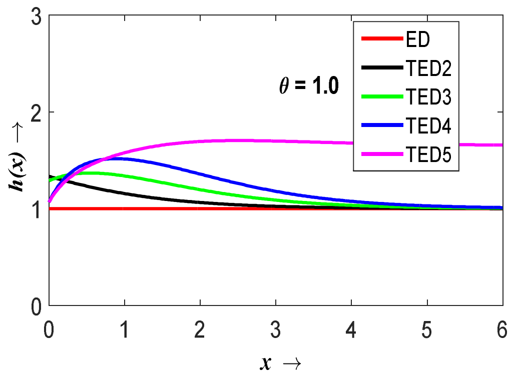

Figure 2 Hazard rate curve $h_{TED_n}\,(n=1,2,3,4,5)$ for $\theta=1.0$

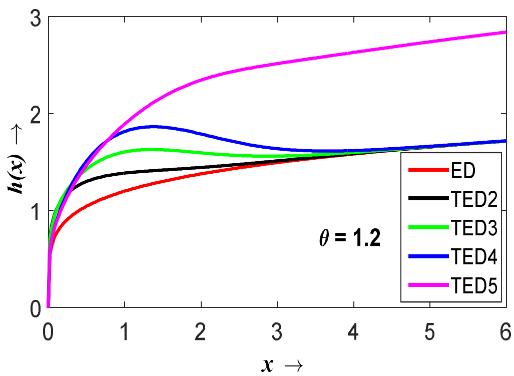

Figure 3 Hazard rate curve $h_{TED_n}\,(n=1,2,3,4,5)$ for $\theta=1.2$

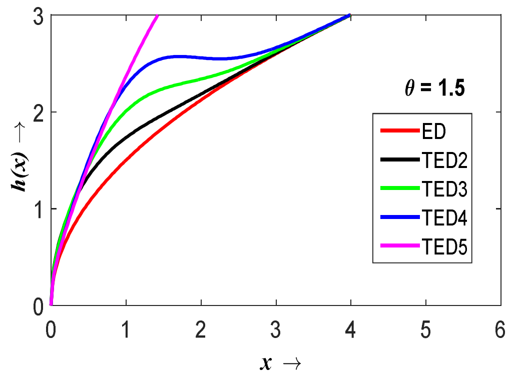

Figure 4 Hazard rate curve $h_{TED_n}\ (n=1,2,3,4,5)$ for $\theta=1.5$

4. Transmuted Exponential Distribution (TED)

There are many situations in which one would expect an exponential distribution to give a useful description of observed variation. One of the most widely quoted is that of events recurring "at random in time". In particular, suppose that the future lifetime of an individual has the same distribution, no matter how old it is at present. There are other situations in which exponential distributions appear to be the most natural. Many of these do, however, have as an essential feature the random recurrence (often in time) of an event.

The exponential distribution is proposed in a very important lifetime distribution and is widely used in many fields. However, the hazard function of the traditional exponential distribution can only be increasing, decreasing or constant. To meet the need of fitting complex modes and the bathtub-shaped hazard rate, researchers have proposed many improved flexible models based on the traditional exponential distribution. To know more about modified or improved models based on the traditional exponential distribution, one may refer to Johnson et al. (1994), Bebbington et al. (2007), Xie et al. (2002), Nassar et al. (2018), Afify et al. (2018) and references therein. Still the available modified exponential models are not enough to represent or fit the data obtained in all cases such as medical and life sciences, engineering, economics, finance and actuarial science. Our proposed transmuted model will be more flexible and will cover such limitation for which data present a good fit for a higher degree of skewness and kurtosis.

A random variable X is said to have traditional exponential distribution (ED) with parameter $\theta(>0)$, if its cdf is given by

$$G(x)=1-e^{-x/\theta};\ x\geq 0, \tag{29}$$

and pdf is given by

$$g(x)=\frac{1}{\theta}e^{-x/\theta};\ x\geq 0, \tag{30}$$

where θ is the scale parameter.

