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Moment Properties of Generalized Order Statistics from Exponentiated Generalized Class of Distributions

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Abstract

Cordeiro (2013) introduced and studied the exponentiated generalized class of distributions. The two classes of Lahman's (2013) alternatives is considered as special cases of this generalized class. In this paper a new exponentiated generalized class of distributions is proposed, which is slightly different from one given by Cardeiro et al. (2013). Moment properties of generalized order statistics in terms of recurrence relations are studied. Further, examples based on some specific distributions are discussed. Results for order statistics and record values are deduced from the main result. In the end, characterization results based on recurrence relations and conditional moment are presented.

Keywords: Order statistics, record values, recurrence relations, exponentiated generalized distributions, characterization.

1. Introduction

The concept of generalized order statistics (gos) have been introduced and extensively studied by Kamps (1995). A variety of ordered models of random variables is contained in this concept with different interpretations. Examples of such models are the order statistics (os), record values, sequential order statistics, progressive type II censored order statistics and Pfeifer's records. These models can be effectively applied, e.g., in reliability theory. The common approach makes it possible to define several distributional properties at once.

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}, k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0 \text{ for } 1 \leq i \leq n-1.$$

The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ are called generalized order statistics from a continuous population having cumulative distribution function (CDF) $F(\cdot)$ with probability density function (PDF) $f(\cdot)$, if their joint PDF has the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$ of $\mathbb{R}^n$.

The particular cases of model (1) are given below:

- If $m_i = m = 0, i = 1,2,...,n-1$ and $k=1$, then $\gamma_r = n-r+1, 1 \le r \le n-1$. In this case the model (1) reduces to the joint density of order statistics.

- By choosing $n = m,\ m_i = R_i,\ \ i = 1,2,\ldots,m-1,$ and $k = R_m + 1,$ then $\gamma_r = m - r + 1 + \sum_{i=r}^{m} R_i,$ $1 \le r \le m,$ where R_i is a set of prefixed integer that shows R_i random removal at i^{th} failure from surviving items of an experiment. In this case the model (1) reduces to the joint density based on progressively type-II censored order statistics.

-If $m_i = m \to -1,\ i = 1,2,...,n-1$ and $k = 1,$ then $\gamma_r = 1, 1 \le r \le n-1$. In this case the model (1) reduces to the joint density of upper record values.

-If $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$ and $k = \alpha_n,$ $\alpha \in R^+,$ $i = 1,2,\ldots n-1,$ then $\gamma_r = (n-r+1)\alpha_r,\ 1 \le r \le n-1.$ In this case the model (1) reduces to the joint density of sequential order statistics.

Cordeiro et al. (2013) proposed a new exponentiated class of distributions

$$F(x) = [1-\{1-H(x)\}^{\alpha}]^{\beta},\ x \in \mathbb{R}, \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ are the shape parameters and $H(x)$ is the CDF of base distribution.

Here in this paper, we define another exponentiated generalized class of distributions, which is slightly different from (2) and also discussed by Corderio and de Castro (2011) with name Kumaraswamy G-family. Thus, for the given CDF $H(x)$ of the base distribution, the CDF of new class is given by

$$F(x) = 1-[1-H^{\alpha}(x)]^{\beta};\ \alpha,\ \beta > 0 \tag{3}$$

and the corresponding PDF is

$$f(x) = \alpha\beta[1-H^{\alpha}(x)]^{\beta-1} H^{\alpha-1}(x)h(x);\ \alpha,\ \beta > 0, \tag{4}$$

where $h(x)$ is the PDF of base distribution.

