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Posterior Analysis of Left Censored Weibull Distribution using Approximate Methods

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Abstract

This paper aims to suggest the various options for the Bayesian estimation of the parameters from the Weibull distribution under left censored samples. Five approximation methods, namely Quadrature method, Gibbs sampler, importance sampling, Lindley's approximation and Tierney and Kadane's approximation, have been used for this purpose. A couple of priors and loss functions have been assumed for the posterior estimation. As the analytical comparison among the different estimators is not possible, we have used the simulated and real life datasets for the numerical comparison among the proposed estimators. Based on these comparisons, it has been assessed that the performance of the importance sampling and Tierney and Kadane's approximation is better as compared to their counterparts with certain constraints on the parameters.

Keywords: Quadrature method, Gibbs sampler, importance sampling, Lindley's approximation, Tierney and Kadane's approximation, posterior distributions, loss functions.

1. Introduction

Due to flexibility in modeling life data of a very widespread multiplicity to multifaceted mechanisms, the Weibull distribution has achieved central place among the life distributions. It is more flexible than exponential and Rayleigh distributions. It is often perceived superior to gamma and lognormal distributions as its distribution function can be presented in compact form which provides extra convenience in modeling life data, especially in case of censored samples. This distribution has increasing/decreasing hazard rate owing to which it is suitable for modeling a range of life datasets. Lawless (2003) had described that the Weibull distribution can also be used to model the data regarding medical, biological and engineering sciences. Sinha and Sloan (1988) showed that for shape parameter greater than unity, the Weibull distribution describes the wear-out of the items. The detailed discussions regarding the applicability of the Weibull model, especially in reliability studies and survival analysis, have been given in Johnson et al. (1995, Chapter 21). As reported by Abernethy (2006), the main advantage of Weibull analysis is its capability to provide accurate life analysis and life forecasts under very small samples. Using this distribution, the remedial measures are possible at the initial indications of the issues. The small sample analysis always provides component testing at lesser costs. A few recent important contributions regarding this distribution are as follows. Bourguignon et al. (2014) proposed a Weibull generalized family of distributions and studied its details. Aryal and Elbatal (2015) introduced a Kumaraswamy modified inverse Weibull distribution and discussed its properties and applications. El-Gohary et al. (2015) developed and estimated an inverse Weibull extension distribution. Ortega et al. (2015) introduced a power series beta Weibull regression model for the prediction of breast carcinoma. Castellares and Lemonte (2015) proposed a gamma exponentiated Weibull model and investigated its properties. Alizadeh et al. (2015) considered the estimation of the PDF and the CDF of the exponentiated Weibull distribution. Kopal et al. (2016) confirmed that the Weibull distribution can be used for modeling the data regarding the temperature dependence of polyurethane storage modulus. Ihaddadene et al. (2016) proposed a hybrid Weibull distribution in order to model the monthly wind speed data. Liu et al. (2017) showed that the Weibull distribution can be used to model the cloud computing data.

The left censoring has very wide range of applications in reliability and survival analysis. For example, due to limitation of the chemical analysis methods, the minute concentrations cannot be measured precisely. The observations for the concentrations which are below the limit of detection are considered left censored (Lambert et al. 1991). Similarly, in environmental and occupational health fields, we often deal with the data having non-detectable levels of contaminant. Hence, the measure of the contaminants below an extraordinarily low level is not possible, which results in the left censored samples (Hornung and Reed 1990). In medical studies, the patients are regularly examined. If a disease is diagnosed, then all we know is that the onset of the disease fell in the period since the last examination. Therefore, the time elapsed since onset of the disease has been left censored. The left censoring also occurs when we have to estimate the functions of exact age with exact date of birth is unknown. A study by Danzon et al. (2005) considered the data regarding 900 firms from 1988 to 2000 for the estimation of effect on the phase specific (phases I, II and III) biotech and pharmaceutical R&D success rates of a firm's overall experience, the diversification of its experience across categories along with some other factors. This study encountered with left censored data, because for example, at the initiation of many phase-II trials for a particular indication, there was no information on the phase-I trial. Mitra and Kundu (2008) had considered the estimation of left censored samples from the generalized exponential distribution using classical methods. However, the Bayesian analysis of the left censored samples from the life distributions using approximate methods is not much frequent in the literature.

We have considered approximate Bayesian estimation for the parameters of the Weibull distribution under left censored samples. Five approximate methods have been used for the posterior estimation. It is worth mentioning here that in case of censored data, the numerical integrations often have convergence issues. Therefore, we need to go for some other options. One way to avoid such complex numerical integrations is to use a quadrature method. The quadrature method can be used efficiently unless the corresponding function does not involve any singularities. Though, the quadrature method has behaved efficiently in our case, but it can produce problems in cases where the concerned functions involve singularities. The other option can be found in shape of Lindley's approximation, due to Lindley (1980). The Lindley's approximation do not have the issues with the functions involving singularities. However, the Lindley's approximation involves evaluation of third order derivatives from the log-likelihood function, which become cumbersome at times. To avoid this issue, Tierney and Kadane (1986) proposed an alternative approximation called Tierney and Kadane's approximation. This approximation does not require the evaluation of third order derivatives. Hence, this approximation is relatively economical in terms of computational times. In addition, this approximation has lesser estimation error as compared to Lindley's approximation

(Danish and Aslam 2013). A computationally more economical option is to use the importance sampling. Unfortunately, all of the above mentioned methods cannot provide the Bayesian interval estimation for the model parameters. The Gibbs sampler can be used to obtain the point as well as the interval Bayesian estimation for the model parameters. We have considered these five approximate methods to develop a comprehensive comparison among these methods to estimate the left censored Weibull model. There are two main aims of the paper. Firstly, to introduce the approximate Bayesian estimation for the left censored Weibull distribution, and secondly, to compare the performance of these approximate Bayesian methods.

The remaining part of the paper is arranged as follows. The model and likelihood function under left censored samples have been given in Section 2. The derivation of posterior distributions has been presented in Section 3. The introduction of the loss functions has been considered in Section 4. The posterior estimation using quadrature method, importance sampling, Gibbs sampler, Tierney and Kadane's approximation and Lindley's approximation has been presented in Section 5. The analysis of the simulated datasets has been reported in Section 6. The real life example has been discussed in Section 7. The conclusions of the study have been given in Section 8.

2. The Model and Likelihood Function

The probability density function (pdf) of the Weibull distribution is

$$f(x)=\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}},x>0,\alpha,\beta>0,$$

where α and β are shape and scale parameters of the model.The cumulative distribution function (CDF) of the distribution is

$$F(x)=1-e^{-\beta x^{\alpha}},x>0,\alpha,\beta>0.$$

Let a sample of size 'n' has been selected from the Weibull distribution and smallest 'r' observations have been censored. This means that the last $x_{r+1},...,x_n$ ordered values can only be observed. Then the likelihood function for the left censored observations $x_1,...,x_n$, as used by Mitra and Kundu (2008) is

$$L(x|\alpha,\beta)\propto\{F(x_{r+1})\}^r\prod_{i=r+1}^{n}f(x_i). \tag{1}$$

Putting entries in (1), we have

$$L(x|\alpha,\beta)\propto\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r\prod_{i=r+1}^{n}\alpha\beta x_i^{\alpha-1}e^{-\beta x_i^{\alpha}}.$$

After simplifications it becomes

$$L(x|\alpha,\beta)\propto\alpha^{n-r}\beta^{n-r}e^{(\alpha-1)\sum_{i=r+1}^{n}\log x_i}e^{-\beta\sum_{i=r+1}^{n}x_i^{\alpha}}\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r. \tag{2}$$

3. Prior and Posterior Distribution

In this section, two priors have been assumed for the construction of joint posterior distributions for the parameters α and β. First, consider the non-informative priors for the parameters α and β as $\pi_1(\alpha)\propto 1,\alpha>0$ and $\pi_2(\beta)\propto 1,\beta>0,$ respectively. Now, using these two priors and assuming independence the joint prior for the parameters α and β can be constructed as

$$\pi_1(\alpha,\beta)\propto 1,\alpha,\beta>0. \tag{3}$$

Combining (2) and (3) the joint posterior distribution, as discussed by Berger (1985), for the parameters α and β under non-informative prior is

$$g_1\left(\alpha,\beta\middle|x\right)=\frac{L\left(x\middle|\alpha,\beta\right)\pi_1\left(\alpha,\beta\right)}{\int_0^{\infty}\int_0^{\infty}L\left(x\middle|\alpha,\beta\right)\pi_1\left(\alpha,\beta\right)d\alpha d\beta}. \quad (4)$$

Secondly, let the informative gamma priors for the parameters α and β are $\pi_3\left(\alpha\right)\propto\alpha^{a-1}e^{-b\alpha},\alpha>0$ and $\pi_4\left(\beta\right)\propto\beta^{c-1}e^{-d\beta},\beta>0$, respectively. Again, combining these priors under the assumption of independence the joint prior for the parameters α and β can be derived as

$$\pi_2\left(\alpha,\beta\right)\propto\alpha^{a-1}\beta^{c-1}e^{-b\alpha}e^{-d\beta},\alpha,\beta>0. \quad (5)$$

The assumption of independent gamma priors for the shape-scale densities is frequent in the literature, for example see Kundu and Howlader (2010). The reason for this choice is that the gamma density is log-concave which often results in mathematical tractability of the corresponding posterior distribution. In addition, the gamma prior is more flexible than exponential prior. In fact exponential prior can be treated as special case of the gamma prior when shape parameter is considered unity in the gamma prior. Now, combining (2) and (5) the joint posterior distribution for the parameters α and β under informative prior is

$$g_2\left(\alpha,\beta\middle|x\right)=\frac{L\left(x\middle|\alpha,\beta\right)\pi_2\left(\alpha,\beta\right)}{\int_0^{\infty}\int_0^{\infty}L\left(x\middle|\alpha,\beta\right)\pi_2\left(\alpha,\beta\right)d\alpha d\beta}. \quad (6)$$

4. Loss Functions

In this section, we have assumed two loss functions for the posterior estimation of the parameters from the Weibull distribution. The introduction of these loss functions is given in the followings.

Squared error loss function (SELF): It is a symmetric loss function proposed by Legendre (1805) and Gauss (1810) it can be defined as $L\left(\theta,\theta_S\right)=\left(\theta-\theta_S\right)^2$, where θ is a parameter. Using this loss function, the Bayes estimator for the parameter θ is $\theta_S=E(\theta|x)$.

