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Point and Interval Estimators of $R = P[Y < X]$ Based on Gompertz Distribution and Ranked Set Sampling Data

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Abstract

The stress-strength model $R = P[Y < X]$ is defined as the probability that the stress variable Y is less than the strength variable X. Although the main use of stress-strength model in physics and engineering fields. It has also more uses in economics, quality control, psychology, medicine, and agricultural. Traditionally, simple random sampling (SRS) is used for estimating the reliability model. In recent years, the ranked set sampling (RSS) is used for estimating reliability model because it is more efficient than (SRS). In this paper, we present the point and interval estimators for reliability model when the strength and stress variables are two independent Gompertz distribution based on (RSS). Monte Carlo simulation study is used to compare the maximum likelihood estimator based on (RSS) with the maximum likelihood estimator based on (SRS) and construct an asymptotic confidence interval (ACI) and bootstrap confidence (BCI). Finally, real data in medicine is used for illustrative our proposed method.

Keywords: Monte Carlo simulation, asymptotic confidence interval, bootstrap confidence interval, maximum likelihood estimator.

1. Introduction

The stress-strength model $R = P[Y < X]$ describes the life of component, which has a random strength variable X and subjected to a random stress variable Y. The component fails at the instant that the $Y > X$ inference for stress-strength model is one of the most popular problem in many fields such as medicine (see Zhou 2008) and quality control (see Ventura and Racugno 2011). For earlier detailed bibliography for stress-strength model (see Kotz et al. 2003). For recent (see Chaturvdi et al. 2016, Hassan 2017). Stress-strength model have also studied in a multi-component started by Bhattacharyya and Johnson (1974) and for recent (see Rao and Kantam 2010, Rao et al. 2017, Hassan 2017, Hassan and Alohali 2018). The ranked set sampling (RSS) method is used in the estimation of R

(see Sengupta and Mukhuti 2008a, Sengupta and Mukhuti 2008b, Muitlak et al. 2010, Dong 2013, Akgul et al. 2018). In this paper, we are considered the estimation for R, when the stress and strength are two independent Gompertz random variables with common scale parameter and different in shap parameters based on (RSS), because it becomes popular in performing statistical inference because the RSS results in more informative sample of the underlying population than a simple random sample of the same size because it contains not only information carried by quantified observations but also information provided by the ranking process. The main reason of using Gompertz distribution, it is commonly used in many engineering applications such that civil, mechanical, and aerospace. Saracolu et al. (2009) obtained the point estimators of R using Gompertz distribution based on SRS. In this paper, we derive the point estimator of R based on Gompertz distribution and (RSS) using maximum likelihood (ML) method. Also, introduce the asymptotic confidence interval (ACI) estimator. Using re-sampling methods proposed by Chen et al. (2004) to introduce bootstrap confidence interval (BCI) in SRS and using the re-sampling method proposed by Modarres et al. (2006) in RSS. So, we organize this paper as, in Section 2, we present overview about Gompertz distribution and point estimator of R using maximum likelihood method based on (SRS). In Section 3, we present brief description of (RSS) and obtain the point estimator for R using maximum likelihood of R based on (RSS). In Section 4, we construct (ACI) and (BCI) of R in both sampling techniques RSS and SRS. In Section 5, a real data is given to illustrate the computations of our newly proposed estimators.

2. Point Estimator for $R = P[Y < X]$ Based on (SRS)

The Gompertz distribution is continuous probability distribution which have many applications in demography, actuaries biology. Also,it has many applications in survival analysis and computer science. The main reason for using Gompertz distribution is it has an increasing hazard rate for life of sysytems. Gompertz distribution, recently has studied by Jeheen (2003), Wu et al. (2003, 2004, 2006). The probability density function (pdf) and cumulative distribution function (cdf) of Gompertz are given respectively as

$$f(x) = \lambda e^{cx} e^{-\frac{\lambda}{c}\left(e^{cx}-1\right)}, \quad x, c, \lambda > 0,$$

and

$$F(x) = 1 - e^{-\frac{\lambda}{c}\left(e^{cx}-1\right)}, \quad x, c, \lambda > 0.$$