1) Quadratic map ranked transmuted exponential distribution (TED2)

Using (29) into (14), (29) and (30) into (15) to get the new quadratic ranked map transmuted exponential distribution cdf as

$$F_{TED2}(x)=\left[1-e^{-x/\theta}\right]\left[1+\tfrac{1}{3}e^{-x/\theta}\right], \tag{31}$$

and the corresponding new quadratic ranked map transmuted exponential distribution pdf is given by

$$f_{TED2}(x) = \frac{2}{3\theta}\left(1 + e^{-x/\theta}\right)e^{-x/\theta}. \quad (32)$$

Theorem 2 *The quadratic generalized transmuted exponential hazard rate function is given by*

$$h_{TED2}(x) = \frac{2}{3\theta}\left[e^{-x/\theta} + e^{-(x/\theta)^2}\right] / \left[1 - \frac{2}{3}e^{-x/\theta} + \frac{1}{3}e^{-(x/\theta)^2}\right].$$

Proof: From (28) and on using (31) and (32) gives the proof of the theorem.

2) Cubic map ranked transmuted exponential distribution (TED3)

Using (29) into (16), (29) and (30) into (17) to get the new cubic ranked map transmuted exponential distribution cdf as

$$F_{TED3}(x) = \frac{1}{7}\left[9 - 2\left(1 - e^{-x/\theta}\right)^2\right]\left(1 - e^{-x/\theta}\right), \quad (33)$$

and the corresponding new cubic ranked map transmuted exponential distribution pdf is given by

$$f_{TED3}(x) = \frac{1}{7\theta}\left[9 - 6\left(1 - e^{-x/\theta}\right)^2\right]e^{-x/\theta}. \quad (34)$$

3) Quartic map ranked transmuted exponential distribution (TED4)

Using (29) into (18), (29) and (30) into (19) to get the new quartic ranked map transmuted exponential distribution cdf as

$$F_{TED4}(x) = \frac{1}{15}\left[16 + 12\left(1 - e^{-x/\theta}\right) - 16\left(1 - e^{-x/\theta}\right)^2 + 3\left(1 - e^{-x/\theta}\right)^3\right]\left(1 - e^{-x/\theta}\right), \quad (35)$$

and the corresponding new quartic ranked map transmuted exponential distribution pdf is given by

$$f_{TED4}(x) = \frac{1}{15\theta}\left[16 + 24\left(1 - e^{-x/\theta}\right) - 48\left(1 - e^{-x/\theta}\right)^2 + 12\left(1 - e^{-x/\theta}\right)^3\right]e^{-x/\theta}. \quad (36)$$

4) Quintic ranked map transmuted exponential distribution (TED5)

Using (29) into (20), (29) and (30) into (21) to get new the quintic ranked map transmuted exponential distribution cdf as

$$F_{TED5}(x) = \frac{1}{31}\left[25 + 50\left(1 - e^{-x/\theta}\right) - 50\left(1 - e^{-x/\theta}\right)^2 + 6\left(1 - e^{-x/\theta}\right)^4\right]\left(1 - e^{-x/\theta}\right), \quad (37)$$

and the corresponding new quartic ranked map transmuted pdf exponential distribution is given by

$$f_{TED5}(x) = \frac{1}{15\theta}\left[25 + 100\left(1 - e^{-x/\theta}\right) - 150\left(1 - e^{-x/\theta}\right)^2 + 18\left(1 - e^{-x/\theta}\right)^4\right]e^{-x/\theta}. \quad (38)$$

In a similar way one can find any desired rank $(n \geq 6)$ map new transmuted cdf of exponential distribution by using (29) into (12); corresponding pdf by using (29) and (30) into (13).

Simulation pdf curves of different ranked map transmuted exponential distribution for some specific sets of parametric values (θ) were plotted in Figures 5 to 8 to observe and compare the change of skewness and the pdf curve shapes with the change of transmutation rank map.