Precautionary loss function (PLF): It is an asymmetric loss function introduced by Norstrom (1996). The expression of this loss function is $L\left(\theta_P,\theta\right)=\theta_P^{-1}\left(\theta_P-\theta\right)^2$. Using PLF the Bayes estimator is $\theta_P=\left[E(\theta^2|x)\right]^{1/2}$.

5. Bayesian Estimation

This section contains the Bayesian estimation of the Weibull distribution under left censored samples. However, the closed form derivations of Bayes estimators, from the models (6) and (8), are not possible. Therefore, we have used five approximation methods, namely quadrature method, Gibbs sampler, importance sampling, Lindley's approximation and Tierney and Kadane's approximation, for the approximate estimation of the said parameters.

5.1. Quadrature method

The Bayes estimator for the parameters α under SELF using non-informative prior is

$$\alpha_{NS} = \int_0^\infty \int_0^\infty \alpha g_1\left(\alpha, \beta \middle| \mathrm{x}\right) d\alpha d\beta. \tag{7}$$

It is clear from (7), that the expression for the Bayes estimator cannot be derived in a closed form; so we have considered the quadrature method for the approximate solution of the estimates. The quadrature method is an alternative to the numerical integration. In the Bayesian quadrature method we choose a set of points between the finite integral in order to ensure the stability of our uncertainty. This method has been proposed by Stoer and Bulirsch (1980). Consider the posterior density $g(\alpha, \beta)$, where α and β are the parameters. We evaluate this density over a number of the points in the entire range as

$$\int_0^\infty \int_0^\infty g\left(\alpha, \beta\right) d\alpha d\beta = \sum_{i=0}^{n-r} \sum_{i=0}^{n-r} w_i g\left(\alpha_i, \beta_i\right),$$

where w_i are the increments and α_i and β_i are the quadrature points. Using this method, we have framed a code in the Mathematica software to obtain Bayes estimates and associated posterior risks for the parameters α and β under the assumption of both priors and both loss functions.

5.2. Gibbs sampler

The Gibbs sampler was introduced by Geman and Geman (1984). Let $g(\alpha, \beta | x)$ is a posterior distribution with two parameters α and β, where 'x' denotes the data. Further suppose that the full conditional densities for the parameters α and β are $g(\alpha | \beta, x)$ and $g(\beta | \alpha, x)$, respectively. However, we intend to obtain the marginal distributions for the parameters α and β as $g(\alpha | x)$ and $g(\beta | x)$, respectively. In Gibbs sampler, first we choose some reasonable values of the parameters as the initial values for the parameters, and then we take draws from the two conditional distributions in the following sequence

$$\alpha_1 \sim g\left(\alpha \middle| \beta_0, x\right), \alpha_2 \sim g\left(\alpha \middle| \beta_1, x\right), \ldots, \alpha_{n-r} \sim g\left(\alpha \middle| \beta_{n-r-1}, x\right),$$
$$\beta_1 \sim g\left(\beta \middle| \alpha_1, x\right), \beta_2 \sim g\left(\beta \middle| \alpha_2, x\right), \ldots, \beta_{n-r} \sim g\left(\beta \middle| \alpha_{n-r}, x\right).$$

This sequence is surely a Markov chain because the sample values at the step $(n-r)$ depend only on the values at the step $(n-r-1)$. In order to implement the Gibbs sampler, we need to extract the conditional distributions of the parameters α and β from the posterior distributions (4) and (6).

The conditional distribution of the parameter β given α from (4) is

$$g_1\left(\beta \middle| \alpha, x\right) \propto \beta^{n-r} e^{-\beta \sum_{i=r+1}^{n} x_i^\alpha} \left[1 - e^{-\beta x_{r+1}^\alpha}\right]^r. \tag{8}$$

The conditional distribution of the parameter α given β from (4) is

$$g_1\left(\alpha \middle| \beta, x\right) \propto \alpha^{n-r} e^{(\alpha-1)\sum_{i=r+1}^{n} \log x_i} e^{-\beta \sum_{i=r+1}^{n} x_i^\alpha} \left[1 - e^{-\beta x_{r+1}^\alpha}\right]^r. \tag{9}$$

Using the conditional distributions in (8) and (9), we have developed a code in Winbugs software to generate the sequence of the values for the parameters α and β. Consider the estimation of the parameter α. Now using the generated values of the parameters, the Bayes estimate under SELF can be obtained by using the formula $\alpha_S = \sum_{i=1}^{n-r} \alpha_i / (n-r)$. The corresponding root mean square error

(RMSE) can be obtained by the formula $RMSE(\alpha_S)=\sum_{i=1}^{n-r}(\alpha_i-\alpha_S)^2/(n-r).$ Similarly the Bayes estimate under PLF can be calculated by using the formula $\alpha_P=\sqrt{\sum_{i=1}^{n-r}\alpha_i^2/(n-r)},$ and corresponding RMSE can be computed as $RMSE(\alpha_P)=\sum_{i=1}^{n-r}(\alpha_i-\alpha_p)^2/(n-r).$ Replacing α by β in these formulae, the Bayes estimators and RMSEs for the parameter β can be obtained. With little modifications the Gibbs sampler can be implemented for the posterior distribution under informative prior.

The graphs regarding the history for generation of Gibbs samples, marginal densities and percentile points, for the parameters $\alpha=0.5$ and $\beta=0.5$ for $n=30$, have been presented in Figures 1-6 in the followings.

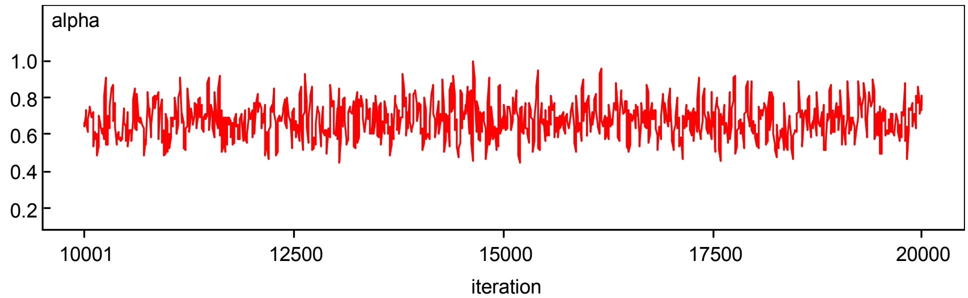

Figure 1 The graph showing the history for generation of Gibbs samples for the parameter α

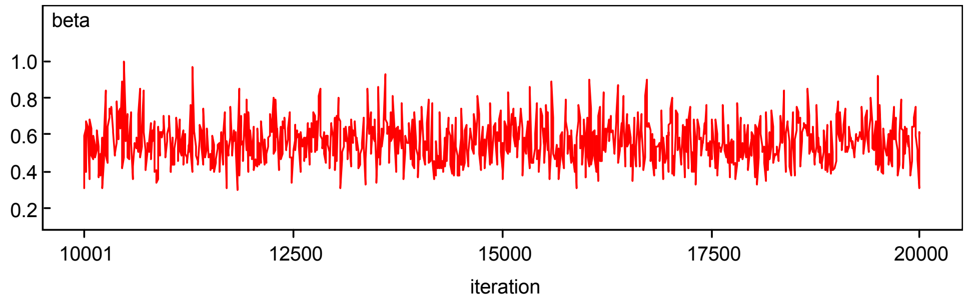

Figure 2 The graph showing the history for generation of Gibbs samples for the parameter β

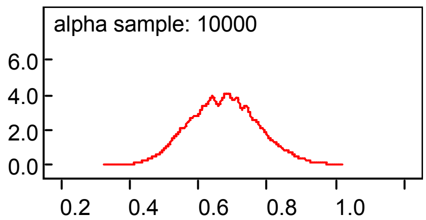

Figure 3 The marginal density for the parameter α

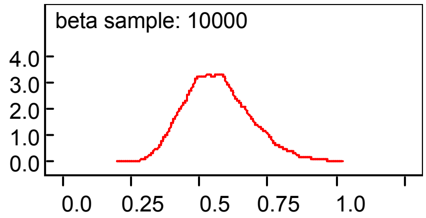

Figure 4 The marginal density for the parameter β

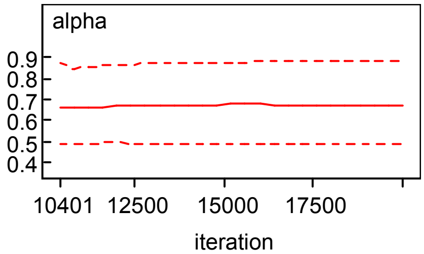

Figure 5 Percentile points graph for the parameter α

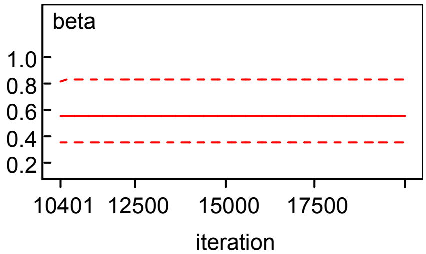

Figure 6 Percentile points graph for the parameter β

5.3. Importance sampling

This section includes the approximate Bayes estimation of the parameters α and β under different priors and loss functions using the importance sampling which was introduced by Kahn (1950). Consider the estimation of the parameter α. Now considering importance sampling, the Bayes estimator for the parameter α under SELF and PLF are respectively presented as

$$\alpha_S = \frac{E'\left[\alpha h(\alpha)\right]}{E'\left[h(\alpha)\right]} \text{ and } \alpha_P = \sqrt{\frac{E'\left\{\alpha^2 h(\alpha)\right\}}{E'\left\{h(\alpha)\right\}}},$$

where $h(\alpha)$ is any function of the parameter α and E' is the expectation with respect to distribution of the parameter α. For simplicity, the likelihood function in (4) can be written as

$$L\left(x|\alpha,\beta\right) \propto \alpha^{n-r}\beta^{n-r} e^{-\alpha\left(-\sum_{i=r+1}^{n}\log x_i\right)} e^{-\beta\sum_{i=r+1}^{n} x_i^{\alpha}} \left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r.$$

Now the posterior distribution under non-informative prior can be written as

$$g_1\left(\alpha,\beta|x\right) \propto \alpha^{n-r}\beta^{n-r}\left\{\psi(x)\right\}^{-n+r} e^{-\alpha\xi(x)} e^{-\beta\psi(x)} \left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r,$$

where $\xi(x)=-\sum_{i=r+1}^{n}\log x_i$ and $\psi(x)=\sum_{i=r+1}^{n}x_i^{\alpha}$. Here α follows gamma distribution with parameters $n-r$ and $\xi(x)$ and the conditional distribution of β given α is $h_1(\beta|\alpha,x)\propto\{\psi(x)\}^{-n+r}\beta^{n-r}e^{-\beta\psi(x)}$. Therefore, the posterior distribution can be re-adjusted to have the following form

$$g_1(\alpha,\beta|x)\propto f_1(\alpha|x)g_1(\beta|\alpha,x)h_1(\alpha,\beta|x),$$

where $f_1(\alpha|x)\sim\alpha^{n-r}e^{-\alpha\xi(x)}$, $g_1(\beta|\alpha,x)\sim\beta^{n-r}e^{-\beta\psi(x)}$ and $h_1(\alpha,\beta|x)\sim\{\psi(x)\}^{-n+r}\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r$.