It may be noted that, if $\beta = 1$ in (3), we get the Lehman type I distributions with CDF

$$F(x) = H^{\alpha}(x), \alpha > 0,$$

and if $\alpha = 1$, then (3) reduces to Lehman type II distributions with CDF

$$F(x) = 1-[1-H(x)]^{\beta}, \beta > 0.$$

There is a large volume of works based on the study of recurrence relations between moments of generalized order statistics and characterizations based on these relations. The moments of ordered random schemes assume considerable importance in the statistical literature. Many authors have investigated and derived several recurrence relations and identities satisfied by the single as well as product moments. Khan et al. (1983a,b) studied the recurrence relations and identities for moments of order statistics for some specific distributions. Recurrence relations for the expected values of certain functions of order statistics are considered by Ali and Khan (1997, 1998). Athar and Islam (2004) investigated the relations between expected values of functions of gos. For more detailed survey, one may refer to Malik et al. (1988), Balakrishnan et al. (1988), Arnold et al. (1992), Keseling (1999), Kamps and Cramer (2001), Anwar et al. (2008), Khan et al. (2010), Athar et al. (2012), Khwaja et al. (2012), Khan and Khan (2016), Nayabuddin and Athar (2017), Singh et al. (2018), Zarrin et al. (2019), Athar et al. (2019a,b) and references therein. The organization of the paper is as follows:

In Section 2, single moment of gos for the distributions considered in (3) is presented and some particular cases and examples are also discussed. Section 3 deals with the product moments of gos for the above said distribution, while Section 4 is related to the characterization theorems. In Section 5 concluding remarks are given.

2. Single Moments

Here we may consider two cases:

Case I: $\gamma_i \neq \gamma_j, i, j = 1, 2, ..., n-1, i \neq j$

In view of (1), the pdf of r^{th} *gos* $X(r,n,\tilde{m},k)$ is given as (Kamps and Cramer 2001)

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i - 1}, \tag{5}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $\gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0$, and $a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, 1 \leq i \leq r \leq n.$

Case II: $m_i = m, i = 1, 2, ..., n-1$

The pdf of r^{th} gos $X(r,n,m,k)$ is given as (Kamps 1995)

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{6}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $\gamma_i = k + (n-i)(m+1)$, $h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, m \neq -1 \\ \log\left(\frac{1}{1-x}\right), \qquad m = -1 \end{cases}$

and $g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m \, dt, \ x \in [0,1).$

Before establishing the main result, we shall prove the following lemma.

Lemma 1 *For the exponentiated generalized class of distribution as given in* (3)*, the relation between survival function and PDF is given as*

$$\bar{F}(x) = \left\{ \frac{1}{\alpha\beta\lambda_1(x)} \sum_{l=1}^{\infty} \binom{[\alpha + l - 1]}{l} [\bar{H}(x)]^l \right\} f(x), \tag{7}$$

where $\bar{F}(x) = 1 - F(x)$, $\bar{H}(x)$ *is the survival function of base distribution,* $\lambda_1(x) = \frac{h(x)}{H(x)}$ *is inverse hazard rate function and* $[\alpha + l - 1]$ *is an integer.*

Proof: We have

$$\frac{\bar{F}(x)}{f(x)} = \frac{[1 - H^{\alpha}(x)]^{\beta}}{\alpha\beta[1 - H^{\alpha}(x)]^{\beta-1} H^{\alpha-1}(x) h(x)} = \frac{1}{\alpha\beta} \left\{ \frac{H^{1-\alpha}(x)}{h(x)} - \frac{H(x)}{h(x)} \right\} = \frac{1}{\alpha\beta\lambda_1(x)} \left[\left\{1 - \bar{H}(x)\right\}^{-\alpha} - 1 \right]$$

$$=\frac{1}{\alpha\beta\lambda_1(x)}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}[\bar{H}(x)]^l$$. Hence the result.

Theorem 1 *Suppose Case I be satisfied. For the exponentiated generalized class of distribution as given in* (3) *and* $n\in N, \tilde{m}\in\mathbb{R}, k>0, 1\le r\le n, j=1,2,...$

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l\{X(r,n,\tilde{m},k)\}\right], \tag{8}$$

where $A^l(x)=\frac{x^{j-1}(\bar{H}(x))^l}{\lambda_1(x)}$ and $\lambda_1(x)=\frac{h(x)}{H(x)}$.