Now, to derive the reliability parameter $R = P[Y < X]$. Saracolu et al. (2009) assumed X is the strength variable Y is the stress variable. Both of them has Gompertz distribution with common known scale parameter c and different shap parameters λ_1 and λ_2, respectively. As follow $X \sim Gompertz(c, \lambda_1)$ and $Y \sim Gompertz(c, \lambda_2)$, Then

$$R = P[Y < X] = \int_0^{\infty} P[Y < X] f(x)\, dx = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Also, Saracoglu et al. (2009) derived the maximum likelihood estimator for R based on (SRS) as follows: Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples taken from Gompertz distribution will (c, λ_1) and (c, λ_2), respectively. As c known, the maximum likelihood estimator for R is given by invariance property of maximum likelihood estimator. Since the maximum likelihood estimator of λ_1 and λ_2 are $\hat{\lambda}_1 = \dfrac{nc}{\sum_{i=1}^{n}\left(e^{cx_i - 1}\right)}$ and $\hat{\lambda}_2 = \dfrac{mc}{\sum_{j=1}^{m}\left(e^{cy_j - 1}\right)}$, respectively, the maximum likelihood estimator of R based on (SRS) is $R_{SRS}^{ML} = \dfrac{\hat{\lambda}_2}{\hat{\lambda}_1 + \hat{\lambda}_2}$.

3. Point Estimator for $R = P[Y < X]$ Based on (RSS)

In this section, we derive the maximum likelihood estimator of R based on (RSS). But first, we introduce a brief of (RSS) as follows: The (RSS) method was introduced by McIntyre (1952) for estimating average yield in agriculture. Review of applications and theoretical work on (RSS) (see Patil et al. 1994, Kaur et al. 1995, Johnson et al. 1993). The (RSS) has applied in many years such as forestry, environmental science and medicine (see Chen et al. 2004, Mahdizadeh and Zamanzade 2018, Ashour and Abdallah 2019). Now, to design (RSS), we use the following algorithm.

1. Select m random samples, each of size m from population under study.
2. Use judgement ordering on the element of sample to identify the smallest element.
3. Actually measure the m identified units in Step 2.
4. Repeat Step 1 to Step 3 r times to get a (RSS) of size $n = rm.$

In this study, we represent our RSS by matrix notation as follows

$$\xleftarrow{\text{Before rank}}\begin{pmatrix} X_{11} & . & . & . & X_{1m} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ X_{m1} & . & . & . & X_{mm} \end{pmatrix}\xleftarrow{\text{After rank}}\begin{pmatrix} X_{1(1)} & . & . & . & X_{1(m)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ X_{m(1)} & . & . & . & X_{m(m)} \end{pmatrix}.$$

The first cycle $X_{1(1)} \ldots X_{m(m)}$ and by repeating the above procedure r-times, we obtain a (RSS) of size $n = rm.$ Now, we obtain the maximum likelihood estimator of $R = P[Y < X]$ based on (RSS).

3.1. The maximum likelihood estimator of $R = P[Y < X]$

Let $X_{ij}, i = 1,\ldots,m_1, j = 1,\ldots,r_1$ denote the ranked set sample of size $n_1 = r_1 m_1$ from Gompertz distribution with parameters (c, λ_1) where m_1 is the set size and r_1 is the number of cycles and $Y_{kl}, k = 1,\ldots,m_2, l = 1,\ldots,r_2$ denote the ranked set sample of size $n_2 = m_2 r_2$ from Gompertz distribution with parameters (c, λ_2) where m_2 is the set size and r_2 is the number of cycles. Then the pdf of X_{ij} and Y_{kl} are given by

$$f_i\left(x_{ij}\right) = \frac{m_1!}{(i-1)!(m_1 - i)!}\left[F\left(x_{ij}\right)\right]^{i-1}\left[1 - F\left(x_{ij}\right)\right]^{m_1 - i} f\left(x_{ij}\right),$$

and

$$g_k\left(y_{kl}\right) = \frac{m_2!}{(k-1)!(m_2 - k)!}\left[F\left(y_{kl}\right)\right]^{k-1}\left[1 - F\left(y_{kl}\right)\right]^{m_2 - k} f\left(y_{kl}\right),$$

respectively. Then the likelihood function based on (RSS) is given by

$$L=\prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i\left(x_{ij}\right)\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k\left(y_{kl}\right)$$

$$=\prod_{i=1}^{r_1}(\prod_{j=1}^{m_1}\frac{m_1!\lambda_1 e^{cx_{ij}}}{(i-1)!(m_1-i)!}\ 1-e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)})^{i-1}(e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)})^{m_1-i+1}$$