Now, using importance sampling, the Bayes estimators for the parameter α under non-informative prior based on SELF and PLF are

$$\alpha_{NS}=\frac{E'\left[\alpha h_1(\alpha,\beta|x)\right]}{E'\left[h_1(\alpha,\beta|x)\right]} \text{ and } \alpha_{NP}=\sqrt{\frac{E'\left[\alpha^2 h_1(\alpha,\beta|x)\right]}{E'\left[h_1(\alpha,\beta|x)\right]}}, \text{ respectively,}$$

where E' is the expectation with respect to $Gamma\left(n-r+1,\xi(x)\right)$.

Again, considering importance sampling, the Bayes estimators for the parameter β under non-informative prior based on SELF and PLF are

$$\beta_{NS}=\frac{E'\left[\beta h_1(\alpha,\beta|x)\right]}{E'\left[h_1(\alpha,\beta|x)\right]} \text{ and } \beta_{NP}=\sqrt{\frac{E'\left[\beta^2 h_1(\alpha,\beta|x)\right]}{E'\left[h_1(\alpha,\beta|x)\right]}}, \text{ respectively,}$$

where E' is the expectation with respect to $Gamma\left(n-r+1,\psi(x)\right)$.

Further, the posterior distribution under informative prior can be written as

$$g_2(\alpha,\beta|x)\propto\alpha^{\nu-1}\beta^{\tau-1}\{\mu(x)\}^{-\tau+1}e^{-\alpha\eta(x)}e^{-\beta\mu(x)}\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r,$$

where $\nu=n-r+a$, $\tau=n-r+c$, $\eta(x)=\xi(x)+b$ and $\mu(x)=\psi(x)+d$. Now α follows gamma distribution with parameters ν and $\eta(x)$ and the conditional distribution of β given α is $h_2(\beta|\alpha,x)\propto\{\mu(x)\}^{-\tau+1}\beta^{\tau-1}e^{-\beta\mu(x)}\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r$. Therefore, the posterior distribution can be reframed to have the following form

$$g_2(\alpha,\beta|x)\propto f_2(\alpha|x)g_2(\beta|\alpha,x)h_2(\alpha,\beta|x),$$

where $f_2(\alpha|x)\sim\alpha^{\nu-1}e^{-\alpha\eta(x)}$, $g_2(\beta|\alpha,x)\sim\beta^{\tau-1}e^{-\beta\mu(x)}$ and $h_2(\alpha,\beta|x)\sim\{\mu(x)\}^{-\tau+1}\left[1-e^{-\beta x_{r+1}^{\alpha}}\right]^r$.

Now, using importance sampling, the Bayes estimators for the parameter α under informative prior based on SELF and PLF are

$$\alpha_{IS}=\frac{E'\left[\alpha h_2(\alpha,\beta|x)\right]}{E'\left[h_2(\alpha,\beta|x)\right]} \text{ and } \alpha_{IP}=\sqrt{\frac{E'\left[\alpha^2 h_2(\alpha,\beta|x)\right]}{E'\left[h_2(\alpha,\beta|x)\right]}}, \text{ respectively,}$$

where E' is the expectation with respect to $Gamma\left(\nu,\eta(x)\right)$.

Finally, considering the importance sampling, the Bayes estimators for the parameter β under informative prior based on SELF and PLF are

$$\beta_{IS} = \frac{E'\left[\beta h_2\left(\alpha,\beta|x\right)\right]}{E'\left[h_2\left(\alpha,\beta|x\right)\right]} \text{ and } \beta_{IP} = \sqrt{\frac{E'\left[\beta^2 h_2\left(\alpha,\beta|x\right)\right]}{E'\left[h_2\left(\alpha,\beta|x\right)\right]}}, \text{ respectively,}$$

where E' is the expectation with respect to $Gamma\left(\tau,\mu\left(x\right)\right)$.

5.4. Lindley's approximation

In this section, the Lindley's approximation, proposed by Lindley (1980), have been used for the approximate Bayes estimation of the parameters from the Weibull model. Lindley (1980) confirmed that for a large sample size, the ratio of the integral of the form

$$I\left(\theta\right) = E\left[g\left(\alpha,\beta\right)\right] = \frac{\int\limits_{(\alpha,\beta)} g\left(\alpha,\beta\right) e^{l(\alpha,\beta|x)+G(\alpha,\beta)} d\left(\alpha,\beta\right)}{\int\limits_{(\alpha,\beta)} e^{l(\alpha,\beta|x)+G(\alpha,\beta)} d\left(\alpha,\beta\right)},$$

where $g\left(\alpha,\beta\right)$ is any function of α or $\beta,$ $l\left(\alpha,\beta|x\right)$ is the log-likelihood function and $G\left(\alpha,\beta\right)$ is the logarithmic of joint prior for the parameters α and $\beta,$ can be evaluated as

$$I\left(\theta\right) = g\left(\hat{\alpha},\hat{\beta}\right) + \left(G_1H_1 + G_2H_2 + H_3 + H_4\right) + \frac{1}{2}\left(A_1B_1 + A_2B_2\right),$$

where $\hat{\alpha}$ and $\hat{\beta}$ are MLEs of the parameters α and $\beta,$ respectively,

$B_i = G_1\delta_{i1} + G_2\delta_{i2},\ A_i = \delta_{11}L_{11i} + \delta_{22}L_{22i} + 2\delta_{12}L_{12i},\ H_i = P_1\delta_{i1} + P_2\delta_{i2},\ i = 1,2,$

$H_3 = G_{12}\delta_{12},\ H_4 = \frac{1}{2}\left(G_{11}\delta_{11} + G_{22}\delta_{22}\right),\ P_i = \frac{\partial G\left(\theta\right)}{\partial\theta_i},\ i = 1,2,\ \theta = \left(\alpha,\beta\right),$

$G_{ij} = \frac{\partial^2 g\left(\theta\right)}{\partial\theta_i\partial\theta_j},\ L_{ij} = \frac{\partial^2 L\left(\theta\right)}{\partial\theta_i\partial\theta_j},\ i,j = 1,2,\ L_{ijk} = \frac{\partial^3 L\left(\theta\right)}{\partial\theta_i\partial\theta_j\partial\theta_k},\ i,j,k = 1,2$

and δ_{ij} is the $(i,j)^{\text{th}}$ element of the inverse of the matrix $\left\{L_{ij}\right\},$ all evaluated at the MLEs of the parameters. Now, the log-likelihood function from (4) can be obtained as

$$L\left(\alpha,\beta|x\right) \propto \left(n-r\right)\log\alpha + \left(n-r\right)\log\beta + \alpha\sum_{i=r+1}^{n}\log x_i - \beta\sum_{i=r+1}^{n} x_i^{\alpha} - r\log\left(1 - e^{-\beta x_{r+1}^{\alpha}}\right). \tag{10}$$

The MLEs for the parameters α and β can be obtained by differentiating (10) with respect to α and β and equating to zero respectively as

$$\frac{n-r}{\alpha} + \sum_{i=r+1}^{n}\log x_i - \beta\sum_{i=r+1}^{n} x_i^{\alpha}\log x_i + \frac{r\beta e^{-\beta x_{r+1}^{\alpha}} x_{r+1}^{\alpha}\log x_{r+1}}{1 - e^{-\beta x_{r+1}^{\alpha}}} = 0, \tag{11}$$

$$\frac{n-r}{\beta} - \sum_{i=r+1}^{n} x_i^{\alpha} + \frac{r e^{-\beta x_{r+1}^{\alpha}} x_{r+1}^{\alpha}}{1 - e^{-\beta x_{r+1}^{\alpha}}} = 0. \tag{12}$$

Since from Equations (11) and (12), the MLEs for the parameters α and β cannot be obtained in closed form, we have obtained the approximate MLEs for the said parameters considering the numerical methods. The second order derivatives from the log-likelihood function shown in (10) are presented in the followings

$$L_{11}=-\frac{n-r}{\hat{\alpha}^2}-\hat{\beta}\sum_{i=r+1}^{n}x_i^{\hat{\alpha}}\left(\log x_i\right)^2+\frac{r\hat{\beta}e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{\hat{\alpha}}\left(\log x_{r+1}\right)^2}{1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}}$$
$$-\frac{r\hat{\beta}^2e^{-2\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}\left(\log x_{r+1}\right)^2}{\left(1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}\right)^2}-\frac{r\hat{\beta}^2e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}\left(\log x_{r+1}\right)^2}{1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}},$$

$$L_{12}=\frac{re^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{\hat{\alpha}}\log x_{r+1}}{1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}}-\frac{r\hat{\beta}e^{-2\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}\log x_{r+1}}{\left(1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}\right)^2}-\frac{r\hat{\beta}e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}\log x_{r+1}}{1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}}-\sum_{i=r+1}^{n}x_i^{\alpha}\log x_i,$$

$$L_{22}=-\frac{n-r}{\hat{\beta}^2}-\frac{re^{-2\hat{\beta}x_r^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}}{\left(1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}\right)^2}-\frac{re^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}x_{r+1}^{2\hat{\alpha}}}{1-e^{-\hat{\beta}x_{r+1}^{\hat{\alpha}}}},$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of the parameters α and β, respectively.