Proof: We have, by Athar and Islam (2004),

$$E[\xi\{X(r,n,\tilde{m},k)\}]-E[\xi\{X(r-1,n,\tilde{m},k)\}]=C_{r-2}\int_{-\infty}^{\infty}\xi'(x)\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\,dx.$$

Let $\xi(x)=x^j$, then

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=j\,C_{r-2}\int_{-\infty}^{\infty}x^{j-1}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\,dx.$$

In view of (7), we have

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]$$

$$=\frac{j}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}C_{r-1}\int_{-\infty}^{\infty}\frac{x^{j-1}}{\lambda_1(x)}(\bar{H}(x))^l\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}f(x)dx$$

$$=\frac{j}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}C_{r-1}\int_{-\infty}^{\infty}A^l(x)\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}f(x)dx,$$

which yields (8).

Corollary 1 *Under the condition as stated in Theorem 1 with* $\beta=1$, *we get the relation between single moments of gos for the Lehman type I distributions as*

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{\gamma_r\alpha}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l\{X(r,n,\tilde{m},k)\}\right]. \tag{9}$$

Corollary 2 *Under the condition as stated in Theorem 1 with* $\alpha=1$, *we get the relation between single moments of gos for the Lehman type II distributions as*

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{\gamma_r\beta}E[\phi\{X(r,n,\tilde{m},k)\}]. \tag{10}$$

Proof: Since $\sum_{l=1}^{\infty}A^l(x)=\frac{x^{j-1}}{\lambda_1(x)}\sum_{l=1}^{\infty}(\bar{H}(x))^l=\frac{\bar{H}(x)}{h(x)}x^{j-1}=\phi(x)$. Thus, the relation (10) can be seen.

Corollary 3 *Let* $m_i=m$, $i=1,2,...,n-1$, *then the recurrence relation for single moments of gos for Case II is given by*

$$E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)] = \frac{j}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l\{X(r,n,m,k)\}\right]. \tag{11}$$

Proof: Khan et al. (2006) have shown that for $\gamma_i \neq \gamma_j$ but at $m_i = m;\ i = 1,2,...,n-1$

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}.$$

Therefore, the PDF of $X(r,n,\tilde{m},k)$ given in (5) reduces to (6). Hence the relation (11) can be established by replacing $\tilde{m}$ with m in (8).

Remark 1 Let $m_i = 0; i = 1,2,...,n-1$ and $k = 1$, then recurrence relation for the single moments of order statistics is

$$E(X_{r:n}^j)-E(X_{r-1:n}^j) = \frac{j}{(n-r+1)\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l(X_{r:n})\right]. \tag{12}$$

Remark 2 Let $m_i \to -1; i = 1,2,...,n-1$, then recurrence relation for the single moments of k^{th} upper record values is

$$E(X_{u(r)}^{(k)})^j - E(X_{u(r-1)}^{(k)})^j = \frac{j}{k\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l(X_{u(r)}^{(k)})\right]. \tag{13}$$

2.1. Examples

2.1.1. Exponentiated generalized power function distribution

Let the base distribution is power function distribution with CDF

$$H(x) = \nu^{-p}x^p,\ 0 \le x \le \nu;\ p,\nu > 0$$

and corresponding PDF

$$h(x) = p\nu^{-p}x^{p-1}, 0 \le x \le \nu;\ p,\nu > 0.$$

Therefore, the CDF of exponentiated generalized power function is given by

$$F(x) = 1-[1-\nu^{-\alpha p}x^{\alpha p}]^{\beta};\ 0 \le x \le \nu; \alpha,\beta > 0. \tag{14}$$

Also we have $\lambda_1(x) = \dfrac{h(x)}{H(x)} = \dfrac{p}{x}$ and $A^l(x) = \dfrac{x^{j-1}(1-\nu^{-p}x^p)^l}{(p/x)} = \dfrac{1}{p}\sum_{u=0}^{l}(-1)^u\binom{l}{u}\nu^{-pu}x^{j+pu}$.