$$\prod_{k=1}^{r_2}(\prod_{l=1}^{m_2}\frac{m_2!\lambda_2 e^{cy_{kl}}}{(k-1)!(m_2-k)!}\ 1-e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)})^{k-1}(e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)})^{m_2-k+1}$$

$$L=W\lambda_1^{n_1}\lambda_2^{n_2}\prod_{i=1}^{r_1}(\prod_{j=1}^{m_1} 1-e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)})^{i-1}(e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)})^{m_1-i+1}e^{cx_{ij}}$$

$$\prod_{k=1}^{r_2}(\prod_{l=1}^{m_2} 1-e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)})^{k-1}(e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)})^{m_2-k+1}e^{cy_{kl}},$$

where $W=\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\frac{m_1!}{(i-1)!(m_1-i)!}\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\frac{m_2!}{(k-1)!(m_2-k)!}$,

$$LogL=\log(W)+n_1\log(\lambda_1)+n_2\log(\lambda_2)+\sum_{i=1}^{r_1}\sum^{m_1}(i-1)\log\left(1-e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)}\right)$$

$$+c\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}-\frac{\lambda_1}{c}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i+1)\left(e^{cx_{ij}}-1\right)+c\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}y_{kl}$$

$$+\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\log\left(1-e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)}\right)-\frac{\lambda_2}{c}\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2-k+1)\left(e^{cy_{kl}}-1\right).$$

The maximum likelihood estimators of λ_1 and λ_2 are the solution of the following equations. For known c

$$\frac{\partial \log L}{\partial\lambda_1}=0=\frac{n_1}{\lambda_1}-\frac{1}{c}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i+1)\left(e^{cx_{ij}}-1\right)+\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\frac{(i-1)\left(e^{cx_{ij}}-1\right)e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)}}{c\left(1-e^{\frac{-\lambda_1}{c}\left(e^{cx_{ij}}-1\right)}\right)},$$

$$\frac{\partial \log L}{\partial\lambda_2}=0=\frac{n_2}{\lambda_2}-\frac{1}{c}\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2-k+1)\left(e^{cy_{kl}}-1\right)+\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\frac{(k-1)\left(e^{cy_{kl}}-1\right)e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)}}{c\left(1-e^{\frac{-\lambda_2}{c}\left(e^{cy_{kl}}-1\right)}\right)}.$$

We use a numerical method to get the value of the maximum likelihood estimators for λ_1 and λ_2 based on (RSS) denoted by λ_{1RSS}^{ML} and λ_{2RSS}^{ML} and using the invariance property of the maximum likelihood estimator, we get the maximum likelihood estimator of reliability parameter R based on (RSS) as

$$R_{RSS}^{ML}=\frac{\lambda_{2RSS}^{ML}}{\lambda_{1RSS}^{ML}+\lambda_{2RSS}^{ML}}.$$

4. Interval Estimator for R Based on RSS

In this section, we construct the interval estimation of $R=P[Y<X]$. Construct an asymptotic confidence interval (ACI) and bootstrap confidence interval (BCI).

4.1. Asymptotic confidence interval (ACI) for R

Since the closed form of maximum likelihood estimator of $R = P[Y < X]$ not exist. It essential to study the asymptotic distribution and construct the (ACI) for R as follows.

Compute the Fisher information matrix as

$$I_{RSS}(\Theta) = -\begin{pmatrix} E\left(\dfrac{\partial^2 \log L}{\partial \lambda_1^2}\right) & E\left(\dfrac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2}\right) \\ E\left(\dfrac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_1}\right) & E\left(\dfrac{\partial^2 \log L}{\partial \lambda_2^2}\right) \end{pmatrix},$$

where $\Theta = (\lambda_1, \lambda_2)$.