As the third order derivatives from (20) with respect to parameters α and β contain long expressions, they have been given in a supplementary file. Based on the second order derivates, the matrix $\{L_{ij}\}$ is

$$\{L_{ij}\}=-\begin{bmatrix}L_{11} & L_{21}\\ L_{12} & L_{22}\end{bmatrix}\text{ and its inverse is }\{L_{ij}\}^{-1}=\begin{bmatrix}\delta_{11} & \delta_{21}\\ \delta_{12} & \delta_{22}\end{bmatrix}.$$

Now using Lindley's approximation under the assumption of non-informative prior and SELF, the Bayes estimators for the parameters α and β are respectively presented as

$$\alpha_{NS}=\hat{\alpha}+\frac{1}{2}\left(\delta_{11}A_1+\delta_{21}A_2\right),\text{ and }\beta_{NS}=\hat{\beta}+\frac{1}{2}\left(\delta_{12}A_1+\delta_{22}A_2\right).$$

Again, considering Lindley's approximation under the assumption of non-informative prior and PLF, the Bayes estimators for the parameters α and β are respectively presented as

$$\alpha_{NP}=\sqrt{\hat{\alpha}^2+\frac{1}{2}\left(2\hat{\alpha}\delta_{11}+\delta_{11}A_1+\delta_{21}A_2\right)},\text{ and }\beta_{NP}=\sqrt{\hat{\beta}^2+\frac{1}{2}\left(2\hat{\beta}\delta_{22}+\delta_{12}A_1+\delta_{22}A_2\right)}.$$

With little modifications, the Bayes estimators under informative priors can be derived in a similar manner.

5.5. Tierney and Kadane's approximation

In this section, the Tierney and Kadane's approximation has been used for the approximate Bayes estimation for the parameters of the left censored Weibull distribution. First, take the case of the non-informative prior and consider $W(\alpha,\beta)=\log\pi_1(\alpha,\beta)+\log L(x|\alpha,\beta)$, where $\log\pi_1(\alpha,\beta)$ is the logarithmic of the joint non-informative prior for the parameters (α,β) and $\log L(x|\alpha,\beta)$ is the logarithmic of likelihood function given in (4). In addition, consider $C(\alpha,\beta)=W(\alpha,\beta)/n$ and $C^*(\alpha,\beta)=\left[\log h(\alpha,\beta)+W(\alpha,\beta)\right]/n$, where $\log h(\alpha,\beta)$ is the logarithmic of the function of the parameter(s) α or β. Then according to Tierney and Kadane (1986), the expression $E\{h(\alpha,\beta|x)\}$ using (6) can be re-expressed as

$$E\left\{h\left(\alpha,\beta\,|\,\mathrm{x}\right)\right\}=\frac{\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}e^{nC^{*}(\alpha,\beta)}d\alpha d\beta}{\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}e^{nC(\alpha,\beta)}d\alpha d\beta}.$$

Now using the Laplace's method, the approximation for $E\left\{h\left(\alpha,\beta\,|\,x\right)\right\}$ can be given as

$$\hat{h}(\alpha,\beta)=\left[\frac{\det\Sigma^{*}}{\det\Sigma}\right]^{1/2}\exp\left[n\left\{C^{*}(\hat{\alpha}^{*},\hat{\beta}^{*})-C(\hat{\alpha},\hat{\beta})\right\}\right], \tag{13}$$

where $(\hat{\alpha}^{*},\hat{\beta}^{*})$ and $(\hat{\alpha},\hat{\beta})$ maximize $C^{*}(\alpha,\beta)$ and $C(\alpha,\beta)$, respectively, and Σ^{*} and Σ are the negatives of the inverse Hessians of $C^{*}(\alpha,\beta)$ and $C(\alpha,\beta)$ evaluated at $(\hat{\alpha}^{*},\hat{\beta}^{*})$ and $(\hat{\alpha},\hat{\beta})$, respectively. Here we have

$$C\left(\alpha,\beta\right)=\frac{1}{n}\left[k+\left(n-r\right)\log\alpha+\left(n-r\right)\log\beta+\alpha\sum_{i=r+1}^{n}\log x_{i}-\beta\sum_{i=r+1}^{n}x_{i}^{\alpha}+r\log\left(1-e^{-\beta x_{r+1}^{\alpha}}\right)\right],$$

$$C^{*}\left(\alpha,\beta\right)=\frac{1}{n}\left[k+\log h\left(\alpha,\beta\right)+\left(n-r\right)\log\alpha+\left(n-r\right)\log\beta+\alpha\sum_{i=r+1}^{n}\log x_{i}-\beta\sum_{i=r+1}^{n}x_{i}^{\alpha}+r\log\left(1-e^{-\beta x_{r+1}^{\alpha}}\right)\right],$$

where k is any constant independent of the parameters α and β.

$$\frac{\partial C\left(\alpha,\beta\right)}{\partial\alpha}=\frac{1}{n}\left[\frac{n-r}{\alpha}+\sum_{i=r+1}^{n}\log x_{i}-\beta\sum_{i=r+1}^{n}x_{i}^{\alpha}\log x_{i}+\frac{r\beta e^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{\alpha}\log x_{r+1}}{1-e^{-\beta x_{r+1}^{\alpha}}}\right]=0. \tag{14}$$

$$\frac{\partial C\left(\alpha,\beta\right)}{\partial\beta}=\frac{1}{n}\left[\frac{n-r}{\beta}-\sum_{i=r+1}^{n}x_{i}^{\alpha}+\frac{re^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{\alpha}}{1-e^{-\beta x_{r+1}^{\alpha}}}\right]=0. \tag{15}$$

Now, $(\hat{\alpha},\hat{\beta})$ can be obtained by solving (14) and (15). The second order derivatives from $C\left(\alpha,\beta\right)$ have been presented in (16)-(18).

$$\begin{aligned}\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\alpha^{2}}=-\frac{1}{n}\Bigg[&\frac{n-r}{\alpha^{2}}+\hat{\beta}\sum_{i=r+1}^{n}x_{i}^{\alpha}\left(\log x_{i}\right)^{2}-\frac{r\beta e^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{\alpha}\left(\log x_{r+1}\right)^{2}}{1-e^{-\beta x_{r+1}^{\alpha}}}\\&+\frac{r\beta^{2}e^{-2\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}\left(\log x_{r+1}\right)^{2}}{\left(1-e^{-\beta x_{r+1}^{\alpha}}\right)^{2}}+\frac{r\beta^{2}e^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}\left(\log x_{r+1}\right)^{2}}{1-e^{-\beta x_{r+1}^{\alpha}}}\Bigg].\end{aligned} \tag{16}$$

$$\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\beta^{2}}=-\frac{1}{n}\left[\frac{n-r}{\beta^{2}}+\frac{re^{-2\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}}{\left(1-e^{-\beta x_{r+1}^{\alpha}}\right)^{2}}+\frac{re^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}}{1-e^{-\beta x_{r+1}^{\alpha}}}\right]. \tag{17}$$

$$\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\alpha\partial\beta}=-\frac{1}{n}\left[-\frac{re^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{\alpha}\log x_{r+1}}{1-e^{-\beta x_{r+1}^{\alpha}}}+\frac{r\beta e^{-2\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}\log x_{r+1}}{\left(1-e^{-\beta x_{r+1}^{\alpha}}\right)^{2}}+\frac{r\beta e^{-\beta x_{r+1}^{\alpha}}x_{r+1}^{2\alpha}\log x_{r+1}}{1-e^{-\beta x_{r+1}^{\alpha}}}+\sum_{i=r+1}^{n}x_{i}^{\alpha}\log x_{i}\right]. \tag{18}$$

The determinant for the negative of the inverse Hessian of $C(\alpha,\beta)$ evaluated at $(\hat{\alpha},\hat{\beta})$ is $\det\Sigma=\left(C_{11}C_{22}-C_{12}^{2}\right)^{-1}$, where $C_{11}=\left.\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\alpha^{2}}\right|_{\hat{\alpha},\hat{\beta}}$, $C_{22}=\left.\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\beta^{2}}\right|_{\hat{\alpha},\hat{\beta}}$, $C_{12}=\left.\frac{\partial^{2}C\left(\alpha,\beta\right)}{\partial\alpha\partial\beta}\right|_{\hat{\alpha},\hat{\beta}}$.

Similarly, the MLEs and the second order derivatives for the $C^*(\alpha,\beta)$ can be obtained. Thereafter, by putting the results in (13) we can find the Bayes estimates under the assumption of non-informative prior using both loss functions. Further, the Bayes estimators and corresponding RMSEs based on informative prior can be obtained in a similar manner.

6. Simulation Study

As the mathematical comparison among the estimators is not possible, a simulation study has been planned to assess and compare the performance of different estimators. The performance of the various estimators has been compared on the basis of amounts of RMSEs associated with these estimates. The choice of different approximation methods, priors, loss functions, sample sizes and true parametric values has been made in this regard. The following parametric space has been considered $\alpha=\beta=0.50$, $\alpha=\beta=1$, $\alpha=\beta=1.50$, $\alpha=\beta=2$, $\alpha=1$, $\beta=10$, $\alpha=10$, $\beta=1$ and $\alpha=10$, $\beta=10$ with sample of sizes $n=20$, 30, 50 and 100. The prior means approach has been used for the elicitation of the hyper-parameters. That means the values of the hyper-parameters, for the informative prior, have been supposed to be so that the mean of the prior distribution is equal to the true parametric values. For example, when $\alpha=\beta=0.50$, we have used $a=c=0.50$ and $b=d$ =1, which satisfy the conditions that $E(\alpha)=\alpha$ and $E(\beta)=\beta$ using the respective prior densities. Similarly for $\alpha=\beta=1$, we have consider $a=c=1$ and $b=d=1$, which also satisfied the said conditions. For other cases, the choice of the hyper-parameters has been made accordingly. Among many others, Soliman (2006) had used the similar method to obtain the values of the hyper-parameters. The observations in the respective samples have been assumed to be 10%, 20% and 30% left censored in different situations. Following abbreviations have been used in the tables below: AM = Approximation Methods, LF = Loss Functions, QM = Quadrature Method, GS = Gibbs Sampler, IS = Importance Sampling, LA = Lindley's Approximation, TKA = Tierney and Kadane's approximation.

The results of the simulation study have been presented in Tables 1-13. From these results, it can be assessed that estimated values of the parameters converge to the true values with increase in the sample size. Further, the amounts of RMSEs associated with these estimates tend to decrease as sample size increases. Hence the proposed estimators can be considered as consistent estimators. This property is true throughout. These trends can also be seen from Figures 7-10, where the Bayes estimates and RMSEs for the parameters using TKA have been presented. From Tables 9-12, it is clear that the performance of the Bayes estimators tend to increase with decrease in the censoring rate. Based on amounts of RMSEs, it can be assessed that the informative prior performs better than the non-informative prior. However, in case of comparison regarding loss functions there are two situations (i) when the true parametric values are less than one and (ii) when the true parametric values are greater than or equal to one. The SELF and PLF perform better than each other in first and second case respectively. From Table 13, it can be observed that the larger choice of the one parameter results into greater RMSEs for the estimates of the other parameter. Finally, in comparison of approximation techniques, again there are above mentioned two conditions. Here the importance sampling and Tierney and Kadane's approximation are superior in performance among all the approximation methods for the first and second condition respectively, because the amounts of RMSEs associated with these estimates are the minimum.