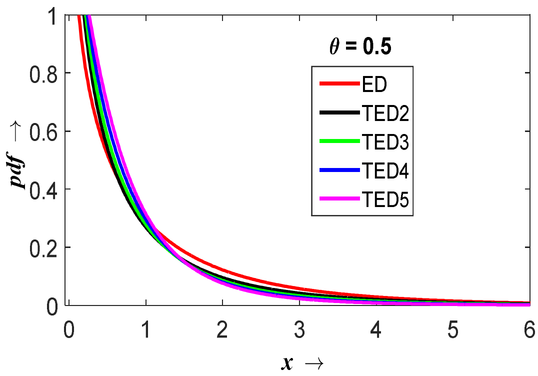

Figure 5 $TED_n (n = 1,2,3,4,5)$ curve for $\theta = 0.5$

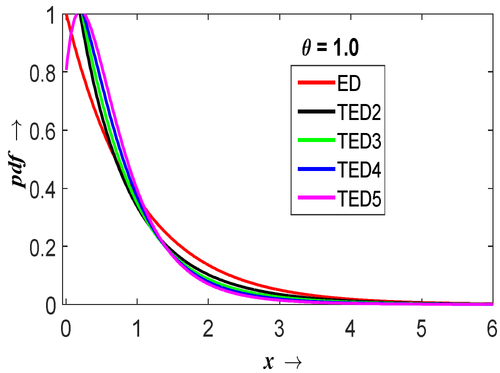

Figure 6 $TED_n (n = 1,2,3,4,5)$ curve for $\theta = 1.0$

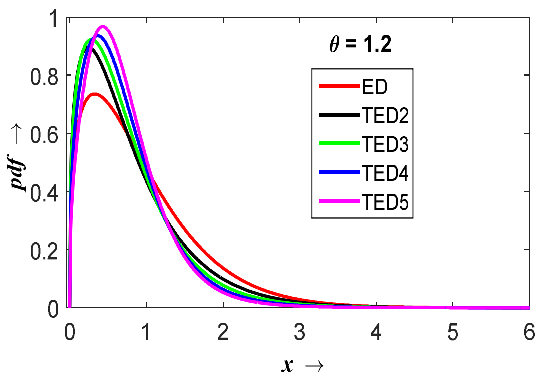

Figure 7 $TED_n (n = 1,2,3,4,5)$ curve for $\theta = 1.2$

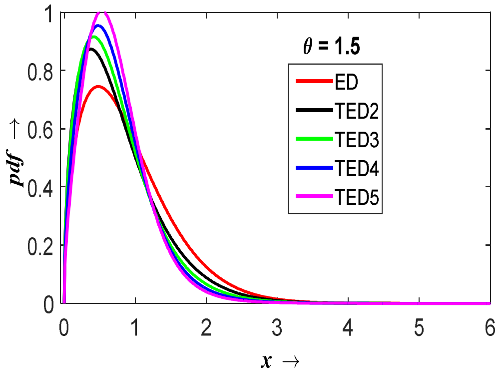

Figure 8 $TED_n(n=1,2,3,4,5)$ curve for $\theta = 1.5$

It is observed from the above figures (Figures 5-8) that the transmuted exponential distributions are more skewed compare to ordinary exponential distribution. The degree of skewness of $TWDn(n$ = 1, 2, 3, 4, 5) curves increases if the degree of rank of transmutation map increases. So, the newly generated transmuted exponential distributions have advantages to fit if the data sets are more skewed in property.

5. Distributional Characteristics of Transmuted Exponential Distribution

The k^{th} $(k = 1, 2, 3, ...)$ raw moment corresponding to generalized n^{th} ranked map transmuted of exponential pdf (30) are given from (13) as

$$\mu_k' = E\left(X^k\right) = \int_0^\infty x^k f(x)\,dx = \sum_{r=1}^{n} k_r, \tag{39}$$

where $k_i = \sum_{j=0}^{n-i} k_r,\ k_{ij} = \dfrac{(-1)^{n-i-j}}{(2^n-1)\,B(i,n-i+1)}\dbinom{n}{i}\dbinom{n-i}{j} M_{k,\,n-1-j}$ and $M_{k,m} = \int_0^\infty x^k g(x)\,G^m(x)\,dx.$

Now on using (29) and (30), we have

$$\begin{aligned} M_{k,m} &= \frac{1}{\theta}\int_0^\infty x^k e^{-x/\theta}\left[1-e^{-x/\theta}\right]^m dx = \frac{1}{\theta}\int_0^\infty x^k e^{-x/\theta}\sum_{i=0}^{m}(-1)^i\binom{m}{i}e^{-ix/\theta}\,dx, \\ &= \frac{1}{\theta}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\int_0^\infty x^k e^{-(i+1)x/\theta}\,dx = k!\theta^k\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{\theta^{k+1}}{(i+1)^{k+1}} = k!\,\theta^k\sum_{i=0}^{m}\delta_{im}, \end{aligned} \tag{40}$$

where $\delta_{im} = (-1)^i\dbinom{m}{i}\dfrac{\theta^{k+1}}{(i+1)^{k+1}}$ and we know $\int_0^\infty x^n e^{-(qx)^m}\,dx = \dfrac{\Gamma\left(\frac{n+1}{m}\right)}{m\,q^{n+1}};\ m,n>0.$ For more it refer to Jeffrey and Dai (2008, p.272).