Proposition 1 *The element of Fisher information matrix are denote by I_{ij}, $i, j = 1, 2$ and given by*

$$I_{11} = -E\left(\frac{\partial^2 \log L}{\partial \lambda_1^2}\right) = \frac{n_1}{\lambda_1^2} + F_1 + F_2, \quad I_{22} = -E\left(\frac{\partial^2 \log L}{\partial \lambda_2^2}\right) = \frac{n_2}{\lambda_2^2} + F_3 + F_4, \text{ and } I_{12} = I_{21} = 0,$$

where

$$F_1 = -\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\sum_{s_1=1}^{i-2}(i-1)(-1)^{i-2-s_1}\binom{i-2}{s_1} \times \frac{-2c^2 + e^{\left(\frac{2-i+m_1+s_1}{c}\right)\lambda_1}\left(2c^2 + 2c\left(-2+i-m_1-s_1\right)\lambda_1 + \left(2-i+m_1+s_1\right)\lambda_1^2\right)}{c^3(2-i+m_1+s_1)^3\lambda_1^2},$$

$$F_2 = -\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\sum_{s_2=1}^{i-2}(i-1)(-1)^{i-3-s_2}\binom{i-3}{s_2} \times \frac{-2c^2 + e^{\left(\frac{4-i+m_1+s_2}{c}\right)\lambda_1}(2c^2 + 2c\left(-4+i-m_1-s_2\right)\lambda_1 + \left(4-i+m_1+s_2\right)^2\lambda_1^2)}{(4-i+m_1+s_2)^3\lambda_1^2},$$

$$F_3 = -\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\sum_{s_3=1}^{k-2}(k-1)(-1)^{k-2-s_3}\binom{k-2}{s_3} \times \frac{-2c^2 + e^{\left(\frac{2-k+m_2+s_3}{c}\right)\lambda_2}\left(2c^2 + 2c\left(-2+k-m_2-s_3\right)\lambda_2 + \left(2-k+m_2+s_3\right)\lambda_2^2\right)}{c^3(2-k+m_2+s_3)^3\lambda_2^2}, \text{ and}$$

$$F_4 = -\sum_{i=1}^{r_2}\sum_{j=1}^{m_2}\sum_{s_4=1}^{k-3}(k-1)(-1)^{k-3-s_4}\binom{k-3}{s_4} \times \frac{-2c^2 + e^{\left(\frac{4-k+m_2+s_4}{c}\right)\lambda_2}(2c^2 + 2c\left(-4+k-m_2-s_4\right)\lambda_2 + \left(4-k+m_2+s_4\right)^2\lambda_2^2)}{(4-k+m_2+s_4)^3\lambda_2^2}.$$

Now, to find the asymptotic distribution of R, we use the following theorems.

Theorem 1 *As $n_1 \to \infty$ and $n_2 \to \infty$ then,* $\left(\sqrt{n_1}\left(\hat{\lambda}_1 - \lambda_1\right), \sqrt{n_2}\left(\hat{\lambda}_2 - \lambda_2\right)\right) \xrightarrow{D} N_2\left(0, A^{-1}(\Theta)\right)$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $a_{11} = \dfrac{I_{11}}{n_1}$, $a_{12} = a_{21} = 0$, $a_{22} = \dfrac{I_{22}}{n_2}$.

Proof: The proof follows from the asymptotic normality of MLE (see, Ferguson 1967).

Theorem 2 *As* $n_1 \to \infty$ *and* $n_2 \to \infty$ *then,* $\sqrt{n_1+n_2}(\hat{R}-R) \xrightarrow{D} N(0,B).$

Proof: From Theorem 1 and delta method we get,

$$B = \left(\frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \lambda_2}\right) A^{-1} \begin{bmatrix} \frac{\partial R}{\partial \lambda_1} \\ \frac{\partial R}{\partial \lambda_2} \end{bmatrix},$$

where $\left(\frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \lambda_2}\right) = \frac{1}{(\lambda_1+\lambda_2)^2}\left(-\lambda_2, \lambda_1\right)$. The estimate of $Var(\hat{R})$ is $\hat{V}ar(\hat{R}) = B|_{\lambda_1=\hat{\lambda}_1, \lambda_2=\hat{\lambda}_2}$, then $\frac{\sqrt{n}(\hat{R}-R)}{Var(\hat{R})} \sim N(0,1)$. Hence, the asymptotic $100\left(1-\alpha\right)\%$ confidence interval for R is

$$\left(\hat{R} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{V}ar(\hat{R})}{n_1+n_2}}, \hat{R} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{V}ar(\hat{R})}{n_1+n_2}}\right),$$

where $Z_{\frac{\alpha}{2}}$ is the upper $\alpha/2$ quantile of the standard normal.