Table 1 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $n = 20$

AM	LF	$\alpha = 0.50$	$\beta = 0.50$	$\alpha = 1.00$	$\beta = 1.00$	$\alpha = 1.50$	$\beta = 1.50$	$\alpha = 2.00$	$\beta = 2.00$
QM	SELF	0.60836	0.63691	1.21912	1.19324	1.75913	1.69216	2.43384	2.34026
		(0.14407)	(0.17880)	(0.21292)	(0.27347)	(0.25125)	(0.29760)	(0.25643)	(0.29757)
	PLF	0.62909	0.65567	1.25632	1.19015	1.73378	1.70934	2.48365	2.36376
		(0.16657)	(0.20979)	(0.20496)	(0.24256)	(0.23384)	(0.27498)	(0.22251)	(0.27427)
GS	SELF	0.60640	0.59083	1.17423	1.10853	1.73308	1.63347	2.23126	2.18066
		(0.07545)	(0.09807)	(0.17575)	(0.23183)	(0.27223)	(0.36143)	(0.36962)	(0.50768)
	PLF	0.60325	0.59545	1.20294	1.13728	1.77280	1.69682	2.25801	2.28255
		(0.10049)	(0.13155)	(0.16702)	(0.22803)	(0.21228)	(0.28924)	(0.25076)	(0.34716)
IS	SELF	0.64947	0.64072	1.19152	1.16195	1.68319	1.64663	2.28632	2.35847
		(0.03790)	(0.03593)	(0.18048)	(0.18627)	(0.21692)	(0.22839)	(0.29391)	(0.31514)
	PLF	0.65054	0.63529	1.22619	1.19828	1.65838	1.70728	2.35436	2.36975
		(0.04780)	(0.04419)	(0.15476)	(0.16908)	(0.18334)	(0.19793)	(0.24793)	(0.29438)
LA	SELF	0.60348	0.57624	1.14432	1.08180	1.70538	1.61081	2.25040	2.25947
		(0.15950)	(0.15310)	(0.25276)	(0.25215)	(0.32097)	(0.35166)	(0.36695)	(0.51102)
	PLF	0.62695	0.60116	1.18099	1.13162	1.75396	1.61812	2.26009	2.26183
		(0.20082)	(0.20723)	(0.22048)	(0.24267)	(0.23953)	(0.27668)	(0.24447)	(0.33124)
TKA	SELF	0.60118	0.60455	1.19895	1.15125	1.76278	1.64617	2.27290	2.29131
		(0.04417)	(0.05042)	(0.11834)	(0.12374)	(0.18915)	(0.20230)	(0.25430)	(0.28630)
	PLF	0.61879	0.59307	1.20951	1.16219	1.74800	1.71045	2.29973	2.28736
		(0.05736)	(0.06129)	(0.11003)	(0.10742)	(0.14400)	(0.14552)	(0.16985)	(0.17497)

Table 2 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $n = 30$

AM	LF	$\alpha = 0.50$	$\beta = 0.50$	$\alpha = 1.00$	$\beta = 1.00$	$\alpha = 1.50$	$\beta = 1.50$	$\alpha = 2.00$	$\beta = 2.00$
QM	SELF	0.58985	0.61891	1.18552	1.14651	1.68732	1.63733	2.34879	2.26818
		(0.13773)	(0.17456)	(0.20813)	(0.26524)	(0.24585)	(0.28424)	(0.25334)	(0.29358)
	PLF	0.60249	0.63841	1.20590	1.17230	1.71060	1.65598	2.37094	2.27905
		(0.16384)	(0.20197)	(0.19702)	(0.23318)	(0.22393)	(0.26489)	(0.21844)	(0.26888)
GS	SELF	0.57837	0.56570	1.14484	1.08541	1.67110	1.58753	2.15499	2.13859
		(0.07273)	(0.09508)	(0.17141)	(0.22808)	(0.26525)	(0.35357)	(0.36012)	(0.48812)
	PLF	0.58273	0.57341	1.15677	1.10828	1.69091	1.62532	2.18323	2.19166
		(0.09627)	(0.12705)	(0.16116)	(0.21958)	(0.20648)	(0.28120)	(0.24667)	(0.33447)
IS	SELF	0.62484	0.62671	1.16658	1.13885	1.63451	1.61163	2.24206	2.28547
		(0.03661)	(0.03435)	(0.17421)	(0.18149)	(0.21246)	(0.22510)	(0.28538)	(0.30504)
	PLF	0.62586	0.62681	1.19200	1.14354	1.63961	1.64803	2.25373	2.30963
		(0.04671)	(0.04365)	(0.15133)	(0.16638)	(0.17913)	(0.19595)	(0.23934)	(0.28695)
LA	SELF	0.58880	0.55541	1.12872	1.06905	1.67603	1.55941	2.18102	2.16742
		(0.15675)	(0.14926)	(0.24143)	(0.24704)	(0.31018)	(0.33898)	(0.35351)	(0.49203)
	PLF	0.60695	0.57406	1.15059	1.09530	1.70144	1.58378	2.20744	2.21687
		(0.19637)	(0.19756)	(0.21820)	(0.23528)	(0.23393)	(0.26787)	(0.23863)	(0.32291)
TKA	SELF	0.59017	0.57725	1.16820	1.10755	1.70521	1.61993	2.19897	2.18224
		(0.04301)	(0.04980)	(0.11567)	(0.11819)	(0.18287)	(0.19791)	(0.24319)	(0.28291)
	PLF	0.59165	0.57899	1.17336	1.11288	1.71435	1.62954	2.21158	2.19647
		(0.05607)	(0.06030)	(0.10590)	(0.10604)	(0.14026)	(0.14190)	(0.16484)	(0.17323)

Table 3 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $n = 50$

AM	LF	$\alpha = 0.50$	$\beta = 0.50$	$\alpha = 1.00$	$\beta = 1.00$	$\alpha = 1.50$	$\beta = 1.50$	$\alpha = 2.00$	$\beta = 2.00$
QM	SELF	0.57590	0.58866	1.10927	1.12022	1.59765	1.60091	2.23092	2.13553
		(0.12223)	(0.13480)	(0.16227)	(0.20688)	(0.18764)	(0.20521)	(0.19743)	(0.20472)
	PLF	0.58796	0.60670	1.12685	1.14362	1.61669	1.61546	2.24775	2.14052
		(0.14609)	(0.16761)	(0.15966)	(0.18854)	(0.17596)	(0.20551)	(0.17455)	(0.20571)
GS	SELF	0.56468	0.53805	1.07120	1.06052	1.58229	1.55221	2.04684	2.03338
		(0.06450)	(0.07342)	(0.13364)	(0.17790)	(0.20246)	(0.25528)	(0.28066)	(0.34038)
	PLF	0.56868	0.54493	1.08094	1.08116	1.59807	1.58555	2.06980	2.06837
		(0.08581)	(0.10546)	(0.13062)	(0.17758)	(0.16224)	(0.21818)	(0.19712)	(0.25588)
IS	SELF	0.61005	0.59608	1.09154	1.11274	1.54765	1.57577	2.12954	2.15181
		(0.03256)	(0.02646)	(0.13583)	(0.14156)	(0.16217)	(0.16251)	(0.22240)	(0.21272)
	PLF	0.61078	0.59568	1.11386	1.11555	1.36160	1.31732	2.13663	2.16923
		(0.04173)	(0.03621)	(0.12262)	(0.13452)	(0.14074)	(0.15204)	(0.19127)	(0.21954)
LA	SELF	0.54792	0.53014	1.04460	1.02611	1.55887	1.53957	2.10279	2.05372
		(0.11942)	(0.11207)	(0.17300)	(0.18665)	(0.22041)	(0.26535)	(0.27597)	(0.34746)
	PLF	0.55933	0.54179	1.05733	1.04208	1.57311	1.56092	2.11926	2.08115
		(0.15567)	(0.15609)	(0.16651)	(0.18345)	(0.17517)	(0.21141)	(0.18838)	(0.24049)
TKA	SELF	0.57621	0.54903	1.09306	1.08216	1.61459	1.58389	2.08862	2.05462
		(0.03821)	(0.03847)	(0.09017)	(0.09220)	(0.13957)	(0.14290)	(0.18950)	(0.19728)
	PLF	0.57738	0.55022	1.09644	1.08565	1.62023	1.58968	2.09667	2.06296
		(0.04999)	(0.04999)	(0.08581)	(0.08575)	(0.11021)	(0.11007)	(0.13168)	(0.13257)

Table 4 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $n = 100$

AM	LF	$\alpha = 0.50$	$\beta = 0.50$	$\alpha = 1.00$	$\beta = 1.00$	$\alpha = 1.50$	$\beta = 1.50$	$\alpha = 2.00$	$\beta = 2.00$
QM	SELF	0.53727	0.54860	1.05921	1.06298	1.52771	1.54911	2.19661	2.09916
		(0.09066)	(0.09586)	(0.11118)	(0.13667)	(0.13073)	(0.14192)	(0.13918)	(0.15323)
	PLF	0.54812	0.56492	1.07442	1.08347	1.54340	1.56047	2.20907	2.09983
		(0.11256)	(0.12567)	(0.10801)	(0.13121)	(0.12555)	(0.13509)	(0.12440)	(0.14479)
GS	SELF	0.52680	0.50142	1.02285	1.00633	1.51303	1.50199	2.01537	2.00901
		(0.04785)	(0.05225)	(0.09154)	(0.11752)	(0.14103)	(0.17655)	(0.19784)	(0.23154)
	PLF	0.53013	0.50741	1.03064	1.02430	1.52563	1.53159	2.03418	2.01930
		(0.06613)	(0.07904)	(0.09236)	(0.11476)	(0.11579)	(0.15517)	(0.14048)	(0.18007)
IS	SELF	0.56913	0.55551	1.04227	1.05588	1.47990	1.52479	2.09680	2.11515
		(0.02408)	(0.01871)	(0.09306)	(0.09354)	(0.11296)	(0.11243)	(0.15678)	(0.15492)
	PLF	0.56938	0.55465	1.06204	1.05689	1.29987	1.27249	2.09987	2.12800
		(0.03216)	(0.02716)	(0.08670)	(0.09358)	(0.10044)	(0.10815)	(0.13634)	(0.15451)
LA	SELF	0.53653	0.52464	1.04283	1.01475	1.51413	1.50492	2.07827	1.98671
		(0.08155)	(0.08210)	(0.12300)	(0.13149)	(0.15408)	(0.18243)	(0.19367)	(0.23700)
	PLF	0.54221	0.53078	1.04938	1.02286	1.52136	1.51551	2.08661	2.00016
		(0.10989)	(0.11337)	(0.11944)	(0.13078)	(0.12473)	(0.14884)	(0.13414)	(0.16840)
TKA	SELF	0.53755	0.51166	1.04372	1.02687	1.54391	1.53264	2.05650	2.01962
		(0.02828)	(0.02739)	(0.06173)	(0.06091)	(0.09721)	(0.09884)	(0.13360)	(0.13420)
	PLF	0.53825	0.51233	1.04542	1.02856	1.54679	1.53557	2.06060	2.02374
		(0.03854)	(0.03746)	(0.06072)	(0.05961)	(0.07865)	(0.07826)	(0.09385)	(0.09325)