5.1. Moments for quadratic transmuted exponential distribution (TED2)

For $n = 2$ in (39), the k^{th} $(k = 1,2,3,4,...)$ raw moment of the quadratic transmuted exponential distribution is given by

$$(\mu_k')_{TED2} = E(X^k) = \int_0^\infty x^k f(x)\,dx = \frac{\theta^k}{3}\left[2\Gamma(k+1) + \Gamma\left(\frac{k+1}{2}\right)\right], \quad (41)$$

and for $k = 1, 2, 3, 4,$ the first four moments of the quadratic transmuted exponential distribution are given by

$$(\mu_1')_{TED2} = \frac{\theta}{3}\left[2\Gamma(2) + \Gamma(1)\right] = \theta = \text{Mean},$$

$$(\mu_2')_{TED2} = \frac{\theta^2}{3}\left[2\Gamma(3) + \Gamma\left(\tfrac{3}{2}\right)\right] = \frac{\theta^2}{3}\left(8 + \sqrt{\pi}\right),$$

$$(\mu_3')_{TED2} = \frac{\theta^3}{3}\left[2\Gamma(4) + \Gamma(2)\right] = \frac{13}{3}\theta^3,$$

and $$(\mu_4')_{TED2} = \frac{\theta^4}{3}\left[2\Gamma(5) + \Gamma\left(\frac{5}{3}\right)\right] = \frac{\theta^4}{3}\left(48 + 0.75\sqrt{\pi}\right).$$

Central moments are given by

$$(Var(X))_{TED2} = (\mu_2)_{TED2} = \frac{\theta^2}{3}\left(8 + \sqrt{\pi}\right) - \theta^2 = \frac{\theta^2}{3}\left(5 + \sqrt{\pi}\right) = 2.257\,\theta^2,$$

$$(\mu_3)_{TED2} = (\mu_3')_{TED2} - 3(\mu_2')_{TED2}(\mu_1')_{TED2} + 2(\mu_1')^3_{TED2} = \theta^3(\sqrt{\pi} - 5/3) = 0.106\,\theta^3,$$

and $$(\mu_4)_{TED2} = (\mu_4')_{TED2} - 4(\mu_3')_{TED2}(\mu_1')_{TED2} + 6(\mu_2')_{TED2}(\mu_1')^2_{TED2} - 3(\mu_1')^4_{TED2}$$
$$= \frac{\theta^4}{3}(35 + 6.75\sqrt{\pi}) = 15.655\,\theta^4.$$

Pearson's four coefficients, based upon the first four central moments are

$$\text{Skewness} = (\beta_1)_{TED2} = \frac{\left[(\mu_3)_{TED2}\right]^2}{\left[(\mu_2)_{TED2}\right]^3} = 9.727 \times 10^{-4},\ (\gamma_1)_{TED2} = +\sqrt{(\beta_1)_{TED2}} = 3.119 \times 10^{-2},$$

and $$\text{Kurtosis} = (\beta_2)_{TED2} = \frac{(\mu_4)_{TED2}}{\left[(\mu_2)_{TED2}\right]^2} = 3.072,\ (\gamma_2)_{TED2} = (\beta_2)_{TED2} - 3 = 7.181 \times 10^{-2}.$$

It may be pointed out that these coefficients are true numbers independent of units of measurement. The p^{th} $(p \in (0,1))$ percentile point of quadratic transmuted exponential distribution (32) is given by

$$x_p = \theta\left[-\ln\{1 - 2(1 - \sqrt{1 - 3p/4})\}\right],$$

and random observation can be generated from the following inverse function

$$X = \theta\left[-\ln\{1 - 2(1 - \sqrt{1 - 3U/4})\}\right], \text{ where } U \sim U(0,1).$$