4.2. Bootstrap confidence interval (BCI) for R

In this section, BCI for R are constructed by two methods, the first method was introduced by Chen et al. (2004a) as follows:

1. Select the (SRS) samples $x_i\left(i=1,...,n_1\right)$ and $y_k\left(k=1,...,n_2\right)$.
2. Re-sample from each subgroups with replacement.
3. Obtain the (SRS) re-samples which is drawn with replacement $x_i^*\left(i=1,...,n_1\right)$ and $y_k^*\left(k=1,...,n_2\right)$.
4. Compute $\hat{R}^*$ using X_i^* and Y_k^*.
5. Repeat Steps 1-4, B times and compute $\left(\hat{R}_{(1)}^*,...,\hat{R}_{(B)}^*\right)$.
6. Arrange them from the smallest to the largest $\left(\hat{R}_{(1)}^*,...,\hat{R}_{(B)}^*\right)$.
7. Construct $100\left(1-\alpha\right)\%$ (BCI) of R as $\left(\hat{R}_{(\alpha/2)B}^*, \hat{R}_{(1-\alpha/2)B}^*\right)$.

The second method was introduced by Modarres et al. (2006) as follows:

1. Collect (RSS) sample $x_{ij}^*\left(i=1,...,r_1, j=1,...,m_1\right)$ and $y_{kl}^*\left(k=1...r_2, l=1...m_2\right)$.
2. Randomly draw m_1 elements from X_{ij} and m_2 elements from y_{kl}, rank them from smallest to largest as $X_{(1)} \le ...X_{(m_1)}$ and $Y_{(1)} \le ...Y_{(m_2)}$, respectively.
3. Repeat Step 2 for $i=1...m_1$ and $j=1...m_2$ respectively.
4. Repeat Steps 2 and 3 r_1 and r_2 times to obtain samples and obtain the bootstrap estimates of R, say $\hat{R}^*$.

5. Repeat Steps 1-4, B times and compute $\left(\hat{R}^*_{(1)},...,\hat{R}^*_{(B)}\right)$.

6. Arrange them from the smallest to the largest $\left(\hat{R}^*_{(1)},...,\hat{R}^*_{(B)}\right)$.

7. Construct $100(1-\alpha)\%$ (BCI) of R as $\left(\hat{R}^*_{(\alpha/2)B},\hat{R}^*_{(1-\alpha/2)B}\right)$.

5. Simulation Study

In this section, the simulation study is divided into two parts:

Part 1: To compare our proposed estimator of R based on RSS with the traditional estimator of R based on SRS using bias and mean square error (MSE) are given by $Bias(\hat{R})=E(\hat{\hat{R}}-R)$ and $MSE(\hat{R})=E(\hat{R}-R)^2$, respectively. The relative efficiency of the estimator of R is computed as $RE=\dfrac{MSE(R_{MLE,SRS})}{MSE(R_{MLE,RSS})}$. If the vale of relative efficiency larger than 1 indicates that the $R_{MLE,RSS}$ is more efficiency than the $R_{MLE,SRS}$. All computations with performed using Mathematica 11. The simulation study is described through the following steps:

1. Generate 1,000 simple random samples of $x_1,...,x_{n_1}$ and $y_1,...,y_{n_2}$ from Gompertz distribution with samples sizes $(n_1,n_2)=(15,15)$, (15,20), (20,20), (20,25), (25,25), (30,30), (30,40), (40,40), (40,50) and (50,50).

2. Generate 1,000 random samples of $x_{11},...,x_{m_1r_1}$ and $y_{11},...,y_{m_2r_2}$ from Gompertz distribution with set sizes $m_1=m_2=3$, 4, 5 and number of cycles $r_1=r_2=5$, 10.

3. The common parameter c will assumed to be one and $\lambda_1=4$ and $\lambda_2=2,4,6$.

4. The biases, MSE's and relative efficiency are computed.

Part 2: To construct ACI and BCI for R, using part I and asymptotic properties of ML estimator in both cases SRS and RSS. Tables 2 and 3 show the results of simulation study in SRS and RSS respectively as follows:

1. The average confidence lengths based on RSS are less than the corresponding confidence lengths based on SRS.

2. The average confidence lengths decrease when sample sizes increase.

3. The average confidence lengths ACLs are less than the corresponding of BCLs in both cases SRS and RSS.

4. The converge probabilities of ACIs are more than the corresponding The converge probabilities of BCIs in both cases SRS and RSS.