Table 5 Bayes estimates and RMSEs (in parenthesis) under informative prior for $n = 20$

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	0.60227	0.63054	1.20693	1.18130	1.74154	1.67525	2.40950	2.31686
		(0.14189)	(0.17611)	(0.20971)	(0.26934)	(0.24744)	(0.29311)	(0.25252)	(0.29308)
	PLF	0.62279	0.64910	1.24376	1.17825	1.71644	1.69224	2.45882	2.34013
		(0.16407)	(0.20663)	(0.20185)	(0.23889)	(0.23030)	(0.27081)	(0.21914)	(0.27014)
GS	SELF	0.60033	0.58493	1.16248	1.09744	1.71575	1.61714	2.20893	2.15886
		(0.07423)	(0.09664)	(0.17307)	(0.22832)	(0.26812)	(0.35596)	(0.36404)	(0.50001)
	PLF	0.59722	0.58950	1.19090	1.12590	1.75507	1.67984	2.23543	2.25972
		(0.09892)	(0.12957)	(0.16450)	(0.22459)	(0.20907)	(0.28489)	(0.24698)	(0.34194)
IS	SELF	0.64297	0.63432	1.17961	1.15034	1.66636	1.63016	2.26346	2.33489
		(0.03733)	(0.03547)	(0.17775)	(0.18345)	(0.21363)	(0.22494)	(0.28947)	(0.31038)
	PLF	0.64403	0.62894	1.21393	1.18629	1.44262	1.38491	2.33081	2.34605
		(0.04712)	(0.04347)	(0.15242)	(0.16653)	(0.18057)	(0.19496)	(0.24417)	(0.28994)
LA	SELF	0.59745	0.57048	1.13288	1.07099	1.68832	1.59469	2.22789	2.23688
		(0.15711)	(0.15078)	(0.24894)	(0.24834)	(0.31611)	(0.34636)	(0.36141)	(0.50329)
	PLF	0.62068	0.59514	1.16918	1.12031	1.73642	1.60194	2.23750	2.23922
		(0.19776)	(0.20409)	(0.21717)	(0.23903)	(0.23592)	(0.27252)	(0.24077)	(0.32622)
TKA	SELF	0.59517	0.59851	1.18696	1.13974	1.74515	1.62971	2.25017	2.26840
		(0.04357)	(0.04960)	(0.11655)	(0.12186)	(0.18630)	(0.19926)	(0.25044)	(0.28198)
	PLF	0.61260	0.58712	1.19741	1.15058	1.73052	1.69335	2.27673	2.26449
		(0.05651)	(0.06033)	(0.10841)	(0.10585)	(0.14182)	(0.14329)	(0.16726)	(0.17232)

Table 6 Bayes estimates and RMSEs (in parenthesis) under informative prior for $n = 30$

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	0.58395	0.61272	1.17367	1.13504	1.67045	1.62096	2.32530	2.24550
		(0.13565)	(0.17193)	(0.20499)	(0.26123)	(0.24212)	(0.27995)	(0.24948)	(0.28915)
	PLF	0.59646	0.63202	1.19385	1.16057	1.69349	1.63942	2.34723	2.25626
		(0.16138)	(0.19893)	(0.19403)	(0.22966)	(0.22054)	(0.26087)	(0.21513)	(0.26483)
GS	SELF	0.57258	0.56005	1.13339	1.07455	1.65439	1.57166	2.13343	2.11721
		(0.07155)	(0.09370)	(0.16879)	(0.22463)	(0.26125)	(0.34822)	(0.35468)	(0.48074)
	PLF	0.57690	0.56768	1.14520	1.09719	1.67400	1.60906	2.16140	2.16974
		(0.09477)	(0.12514)	(0.15873)	(0.21627)	(0.20336)	(0.27697)	(0.24295)	(0.32944)
IS	SELF	0.61859	0.62045	1.15492	1.12747	1.61817	1.59551	2.21964	2.26262
		(0.03606)	(0.03391)	(0.17158)	(0.17875)	(0.20924)	(0.22170)	(0.28107)	(0.30043)
	PLF	0.61960	0.62054	1.18009	1.13210	1.42629	1.33685	2.23119	2.28653
		(0.04604)	(0.04294)	(0.14904)	(0.16387)	(0.17642)	(0.19300)	(0.23571)	(0.28262)
LA	SELF	0.58291	0.54986	1.11744	1.05836	1.65927	1.54381	2.15921	2.14575
		(0.15440)	(0.14700)	(0.23778)	(0.24331)	(0.30548)	(0.33387)	(0.34817)	(0.48459)
	PLF	0.60088	0.56831	1.13909	1.08435	1.68442	1.56795	2.18538	2.19471
		(0.19337)	(0.19456)	(0.21492)	(0.23175)	(0.23040)	(0.26384)	(0.23502)	(0.31802)
TKA	SELF	0.58427	0.57148	1.15652	1.09647	1.68816	1.60373	2.17698	2.16042
		(0.04243)	(0.04899)	(0.11393)	(0.11640)	(0.18011)	(0.19494)	(0.23950)	(0.27864)
	PLF	0.58573	0.57319	1.16162	1.10176	1.69720	1.61325	2.18946	2.17451
		(0.05524)	(0.05936)	(0.10434)	(0.10449)	(0.13813)	(0.13972)	(0.16233)	(0.17061)

Table 7 Bayes estimates and RMSEs (in parenthesis) under informative prior for $n = 50$

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	0.57014	0.58278	1.09818	1.10902	1.58167	1.58490	2.20861	2.11418
		(0.12042)	(0.13274)	(0.15984)	(0.20374)	(0.18480)	(0.20211)	(0.19445)	(0.20164)
	PLF	0.58208	0.60063	1.11558	1.13218	1.60051	1.59931	2.22527	2.11911
		(0.14387)	(0.16508)	(0.15727)	(0.18570)	(0.17329)	(0.20240)	(0.17189)	(0.20258)
GS	SELF	0.55904	0.53267	1.06049	1.04991	1.56647	1.53669	2.02637	2.01305
		(0.06356)	(0.07232)	(0.13161)	(0.17521)	(0.19942)	(0.25142)	(0.27642)	(0.33523)
	PLF	0.56298	0.53949	1.07013	1.07035	1.58208	1.56969	2.04910	2.04768
		(0.08455)	(0.10380)	(0.12865)	(0.17488)	(0.15979)	(0.21487)	(0.19414)	(0.25201)
IS	SELF	0.60395	0.59012	1.08063	1.10162	1.53217	1.56002	2.10824	2.13029
		(0.03209)	(0.02608)	(0.13375)	(0.13939)	(0.15969)	(0.16006)	(0.21904)	(0.20950)
	PLF	0.60467	0.58971	1.10272	1.10441	1.34798	1.30415	2.11526	2.14754
		(0.04111)	(0.03564)	(0.12081)	(0.13249)	(0.13861)	(0.14973)	(0.18835)	(0.21620)
LA	SELF	0.54244	0.52484	1.03415	1.01585	1.54328	1.52417	2.08176	2.03318
		(0.11764)	(0.11041)	(0.17038)	(0.18382)	(0.21709)	(0.26132)	(0.27181)	(0.34221)
	PLF	0.55373	0.53637	1.04675	1.03165	1.55738	1.54531	2.09807	2.06034
		(0.15331)	(0.15375)	(0.16400)	(0.18069)	(0.17252)	(0.20824)	(0.18553)	(0.23686)
TKA	SELF	0.57045	0.54353	1.08213	1.07134	1.59845	1.56805	2.06773	2.03407
		(0.03768)	(0.03782)	(0.08883)	(0.09083)	(0.13744)	(0.14075)	(0.18665)	(0.19429)
	PLF	0.57161	0.54472	1.08548	1.07479	1.60403	1.57378	2.07570	2.04233
		(0.04927)	(0.04927)	(0.08455)	(0.08449)	(0.10853)	(0.10843)	(0.12972)	(0.13051)

Table 8 Bayes estimates and RMSEs (in parenthesis) under informative prior for $n = 100$

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	0.53189	0.54311	1.04861	1.05235	1.51243	1.53361	2.17465	2.07817
		(0.08927)	(0.09439)	(0.10950)	(0.13461)	(0.12872)	(0.13982)	(0.13708)	(0.14792)
	PLF	0.54263	0.55927	1.06368	1.07263	1.52797	1.54488	2.18698	2.07883
		(0.11086)	(0.12374)	(0.10624)	(0.12920)	(0.12366)	(0.14398)	(0.12253)	(0.14255)
GS	SELF	0.52154	0.49641	1.01262	0.99626	1.49790	1.48697	1.99522	1.98892
		(0.04722)	(0.05148)	(0.09011)	(0.11580)	(0.13889)	(0.17390)	(0.19486)	(0.22804)
	PLF	0.52483	0.50233	1.02033	1.01405	1.51037	1.51627	2.01384	1.99911
		(0.06512)	(0.07787)	(0.09097)	(0.10801)	(0.11400)	(0.15284)	(0.13839)	(0.17735)
IS	SELF	0.56344	0.54995	1.03185	1.04533	1.46510	1.50954	2.07583	2.09400
		(0.02387)	(0.01844)	(0.09165)	(0.09209)	(0.11127)	(0.11072)	(0.15440)	(0.15291)
	PLF	0.56369	0.54910	1.05141	1.04632	1.28687	1.25976	2.07887	2.10672
		(0.03168)	(0.02678)	(0.08539)	(0.09220)	(0.09895)	(0.10648)	(0.13426)	(0.15214)
LA	SELF	0.53116	0.51939	1.03241	1.00460	1.49899	1.48987	2.05749	2.01327
		(0.08031)	(0.08087)	(0.12112)	(0.12954)	(0.15176)	(0.17967)	(0.19074)	(0.23341)
	PLF	0.53679	0.52548	1.03888	1.01263	1.50615	1.50035	2.06574	2.02453
		(0.10824)	(0.11169)	(0.11763)	(0.12877)	(0.12287)	(0.14662)	(0.13210)	(0.16583)
TKA	SELF	0.53218	0.50654	1.03329	1.01660	1.52847	1.51731	2.03593	1.99942
		(0.02793)	(0.02683)	(0.06083)	(0.06000)	(0.09576)	(0.09737)	(0.13157)	(0.13217)
	PLF	0.53286	0.50721	1.03497	1.01827	1.53132	1.52021	2.03999	2.00350
		(0.03787)	(0.03691)	(0.05979)	(0.05875)	(0.07747)	(0.07714)	(0.09248)	(0.09186)