5.2. Moments for cubic transmuted exponential distribution (TED3)

For $n = 3$ in (39), the k^{th} $(k = 1, 2, 3, 4, ...)$ raw moment of the cubic transmuted exponential distribution TED3 is given by

$$(\mu_k')_{TED3} = E(X^k) = \frac{\theta^k}{7}\left[3\Gamma(k+1) + 6\Gamma\left(\frac{k+1}{2}\right) - 2\Gamma\left(\frac{k+1}{3}\right)\right], \quad (42)$$

$$(\mu_1')_{TED3} = \text{Mean} = \frac{\theta}{7}\left[3\Gamma(2) + 6\Gamma(1) - 2\Gamma\left(\frac{2}{3}\right)\right] = 0.899\theta,$$

$$(\mu_2')_{TED3} = \frac{\theta^2}{7}\left[3\Gamma(3)+6\Gamma\left(\frac{3}{2}\right)-2\Gamma(1)\right] = \frac{\theta^2}{7}[4+3\sqrt{\pi}] = 1.331\theta^2,$$

$$(\mu_3')_{TED3} = \frac{\theta^3}{7}\left[3\Gamma(4)+6\Gamma(2)-2\Gamma\left(\frac{4}{3}\right)\right] = 3.173\theta^3,$$

$$(\mu_4')_{TED3} = \frac{\theta^4}{7}\left[3\Gamma(5)+6\Gamma\left(\frac{5}{2}\right)-2\Gamma\left(\frac{5}{3}\right)\right] = 11.167\theta^4,$$

and

$$(Var(X))_{TED3} = (\mu_2)_{TED3} = \frac{\theta^2}{7}[3\Gamma(3)+6\Gamma(3/2)] - \frac{\theta^2}{49}[3\Gamma(2)+6\Gamma(1)-2\Gamma(2/3)]^2,$$

$$= \frac{\theta^2}{7}\left[4+3\sqrt{\pi}\right] - \frac{\theta^2}{49}[9-2\Gamma(2/3)]^2 = 0.523\theta^2.$$

$$\text{Skewness} = (\beta_1)_{TED3} = 4.270 \text{ and Kurtosis} = (\beta_2)_{TED3} = 6.303.$$

Other central moments and Pearson's four coefficients can be obtained from the above by simple algebraic manipulation.

5.3. Moments for quartic transmuted exponential distribution (TED4)

For $n = 4$ in (39), the k^{th} $(k = 1,2,3,4,...)$ raw moment of the TED4 is given by

$$(\mu_k')_{TED4} = E(X^k) = \frac{\theta^k}{15}\left[4\Gamma(k+1)+18\Gamma\left(\frac{k+1}{2}\right)-4\Gamma\left(\frac{k+1}{3}\right)-3\Gamma\left(\frac{k+1}{4}\right)\right]. \quad (43)$$

5.4. Moments for quintic transmuted exponential distribution (TED5)

For $n = 5$ in (39), the k^{th} $(k = 1,2,3,4,...)$ raw moment of the TED5 is given by

$$(\mu_k')_{TED5} = E(X^k) = \frac{\theta^k}{31}\left[5\Gamma(k+1)+40\Gamma\left(\frac{k+1}{2}\right)+10\Gamma\left(\frac{k+1}{3}\right)-30\Gamma\left(\frac{k+1}{4}\right)+6\Gamma\left(\frac{k+1}{5}\right)\right]. \quad (44)$$

For example, some distributional properties like mean, variance, skewness and kurtosis are simulated from above and presented below in Table 1. Some specific values of the parameter θ are given to observe and compare the differentiation of traditional exponential distribution (30) along with some other different degree of ranked map transmuted exponential distribution $(TED_n, n = 2,3,4,5)$, where n indicates n^{th} rank map as seen in (31)-(38). From Table 1, it is observed that skewness of transmuted distribution is more flexible as degree of rank of transmutation increases. So, one can use more flexible desired degree rank map transmuted distribution to fit desired skewed data set. For all simulation work is performed on MATLAB version R2018*a*.