All results show in Table 1. It shows that, the biases and MSE's in RSS case are smaller than the corresponding biases and MSE's in SRS. Also, All relative efficiencies more than 1. So, the estimators of R in RSS are more efficient than the corresponding estimators in SRS. Tables 2 and 3 show that the ACLs of confidence interval based on RSS are significantly shorter than their SRS counterparts. Also, the ACLs of confidence interval of ACI based on RSS are significantly shorter than the ACLs of confidence interval of BCI but the opposite in the case of SRS. Converge probability of ACI in both cases SRS and RSS are mor than the corresponding of BCI. So, we can conclude ACI is better than BCI.

Table 1 Biases, MSE's and RE of R under SRS and RSS when $c=1$, $\lambda_1=4$ and $r_1=r_2=5,10$

			SRS				RSS			
						$r_1=r_2=5$				
λ_2	(n_1,n_2)	(m_1,m_2)	R_{True}	$R_{MLE,SRS}$	Bias	MSE	$R_{MLE,RSS}$	Bias	MSE	RE
2	(15,15)	(3,3)	0.3333	0.3444	0.0111	0.00020	0.3171	-0.0162	0.00010	2.000
	(15,20)	(3,4)		0.4142	0.0809	0.00060	0.3114	-0.0218	0.00040	1.500
	(20,20)	(4,4)		0.3475	0.0142	0.00030	0.3886	0.0053	0.00020	1.500
	(20,25)	(4,5)		0.3941	0.0608	0.00360	0.4183	0.0085	0.00220	1.630
	(25,25)	(5,5)		0.3458	0.1250	0.00010	0.3463	0.0130	0.00010	1.000
4	(15,15)	(3,3)	0.5000	0.4965	−0.0035	0.00001	0.5000	0.0000	0.00001	1.000
	(15,20)	(3,4)		0.5702	0.0702	0.00890	0.5065	0.0065	0.00400	1.225
	(20,20)	(4,4)		0.4975	−0.0025	0.00010	0.5000	0.0001	0.00010	1.000
	(20,25)	(4,5)		0.5531	0.0531	0.00980	0.5758	0.0075	0.00170	1.640
	(25,25)	(5,5)		0.4976	−0.0024	0.00010	0.5000	0.0001	0.00010	1.000
6	(15,15)	(3,3)	0.6000	0.5946	−0.0054	0.00200	0.5591	-0.0060	0.00160	1.250
	(15,20)	(3,4)		0.6600	0.0060	0.0036	0.6052	0.0052	0.00210	1.710
	(20,20)	(4,4)		0.5988	−0.0012	0.01400	0.5987	-0.0013	0.00010	1.400
	(20,25)	(4,5)		0.6473	0.0673	0.00420	0.6594	0.0594	0.00350	1.200
	(25,25)	(5,5)		0.5999	−0.0001	0.00010	0.5879	-0.0120	0.00010	1.000
						$r_1=r_2=10$				
2	(15,15)	(3,3)	0.3333	0.3456	0.0123	0.0015	0.3633	0.0080	0.00090	1.670
	(15,20)	(3,4)		0.4093	0.0760	0.0057	0.3721	0.00388	0.00150	3.800
	(20,20)	(4,4)		0.3449	0.0116	0.0013	0.3631	0.0029	0.00080	1.620
	(20,25)	(4,5)		0.3971	0.0638	0.0040	0.3881	0.0548	0.00300	1.330
	(25,25)	(5,5)		0.3474	0.0141	0.0019	0.2976	−0.0356	0.00120	1.158
4	(15,15)	(3,3)	0.5000	0.4986	−0.0014	0.0002	0.5000	0.0000	0.00010	2.00
	(15,20)	(3,4)		0.5704	0.0704	0.0049	0.5096	0.0066	0.00360	1.360
	(20,20)	(4,4)		0.5014	0.0014	0.0002	0.5000	0.0000	0.00020	1.000
	(20,25)	(4,5)		0.5531	0.0531	0.0088	0.5994	0.0094	0.00780	1.120
	(25,25)	(5,5)		0.4986	−0.0014	0.0001	0.5000	0.0000	0.00010	1.000
6	(15,15)	(3,3)	0.6000	0.5989	−0.0011	0.0040	0.5384	−0.0610	0.00370	1.080
	(15,20)	(3,4)		0.6637	0.0637	0.0040	0.6379	0.0379	0.00140	2.850
	(20,20)	(4,4)		0.5938	−0.0062	0.0009	0.6289	0.0289	0.00080	1.125
	(20,25)	(4,5)		0.6451	0.0451	0.0050	0.6668	0.0668	0.00440	1.136
	(25,25)	(5,5)		0.5210	−0.0079	0.0008	0.5721	−0.0278	0.00070	1.140