Table 9 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $\alpha = \beta = 0.5$ and $n = 100$

AM	LF	10% censoring		20% censoring		30% censoring	
		α	β	α	β	α	β
QM	SELF	0.52984 (0.08556)	0.54058 (0.09039)	0.53727 (0.09066)	0.54860 (0.09586)	0.54574 (0.09423)	0.56221 (0.10050)
	PLF	0.54083 (0.10619)	0.55739 (0.11845)	0.54812 (0.11256)	0.56492 (0.12567)	0.55705 (0.11697)	0.57968 (0.13172)
GS	SELF	0.52014 (0.04506)	0.49487 (0.04919)	0.52680 (0.04785)	0.50142 (0.05225)	0.53574 (0.04970)	0.51467 (0.05477)
	PLF	0.52531 (0.06222)	0.50186 (0.07444)	0.53013 (0.06613)	0.50741 (0.07904)	0.54107 (0.06856)	0.52194 (0.08277)
IS	SELF	0.56494 (0.02258)	0.55326 (0.01761)	0.56913 (0.02408)	0.55551 (0.01871)	0.58189 (0.02490)	0.57539 (0.01949)
	PLF	0.56535 (0.03019)	0.55297 (0.02560)	0.56938 (0.03216)	0.55465 (0.02716)	0.58231 (0.03326)	0.57509 (0.02844)
LA	SELF	0.53338 (0.07662)	0.52374 (0.07720)	0.53653 (0.08155)	0.52464 (0.08210)	0.54938 (0.08438)	0.54469 (0.08591)
	PLF	0.53980 (0.10321)	0.52998 (0.10657)	0.54221 (0.10989)	0.53078 (0.11337)	0.55600 (0.11369)	0.55117 (0.11854)
TKA	SELF	0.53040 (0.02665)	0.50484 (0.02588)	0.53755 (0.02828)	0.51166 (0.02739)	0.54631 (0.02933)	0.52503 (0.02864)
	PLF	0.53144 (0.03635)	0.50564 (0.03535)	0.53825 (0.03854)	0.51233 (0.03746)	0.54739 (0.03997)	0.52587 (0.03919)

Table 10 Bayes estimates and RMSEs (in parenthesis) under informative prior for $\alpha = \beta = 0.50$ and $n = 100$

AM	LF	10% censoring		20% censoring		30% censoring	
		α	β	α	β	α	β
QM	SELF	0.52454 (0.08426)	0.53517 (0.08899)	0.53189 (0.08927)	0.54311 (0.09439)	0.54027 (0.09279)	0.55658 (0.09894)
	PLF	0.53541 (0.10459)	0.55181 (0.11662)	0.54263 (0.11086)	0.55927 (0.12374)	0.55147 (0.11521)	0.57389 (0.12968)
GS	SELF	0.51495 (0.04450)	0.50293 (0.04848)	0.52154 (0.04722)	0.49641 (0.05148)	0.53039 (0.04899)	0.50953 (0.05394)
	PLF	0.52006 (0.06131)	0.50184 (0.07333)	0.52483 (0.06512)	0.50233 (0.07787)	0.53566 (0.06751)	0.51671 (0.08153)
IS	SELF	0.55930 (0.02236)	0.54772 (0.01732)	0.56344 (0.02387)	0.54995 (0.01844)	0.57607 (0.02470)	0.56963 (0.01924)
	PLF	0.55970 (0.02985)	0.54744 (0.02520)	0.56369 (0.03168)	0.54910 (0.02678)	0.57649 (0.03279)	0.56933 (0.02808)
LA	SELF	0.52804 (0.07543)	0.51850 (0.07609)	0.53116 (0.08031)	0.51939 (0.08087)	0.54388 (0.08313)	0.53924 (0.08462)
	PLF	0.53441 (0.10166)	0.52468 (0.10498)	0.53679 (0.10824)	0.52548 (0.11169)	(0.55044 0.11201)	0.54567 (0.11675)
TKA	SELF	0.52510 (0.02627)	0.50079 (0.02530)	0.53218 (0.02793)	0.50654 (0.02683)	(0.54085 0.02898)	0.51978 (0.02811)
	PLF	0.52612 (0.03564)	0.50059 (0.03476)	0.53286 (0.03787)	0.50721 (0.03691)	0.54191 (0.03933)	0.52061 (0.03867)

Table 11 Bayes estimates and RMSEs (in parenthesis) under non-informative prior for $\alpha = \beta = 2$ and $n = 100$

AM	LF	10% censoring		20% censoring		30% censoring	
		α	β	α	β	α	β
QM	SELF	2.16624 (0.13134)	2.06849 (0.14443)	2.19661 (0.13918)	2.09916 (0.15323)	2.23123 (0.14471)	2.15123 (0.16062)
	PLF	2.17967 (0.11737)	2.07183 (0.13645)	2.20907 (0.12440)	2.09983 (0.14479)	2.24506 (0.12928)	2.15471 (0.15177)
GS	SELF	2.01989 (0.18647)	2.02278 (0.21811)	2.01537 (0.19784)	2.00901 (0.23154)	2.04959 (0.20540)	2.06209 (0.24255)
	PLF	2.02568 (0.13226)	2.02722 (0.16955)	2.03418 (0.14048)	2.01930 (0.18007)	2.07615 (0.14567)	2.07711 (0.18854)
IS	SELF	2.08138 (0.14744)	2.10658 (0.14584)	2.09680 (0.15678)	2.11515 (0.15492)	2.14382 (0.16239)	2.19085 (0.16217)
	PLF	2.08500 (0.12813)	2.12156 (0.14539)	2.09987 (0.13634)	2.12800 (0.15451)	2.14755 (0.14114)	2.20642 (0.16167)
LA	SELF	2.06606 (0.18196)	2.02330 (0.22296)	2.07827 (0.19367)	2.01671 (0.23700)	2.12804 (0.20042)	2.06263 (0.24793)
	PLF	2.07734 (0.12599)	2.02713 (0.15831)	2.08661 (0.13414)	2.01016 (0.16840)	2.13966 (0.13880)	2.07701 (0.17604)
TKA	SELF	2.02913 (0.12606)	2.01269 (0.12649)	2.05650 (0.13360)	2.01962 (0.13420)	2.09000 (0.13885)	2.07240 (0.14064)
	PLF	2.03455 (0.08846)	2.01973 (0.08782)	2.06060 (0.09385)	2.02374 (0.09325)	2.09558 (0.09744)	2.07721 (0.09770)

Table 12 Bayes estimates and RMSEs (in parenthesis) under informative prior for $\alpha = \beta = 2$ and $n = 100$

AM	LF	10% censoring		20% censoring		30% censoring	
		α	β	α	β	α	β
QM	SELF	2.14459 (0.12938)	2.04780 (0.13943)	2.17465 (0.13708)	2.07817 (0.14792)	2.20892 (0.14251)	2.12972 (0.15505)
	PLF	2.15787 (0.11561)	2.05111 (0.13433)	2.18698 (0.12253)	2.07883 (0.14255)	2.22261 (0.12733)	2.13316 (0.14939)
GS	SELF	2.03199 (0.18366)	2.02295 (0.21480)	2.00522 (0.19486)	2.01892 (0.22804)	2.02909 (0.20229)	2.04147 (0.23887)
	PLF	2.03552 (0.13027)	2.02725 (0.16700)	2.01384 (0.13839)	2.02911 (0.17735)	2.05539 (0.14351)	2.05634 (0.18570)
IS	SELF	2.06056 (0.14519)	2.08552 (0.14394)	2.07583 (0.15440)	2.09400 (0.15291)	2.12238 (0.15994)	2.16894 (0.16006)
	PLF	2.06415 (0.12620)	2.10034 (0.14316)	2.07887 (0.13426)	2.10672 (0.15214)	2.12607 (0.13898)	2.18435 (0.15918)
LA	SELF	2.04540 (0.17919)	2.00981 (0.21957)	2.05749 (0.19074)	2.01327 (0.23341)	2.10676 (0.19738)	2.09020 (0.24417)
	PLF	2.05656 (0.12411)	2.02146 (0.15590)	2.06574 (0.13210)	2.02453 (0.16583)	2.11826 (0.13668)	2.10232 (0.17335)
TKA	SELF	2.00883 (0.12414)	2.00276 (0.12458)	2.03593 (0.13157)	2.00142 (0.13217)	2.06910 (0.13671)	2.05167 (0.13853)
	PLF	2.01420 (0.08717)	2.00734 (0.08653)	2.03999 (0.09248)	2.00350 (0.09186)	2.07462 (0.09601)	2.05644 (0.09622)

Table 13 Bayes estimates and RMSEs (in parenthesis) under informative prior using different parametric values and $n = 100$