6. Order Statistics

Order statistics (os) and functions of order statistics play an important role in statistical theory and methodology. Floods and droughts, longevity, breaking strength, aeronautics, oceanography, duration of humans, organisms, components and devices of various kinds can be studied by the theory extreme values. Life tests provide an ideal illustration of the advantage of order statistics in censored data. Since such an experiment may take a long time to complete, it is often advantageous to stop after failure of the first r out of n similar items under test. For more detail survey one may refer to Cohen (1963, 1966, 1991), Ali (1994), Balakrishnan and Aggarwala (2000), Athar and Akhter (2016), Akhter et al. (2019) and references therein.

Table 1 Distributional characteristics of transmuted exponential distribution $(TED_n, n = 2,3,4,5)$

Distributional Characteristics		Different combination of parameter θ values				
		$\theta = 0.2$	$\theta = 0.6$	$\theta = 1.0$	$\theta = 1.4$	$\theta = 1.8$
Means	ED	0.2000	0.600	1.000	1.400	1.800
	TED2	0.2000	0.600	1.000	1.400	1.800
	TED3	0.1798	0.539	0.899	1.258	1.618
	TED4	0.1502	0.451	0.751	1.052	1.352
	TED5	0.1205	0.362	0.603	0.844	1.084
Variances	ED	0.0400	0.360	1.000	1.960	3.240
	TED2	0.0251	0.226	0.629	1.232	2.037
	TED3	0.0209	0.188	0.523	1.025	1.695
	TED4	0.0208	0.188	0.521	1.021	1.688
	TED5	0.0131	0.118	0.327	0.641	1.059
Skewness	ED	0.0022	1.633	35.000	263.534	1,190.428
	TED2	0.0012	0.865	18.529	139.517	630.220
	TED3	0.0006	0.463	9.928	74.749	337.656
	TED4	0.0003	0.254	5.437	40.939	184.927
	TED5	0.0002	0.150	3.219	24.234	109.467
Kurtosis	ED	24.000	24.000	24.000	24.000	24.000
	TED2	41.595	41.595	41.595	41.595	41.595
	TED3	40.800	40.800	40.800	40.800	40.800
	TED4	27.907	27.907	27.907	27.907	27.907
	TED5	48.601	48.601	48.601	48.601	48.601

The pdf of r^{th} order statistic for the TED2 in (15) is given by

$$(f_{r:n})_{TD2} = B(r, n-r+1)[F_{TD2}(x)]^{r-1}[1-F_{TD2}(x)]^{n-r} f_{TD2}(x). \tag{45}$$

The pdf of extreme os follows from (45) at $r = 1$ and $r = n$, respectively,

$$(f_{1:n})_{TD2} = n[1-F_{TD2}(x)]^{n-1} f_{TD2}(x), \tag{46}$$

$$(f_{n:n})_{TD2} = n[F_{TD2}(x)]^{n-1} f_{TD2}(x). \tag{47}$$

Theorem 3 *For $n = 2,3,...,$ the recurrence relation between pdf of largest os of quadratic rank transmuted distribution as given in (47) and pdf of largest os of any arbitrary distribution is given by*

$$\left(f_{n:n}\right)_{TD2}(x) = n(4/3)^n \sum_{i=0}^{n-1}(-1/4)^i \left[\frac{g_{n+i:n+i}}{n+i} - \frac{g_{n+i+1:n+i+1}}{2(n+i+1)}\right], \tag{48}$$

where $g_{m:m} = mG^{m-1}(x)g(x)$. $G(x)$ and $g(x)$ are *cdf* and *pdf* of any continuous arbitrary distribution respectively.

Proof: From (47) and on using (14) and (15) we have

$$\left(f_{n:n}\right)_{TD2}(x)=\frac{n}{3}\left[4-2G(x)\right]\left[\frac{1}{3}\left\{4G(x)-G^2(x)\right\}\right]^{n-1}g(x),$$

$$=n\left(4/3\right)^n G^{n-1}(x)\left[1-\frac{1}{4}G(x)\right]^{n-1}\left[1-\frac{1}{2}G(x)\right]g(x).$$

Expanding the term $\left[1-\frac{1}{4}G(x)\right]^{n-1}$ binomially, we get

$$\left(f_{n:n}\right)_{TD2}(x)=n\left(4/3\right)^n\sum_{i=0}^{n-1}\left(-1/4\right)^i\binom{n-1}{i}\left[G^{n+i-1}(x)-\frac{1}{2}G^{n+i}(x)\right]g(x).$$

Now on using $g_{m:m}(x)=m\,G^{m-1}\,g(x),$ we get the required result.