Table 2 Average confidence length (ACL) and converge probability (CP) of ACI , BCI-I in SRS case

		ACI		BCI-I	
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 2$					
n_1	n_2	ACL	CP	ACL	CP
15	15	0.3282	0.9500	0.3797	0.8411
15	20	0.3268	0.9424	0.3684	0.8379
20	20	0.3257	0.9374	0.3606	0.8388
20	25	0.3168	0.9330	0.3395	0.8388
25	25	0.3042	0.8798	0.3770	0.8292
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 4$					
15	15	0.4792	0.9500	0.4785	0.8410
15	20	0.4690	0.9460	0.3266	0.8409
20	20	0.4576	0.9401	0.4109	0.8406
20	25	0.4167	0.9341	0.3731	0.8396
25	25	0.4094	0.9030	0.4364	0.8403
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 6$					
15	15	0.3578	0.9500	0.3798	0.8411
15	20	0.3301	0.9500	0.3913	0.8411
20	20	0.3244	0.9085	0.3833	0.8403
20	25	0.3205	0.8500	0.3514	0.8399
25	25	0.3199	0.9000	0.3451	0.8406
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 2$					
30	30	0.3455	0.9500	0.2750	0.8025
30	40	0.3992	0.9400	0.2859	0.7987
40	40	0.3448	0.9300	0.3546	0.8018
40	50	0.3370	0.9250	0.2862	0.7964
50	50	0.3361	0.9200	0.2846	0.7458
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 4$					
30	30	0.4985	0.9080	0.4177	0.8381
30	40	0.4703	0.8862	0.3835	0.8282
40	40	0.4501	0.8899	0.4270	0.8379
40	50	0.4215	0.7434	0.3885	0.8097
50	50	0.4197	0.7321	0.3781	0.8068
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 6$					
30	30	0.5989	0.9411	0.4503	0.8367
30	40	0.6036	0.9302	0.4034	0.8382
40	40	0.5934	0.8707	0.4711	0.8364
40	50	0.5944	0.8669	0.4382	0.8281
50	50	0.5910	0.8716	0.4839	0.7998

Table 3 Average confidence length (ACL) and converge probability (CP) of ACI , BCI-II in RSS case

		ACI		BCI-II	
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 2$					
n_1	n_2	ACL	CP	ACL	CP
15	15	0.1776	0.8852	0.3411	0.8409
15	20	0.1583	0.8939	0.2997	0.8396
20	20	0.1974	0.8948	0.3538	0.8393
20	25	0.2099	0.8963	0.3484	0.8379
25	25	0.1737	0.8966	0.2980	0.8189
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 4$					
15	15	0.2513	0.8950	0.4007	0.8412
15	20	0.2538	0.8964	0.3711	0.8405
20	20	0.2503	0.8986	0.4513	0.8401
20	25	0.2882	0.8990	0.4575	0.8403
25	25	0.2500	0.8999	0.4549	0.8405
$r_1 = r_2 = 5,\ \lambda_1 = 4$ and $\lambda_2 = 6$					
15	15	0.2806	0.8966	0.4733	0.8411
15	20	0.3037	0.8968	0.4520	0.8411
20	20	0.2996	0.8991	0.4263	0.8406
20	25	0.3300	0.8992	0.4553	0.8405
25	25	0.2939	0.8998	0.4181	0.8401
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 2$					
30	30	0.2383	0.7016	0.3101	0.8255
30	40	0.2157	0.8500	0.2764	0.8181
40	40	0.2101	0.8519	0.3050	0.8123
40	50	0.2149	0.8996	0.3023	0.8023
50	50	0.1629	0.9091	0.3036	0.7960
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 4$					
30	30	0.2569	0.8756	0.4153	0.8381
30	40	0.2601	0.8915	0.3984	0.8359
40	40	0.2527	0.8901	0.3999	0.8282
40	50	0.3019	0.8936	0.3743	0.8070
50	50	0.2501	0.8996	0.4186	0.8237
$r_1 = r_2 = 10,\ \lambda_1 = 4$ and $\lambda_2 = 6$					
30	30	0.2747	0.8920	0.4649	0.8395
30	40	0.3247	0.8952	0.4011	0.8388
40	40	0.3167	0.8939	0.4290	0.8397
40	50	0.3357	0.8943	0.4815	0.8383
50	50	0.2861	0.8997	0.4510	0.8075

6. Application

Samawi et al (2009) used the RSS in medicine. He compared bilirubin level between male and female jaundice babies to perform this study. Blood sample must be taken from the sampled babies and tested in a laboratory. Table 4 shows the results of 15 measurements for male and female babies collected by RSS with set size 3 and cycle size 5. Let X and Y represent the response variable for male and female babies, respectively. Then the stress-strength model can be used to decide whether male babies are more likely to experience jaundice.