Thus, in view of (8), we get

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]$$

$$= \frac{j}{\gamma_r p\alpha\beta}\sum_{l=1}^{\infty}\sum_{u=0}^{l}(-1)^u\binom{[\alpha+l-1]}{l}\binom{l}{u}\nu^{-pu}E[X^{j+pu}(r,n,\tilde{m},k)]. \tag{15}$$

Now at $\alpha = 1$, we have

$$\sum_{l=1}^{\infty}A^l(x) = \phi(x) = \frac{1-H(x)}{h(x)}x^{j-1} = \frac{1}{p}[\nu^p x^{j-p} - x^j].$$

Thus, in view of (10), we get the relation for moments of gos from exponentiated power function distribution as

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{p\beta\gamma_r}\Big[\nu^p E[X^{j-p}(r,n,\tilde{m},k)]-E[X^j(r,n,\tilde{m},k)]\Big]. \quad (16)$$

Futher, at $\beta=1$ with $j=p$, we have the relation for moments of gos from power function distribution

$$E[X^p(r,n,\tilde{m},k)]=\frac{\gamma_r}{\gamma_r+1}E[X^p(r-1,n,\tilde{m},k)]+\frac{\nu^p}{\gamma_r+1} \quad (17)$$

as obtained by Kamps (1995).

2.1.2. Exponentiated generalized Pareto distribution

Let the base distribution is the Pareto distribution with CDF

$$H(x)=1-\lambda^p x^{-p},\ \lambda\le x<\infty;\ \lambda,p>0$$

and corresponding PDF

$$h(x)=p\lambda^p x^{-(p+1)},\lambda\le x<\infty;\ \lambda,p>0.$$

Thus, the CDF of exponentiated generalized Pareto distribution can be obtained as

$$F(x)=1-[1-(1-\lambda^p x^{-p})^{\alpha}]^{\beta};\ \lambda\le x<\infty;\alpha,\beta>0. \quad (18)$$

Now we have $\lambda_1(x)=\frac{h(x)}{H(x)}=\frac{p\lambda^p x^{-(p+1)}}{1-\lambda^p x^{-p}}$ and

$$A^l(x)=\frac{x^{j-1}(\lambda^p x^{-p})^l(1-\lambda^p x^{-p})}{p\lambda^p x^{-(p+1)}}=\frac{1}{p}[\lambda^{p(l-1)}x^{j+p(1-l)}-\lambda^{lp}x^{j-lp}].$$

Therefore, in view of (8), we have

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]$$

$$=\frac{j}{\gamma_r p\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}\lambda^{lp}\left[\frac{1}{\lambda^p}E[X^{j+(1-l)p}(r,n,\tilde{m},k)]-E[X^{j-lp}(r,n,\tilde{m},k)\right].$$

Further, at $\alpha=1$,

$$\sum_{l=1}^{\infty}A^l(x)=\phi(x)=\frac{1-H(x)}{h(x)}x^{j-1}=\frac{x^j}{p}$$

Thus using (10) we get,

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{p\beta\gamma_r}E[X^j(r,n,\tilde{m},k)]$$

or

$$E[X^j(r,n,\tilde{m},k)]=\frac{p\beta\gamma_r}{p\beta\gamma_r-j}E[X^j(r-1,n,\tilde{m},k)]. \quad (19)$$

The above relation (19) is the relation for single moments of gos from exponentiated Pareto distribution.

At $\beta=1$ in (19), we get the relation for single moments of gos from Pareto distribution as

$$E[X^j(r,n,\tilde{m},k)]=\frac{p\gamma_r}{p\gamma_r-j}E[X^j(r-1,n,\tilde{m},k)]. \quad (20)$$

The above relation (20) is also obtained by Athar et al. (2012).