Theorem 4 *For* $n=2,3,...,$ *the recurrence relation between k^th^ order moment of largest os for the pdf (47) of quadratic rank transmuted distribution and k^th^ order moment of largest os for the pdf of arbitrary continuous distribution is given by*

$$\left(\mu_{n:n}^{(k)}\right)_{TD2}(x)=n\,(4/3)^n\sum_{i=0}^{n-1}(-1/4)^i\binom{n-1}{i}\left[\frac{\mu_{n+i:n+i}^{(k)}}{n+i}-\frac{\mu_{n+i+1:n+i+1}^{(k)}}{2(n+i+1)}\right]. \tag{49}$$

Proof: Multiplying both sides of (48) by X^k and then take expectation to get the result of the theorem.

Theorem 5 *For* $n=2,3,...,$ *the recurrence relation between pdf of largest os of quadratic rank transmuted distribution as given in (47) and pdf of largest os of exponential distribution (30) is given by*

$$\left(f_{n:n}\right)_{TED2}(x)=\frac{n}{\theta}(4/3)^n\left[\sum_{i=0}^{n-1}\sum_{j=0}^{n+i-1}\frac{(-1)^{i+j}}{4^i}\binom{n-1}{i}\left\{\binom{n+i-1}{j}-\frac{1}{2}\binom{n+i}{j}\right\}e^{-(x/\theta)(j+1)}\right.$$
$$\left.-\frac{1}{2}\sum_{i=0}^{n-1}\frac{(-1)^{n+2i}}{4^i}\binom{n-1}{i}e^{-(x/\theta)(n+i+1)}\right]. \tag{50}$$

Proof: We have $g_{m:m}(x)=m\,G^{m-1}(x)\,g(x).$ Now using (29), (30) in the above expression, we get

$$g_{m:m}(x)=\frac{m}{\theta}e^{-x/\theta}\left[1-e^{-x/\theta}\right]^{m-1}=\frac{m}{\theta}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}e^{-(x/\theta)(j+1)}.$$

Now using above expression in (48) and on algebraic manipulation gives the proof of the required theorem.

Theorem 6 *For* $n=2,3,...,$ *the* k^{th} *order moment of largest os for the pdf (32) of quadratic rank transmuted exponential distribution is given by*

$$\left(\mu_{n:n}^{(k)}\right)_{TED2}=n\,k!\,\theta^k\left(\frac{4}{3}\right)^n\sum_{i=0}^{n-1}(-1/4)^i\binom{n-1}{i}\left[\sum_{j=0}^{n+i-1}\frac{(-1)^j}{(j+1)^{k+1}}\binom{n+i-1}{j}-\frac{1}{2}\sum_{j=0}^{n+i}\frac{(-1)^j}{(j+1)^{k+1}}\binom{n+i}{j}\right].$$

Proof: For exponential distribution defined in (29) and (30), we have

$$g_{m:m}(x)=m\,G^{m-1}(x)\,g(x)=\frac{m}{\theta}e^{-x/\theta}\left[1-e^{-x/\theta}\right]^{m-1}=\frac{m}{\theta}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}e^{-(j+1)x/\theta}. \quad (51)$$

Thus, the k^{th} order moment of largest os of exponential distribution (30) is given by

$$\mu_{m:m}^{(k)}=\int_0^\infty x^k g_{m:m}(x)\,dx=\frac{m}{\theta}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\int_0^\infty x^k e^{-(j+1)x/\theta}\,dx.$$

Since $\int_0^\infty x^n e^{-qx}\,dx=\dfrac{\Gamma(n+1)}{q^{n+1}}$. Therefore,

$$\mu_{m:m}^{(k)}=m\,k!\,\theta^k\sum_{j=0}^{m-1}\frac{(-1)^j}{(j+1)^{k+1}}\binom{m-1}{j}. \quad (52)$$

Now using (52) in (49) for $m=n+i$ and $m=n+i+1$, the k^{th} order moment of largest os for the pdf (32) of quadratic rank transmuted exponential distribution is obtained. This completes the proof.