Table 4 Ranked set sample data of bilirubin level in jaundice babies

Group	Cycle	Rank (1)	Rank (2)	Rank (3)
Male (X)	1	7.50	10.50	7.30
	2	7.50	15.00	8.60
	3	8.90	14.60	13.53
	4	7.00	11.90	15.70
	5	10.24	13.18	18.47
Female (Y)	1	1.20	8.94	15.00
	2	7.50	12.82	10.80
	3	8.00	8.82	10.70
	4	8.90	8.94	14.59
	5	8.53	8.20	18.29

1. Check the validity of Gompertz distribution for given data using Q-Q plots. Figure 1 shows that the Gompertz distribution for the data of bilirubin level in jaundice babies. Also, we used Anderson-Darling and Cramér-von Mises goodness of fit tests the p-value for data set X are 0.79, 0.80 also for data set Y are 0.46 and 0.36, respectively.

2. Compute $MSE\left[\hat{R}_{SRS}^{ML}\right]$, $MSE\left[\hat{R}_{RSS}^{ML}\right]$ and $RE = 2,18;$ so the point estimator of R based on RSS is more efficiency than the point estimator based on SRS.

3. Compute the interval estimator of R in both cases SRS and RSS and put the results in Table 5, which shows that the length of confidence interval in RSS is less than the length of confidence interval in SRS.

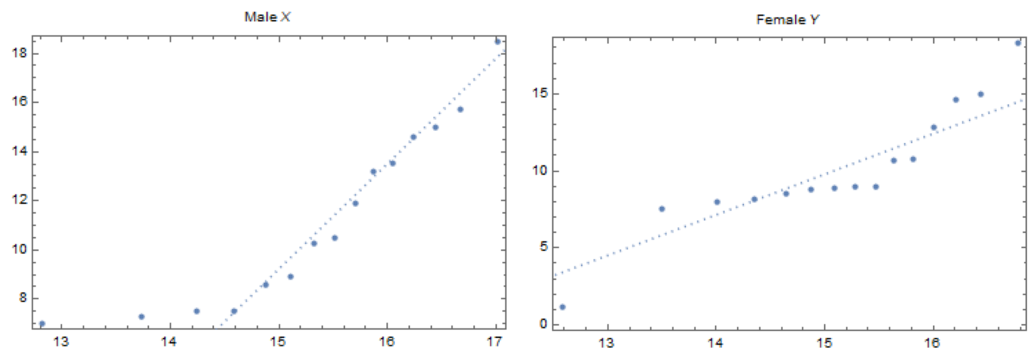

Figure 1 Fitted Gompertz distribution for data of bilirubin level in jaundice babies

Table 5 Interval estimators for data of bilirubin level in jaundice babies

Sampling Technique	Interval Estimator	Lower	Upper	Length
SRS	ACI	0.6320	0.7679	0.3878
	BCI-I	0.2825	0.9670	0.3317
RSS	ACI	0.5229	0.8170	0.2516
	BCI-II	0.1902	0.9705	0.2442

7. Conclusions

In this study, the problem of estimation the reliability system $R = P[Y < X]$ when the stress Y and strength X are independent Gompertz random variables based on RSS. The maximum likelihood estimator of R is derived, ACI and BCI are constructed of R in both cases SRS and RSS. Monte Carlo simulation study is performed to compare between point and interval estimators of R in both cases SRS and RSS. Relative efficiency is used to compare point estimator which verified the maximum likelihood estimator of R based on RSS is more efficient than the maximum likelihood estimator of R based on SRS. Also, in order to interval estimator, we get ACLs of ACI and BCI in RSS case is less than ACLs of ACI and BCI in SRS. Finally, We test the performance of our estimator ion real data. In our future work, we try to estimate the stress strength model in multicomponent case based on RSS.

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