AM	LF	$\alpha = 1$	$\beta = 10$	$\alpha = 10$	$\beta = 1$	$\alpha = 10$	$\beta = 10$
QM	SELF	1.08407 (0.06779)	10.50184 (0.42848)	11.05135 (0.49337)	1.05512 (0.07843)	11.24241 (0.51218)	10.94335 (0.47738)
	PLF	1.09414 (0.06073)	10.56260 (0.40823)	11.19963 (0.43685)	1.05613 (0.07442)	11.38443 (0.45642)	10.93970 (0.45576)
GS	SELF	1.04547 (0.09672)	10.29651 (0.65956)	10.17439 (0.69821)	1.02293 (0.12003)	10.38793 (0.71952)	10.32310 (0.73562)
	PLF	1.04364 (0.06791)	10.43203 (0.50821)	10.25669 (0.49566)	1.02797 (0.09322)	10.53716 (0.51681)	10.48014 (0.57326)
IS	SELF	1.04923 (0.07675)	10.56399 (0.44203)	10.62835 (0.55079)	1.07704 (0.07992)	10.81416 (0.57237)	11.04793 (0.49445)
	PLF	1.05825 (0.06585)	10.75920 (0.43767)	10.50973 (0.47757)	1.08371 (0.07936)	10.90259 (0.49233)	11.23848 (0.48474)
LA	SELF	1.04876 (0.09403)	10.16642 (0.66928)	10.48116 (0.67884)	1.02011 (0.12150)	10.66895 (0.70721)	10.70799 (0.74320)
	PLF	1.04021 (0.06498)	10.29792 (0.47411)	10.47661 (0.46757)	1.02673 (0.08653)	10.73185 (0.48917)	10.63070 (0.53115)
TKA	SELF	1.03016 (0.06518)	10.22239 (0.37821)	10.32827 (0.47314)	1.01426 (0.06993)	10.63228 (0.49077)	10.42094 (0.42070)
	PLF	1.02316 (0.04544)	10.17169 (0.26727)	10.47066 (0.33111)	1.02557 (0.04852)	10.66936 (0.34498)	10.47489 (0.29440)

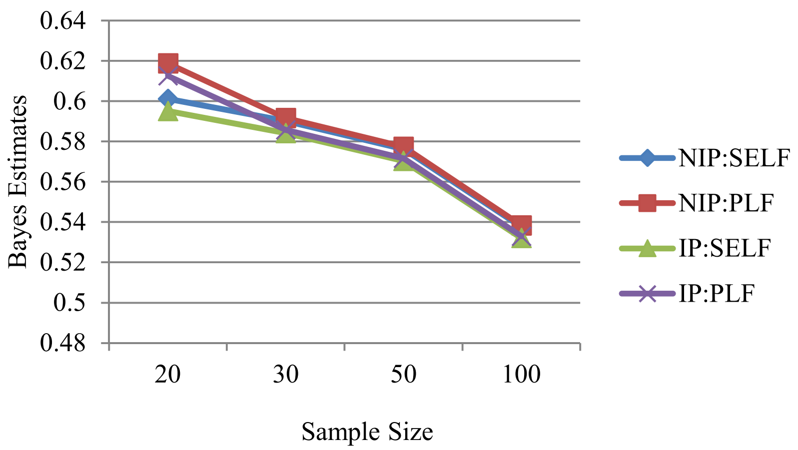

Figure 7 Bayes estimators for the parameter α

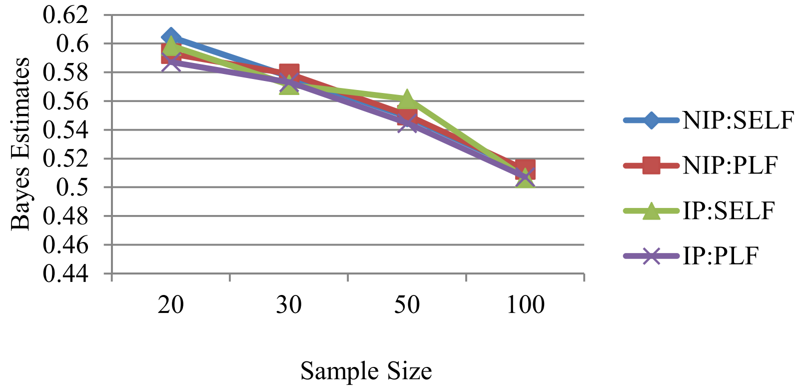

Figure 8 Bayes Estimators for the parameter β

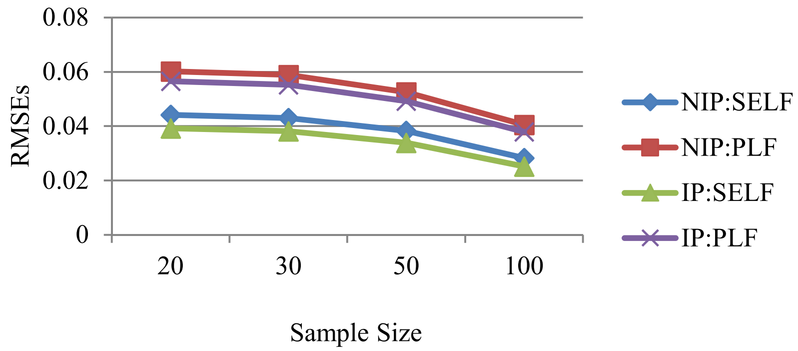

Figure 9 RMSEs associated with estimators for the parameter α

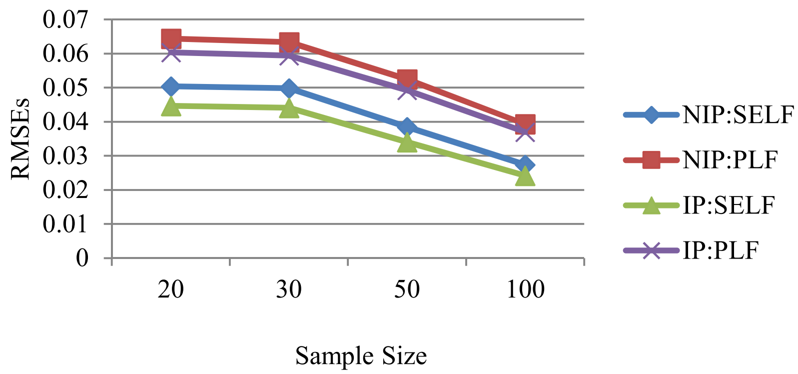

Figure 10 RMSEs associated with Estimators for the parameter β

7. Real Life Example

In this section, we have used the real life dataset regarding NH3-N concentrations, as mg/l, measured in the Skagit River, Washington State, from January 1978 to December 1982 arranged in an ascending order, for which all censored observations were censored at 0.01 mg/l, that is, the values which are less than 0.01 mg/1 have assumed to be left censored. The detail of data is as follows <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01,

<0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.04, 0.04, 0.04, 0.04, 0.04, 0.04, 0.05, 0.05, 0.05, 0.06 and 0.47. The data is available at www.public.iastate.edu/~pdixon/stat505/Chapter%2011.pdf. The chi-square and the Kolomogorov-Smirnov tests, with significant p-values, confirmed that the data comes from the Weibull distribution. The results based on analysis of this real dataset have been given in the Table 14. For convenience, the values of the hyper-parameters have been assumed to be $a = b = c = d = 1$ for the estimation.

Table 14 contains the results for the Bayes estimates, RMSEs, Akaike information criteria (AIC) and Bayesian information criteria (BIC) under different situations using real dataset. Based on values of RMSEs, the parameter (α) has been better estimated using IS, while parameter (β) has been better estimated under TKA. The performance of the informative prior is better than non-informative prior, as the RMSEs are smaller in case of informative priors. Hence, the conclusions from real life analysis and simulation study are in close agreement. However, the overall suitability of the approximate methods for estimating left censored Weibull model has been discussed under information criteria. It is evident from the results that the values of AIC and BIC are the minimum for TKA. According to these criteria, the importance sampling technique stood second in terms of performance. Hence, TKA can be preferred for the estimation of left censored Weibull distribution.

Table 14 Bayes estimates, RMSEs (in parenthesis), AIC and BIC under non-informative and informative priors using real dataset

AM	LF	Non-informative Prior				Informative Prior			
		α	β	AIC	BIC	α	β	AIC	BIC
QM	SELF	0.7302	6.8748	−131.7540	−128.1860	0.7201	6.9776	−134.7130	−131.1450
		(0.1124)	(0.3363)			(0.1068)	(0.3182)		
	PLF	0.7461	7.1054	−130.8900	−127.3220	0.7385	7.2251	−133.5730	−130.0050
		(0.1398)	(0.3178)			(0.1324)	(0.3066)		
GS	SELF	0.6859	6.9980	−140.7900	−137.2220	0.6768	7.1118	−143.1310	−139.5620
		(0.0592)	(0.5069)			(0.0563)	(0.4887)		
	PLF	0.7009	7.2327	−140.4140	−136.8450	0.6932	7.3527	−142.6400	−139.0720
		(0.0821)	(0.3951)			(0.0776)	(0.3808)		
IS	SELF	0.6759	7.1084	−143.2330	−139.6650	0.6674	7.2260	−145.3450	−141.7770
		(0.0297)	(0.3404)			(0.0283)	(0.3273)		
	PLF	0.6907	7.3468	−142.9750	−139.4070	0.6817	7.4606	−145.1780	−141.6100
		(0.0398)	(0.3385)			(0.0378)	(0.3266)		
LA	SELF	0.6878	7.0641	−141.0660	−137.4980	0.6815	7.1965	−143.1350	−139.5660
		(0.1010)	(0.5186)			(0.0954)	(0.5002)		
	PLF	0.7019	7.2744	−140.6060	−137.0380	0.6912	7.3807	−143.1710	−139.6030
		(0.1357)	(0.3686)			(0.1288)	(0.3558)		
TKA	SELF	0.6765	7.1361	−143.3730	−139.8040	0.6715	7.3205	−145.5180	−141.9490
		(0.0351)	(0.2944)			(0.0332)	(0.2836)		
	PLF	0.6899	7.2788	−142.5450	−138.9760	0.6779	7.3816	−145.1080	−141.5400
		(0.0477)	(0.2048)			(0.0450)	(0.1971)		

8. Conclusions

This paper considers the Bayesian parameter estimation from the Weibull distribution based on left censored samples. We have assumed the combination of informative and non-informative priors along with symmetric (SELF) and asymmetric (PLF) loss functions for the posterior estimation of the model parameters. In addition, we have considered five approximation methods for estimation in order to assess and compare different options for the estimation of the parameters from the Weibull distribution. From the simulation study, it has been assessed that there are two important situations for conclusion (i) when the true parametric values are less than one and (ii) when the true parametric values are greater than or equal to one. In first situation, the use of informative prior, SELF and importance sampling gives the best results. While in the second situation, the employment of informative prior, PLF and Tierney and Kadane's approximation produces the superior estimation. In both situations, the proposed estimators are consistent in nature and are capable of producing efficient results even in the moderate samples. The real dataset has also been analyzed to illustrate the applicability of the results obtained under the study.

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