Table 2 Simulated distributional characteristics of largest order statistic of TED2 (48) for different parametric values and sample sizes $n=2,4,\ldots,20$

	$\theta=0.2$				$\theta=0.5$				$\theta=1.0$			
n	Mean	Var	Skew	Kurt	Mean	Var	Skew	Kurt	Mean	Var	Skew	Kurt
2	0.085	0.096	3.975	4.547	0.213	0.602	3.975	4.547	0.427	2.408	3.975	4.547
4	0.148	0.236	1.295	1.655	0.371	1.476	1.295	1.655	0.741	5.904	1.295	1.655
6	0.167	0.384	0.850	1.062	0.418	2.402	0.850	1.062	0.835	9.610	0.850	1.062
8	0.172	0.552	0.669	0.797	0.430	3.447	0.669	0.797	0.860	13.788	0.669	0.797
10	0.171	0.738	0.571	0.648	0.427	4.620	0.573	0.648	0.854	18.439	0.573	0.648
12	0.166	0.941	0.516	0.556	0.414	5.882	0.516	0.556	0.828	23.526	0.516	0.556
14	0.158	1.161	0.481	0.495	0.394	7.254	0.481	0.495	0.789	29.016	0.481	0.495
16	0.148	1.395	0.458	0.452	0.369	8.720	0.458	0.452	0.739	34.879	0.458	0.452
18	0.136	1.644	0.445	0.423	0.340	10.273	0.445	0.423	0.680	41.093	0.445	0.423
20	0.123	1.906	0.437	0.401	0.308	11.909	0.437	0.401	0.615	47.637	0.437	0.401

Table 3 Simulated distributional characteristics of largest os of TED2 (48) for different parametric values and sample size $n=3,5,\ldots,19$.

	$\theta=0.2$				$\theta=0.5$				$\theta=1.0$			
n	Mean	Var	Skew	Kurt	Mean	Var	Skew	Kurt	Mean	Var	Skew	Kurt
3	0.128	0.152	2.678	3.062	0.321	0.952	2.678	3.062	0.642	3.807	2.678	3.062
5	0.190	0.263	1.198	1.504	0.475	1.642	1.198	1.504	0.950	6.569	1.198	1.504
7	0.253	0.391	0.515	0.800	0.634	2.444	0.515	0.800	1.267	9.775	0.515	0.800
9	0.316	0.532	0.193	0.457	0.789	3.325	0.193	0.457	1.578	13.299	0.193	0.457
11	0.377	0.684	0.050	0.286	0.941	4.275	0.050	0.286	1.882	17.099	0.050	0.286
13	0.436	0.846	0.002	0.206	1.090	5.288	0.002	0.206	2.180	21.150	0.002	0.206
15	0.494	1.017	0.010	0.179	1.236	6.359	0.010	0.179	2.472	25.434	0.010	0.179
17	0.552	1.197	0.053	0.183	1.379	7.484	0.053	0.183	2.758	29.935	0.053	0.183
19	0.608	1.386	0.116	0.207	1.520	8.660	0.116	0.207	3.040	34.640	0.116	0.207

7. Conclusions

In this paper, new generalized transmuted family of distributions (TDn) has been generated. Some generalized transmuted distributions available in literature are found as particular cases from our transmuted family of distributions. These new generalized transmuted families of distributions are applied to exponential distribution to find generalized rank map transmuted exponential distributions (TEDn). Simulated hazard function, pdf curves and some other distributional characteristics such as mean, variance, skewness and kurtosis for some specific parametric values of generalized transmuted families of exponential distribution are presented in Figures 1-4, Figures 5-8 and in Table 1, respectively to make a comparative study among changes of degree of rank maps. Also simulated quadratic ranked transmuted largest os distributional characteristics are studied and presented in Tables 2 and 3. These new distributions are more flexible and skewed compared to ordinary exponential distribution as well as degree of ranked transmuted distributions. Flexibility prominently increases as degree of rank of transmutation map increases. These are observed in pdf curves plotting (Figures 5-8) as well as in distributional characteristics presented in Table 1. It is observed that the transmuted distributions are more flexible to model real data, since the data can present a high degree of skewness and kurtosis. If someone is interested to locate more flexible and higher degree of skewed distribution, can explore this generalized transmuted family of distributions for future use.

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