2.1.3. Exponentiated generalized inverse Weibull distribution

Suppose the base distribution is inverse Weibull distribution with CDF

$$H(x) = e^{-\lambda x^{-p}},\ 0 \le x < \infty;\ \lambda, p > 0$$

and corresponding PDF

$$h(x) = \lambda p x^{-(p+1)} e^{-\lambda x^{-p}},\ 0 \le x < \infty;\ \lambda, p > 0.$$

Therefore, the CDF of exponentiated generalized power function is given by

$$F(x) = 1 - [1 - e^{-\alpha\lambda x^{-p}}]^{\beta};\ 0 \le x < \infty; \alpha, \beta > 0. \tag{21}$$

Now we have $\lambda_1(x) = \dfrac{h(x)}{H(x)} = \lambda p x^{-(p+1)}$

and $A^l(x) = \dfrac{x^{j-1}(1-e^{-\lambda x^{-p}})^l}{\lambda p x^{-(p+1)}} = \dfrac{1}{\lambda p} x^{j+p} \sum_{v=0}^{l} (-1)^v \binom{l}{v} e^{-\lambda v x^{-p}}$

$$= \frac{1}{\lambda p} x^{j+p} \sum_{v=0}^{l} (-1)^v \binom{l}{v} \left\{ \sum_{t=0}^{\infty} (-1)^t \frac{(\lambda v x^{-p})^t}{t!} \right\} = \frac{1}{p} \sum_{v=0}^{l} \sum_{t=0}^{\infty} (-1)^{v+t} \binom{l}{v} \frac{v^t}{t!} \lambda^{t-1} x^{j+p(1-t)}$$

Therefore from (8), we get

$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)]$

$$= \frac{j}{p\alpha\beta\gamma_r} \sum_{l=1}^{\infty} \sum_{v=0}^{l} \sum_{t=0}^{\infty} (-1)^{v+t} \binom{[\alpha+l-1]}{l} \binom{l}{v} \frac{v^t}{t!} \lambda^{t-1} E[X^{j+p(1-t)}(r,n,\tilde{m},k)].$$

At $\alpha = 1$, we have

$$\sum_{l=1}^{\infty} A^l(x) = \phi(x) = \frac{1-H(x)}{h(x)} x^{j-1} = \frac{1}{p} \sum_{s=1}^{\infty} \frac{\lambda^{s-1}}{s!} x^{p(1-s)+1}.$$

Now using (10), we get

$$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = \frac{j}{p\beta\gamma_r} \sum_{s=1}^{\infty} \frac{\lambda^{s-1}}{s!} E[X^{j+p(1-s)+1}(r,n,\tilde{m},k)], \tag{22}$$

which is the relation between moments of gos for exponentiated inverse Weibull distribution. Further, at $\beta = 1$ in (22), we get the relation for inverse Weibull distribution.

3. Product Moments

Case I: $\gamma_i \ne \gamma_j;\ i, j = 1, 2, \ldots, n-1, i \ne j$

The joint pdf of $X(r,n,\tilde{m},k) = x$ and $X(s,n,\tilde{m},k) = y$, $1 \le r < s \le n$, is given as (Kamps and Cramer 2001)

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \left[\sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)},\ x < y, \tag{23}$$

where $a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \dfrac{1}{(\gamma_j - \gamma_i)}$, $\quad r+1 \le i \le s \le n.$

Case II: $m_i = m, i = 1, 2, \ldots, n-1$

The joint pdf of $X(r,n,m,k) = x$ and $X(s,n,m,k) = y$, $1 \le r < s \le n$, is given as (Pawlas and Syznal 2001)

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(x)]^{m}\,g_{m}^{r-1}(F(x))$$
$$\times[h_{m}(F(y))-h_{m}(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_{s}-1}f(x)f(y),\ -\infty\le x<y\le\infty. \tag{24}$$

Theorem 2 *Let Case I be satisfied. For the exponentiated generalized class of distribution as given in* (3) *and* $n\in N,\tilde{m}\in\mathbb{R},k>0,1\le r<s\le n,i,j=1,2,...$

$$E[X^{i}(r,n,\tilde{m},k)X^{j}(s,n,\tilde{m},k)]-E[X^{i}(r,n,\tilde{m},k)X^{j}(s-1,n,\tilde{m},k)]$$
$$=\frac{j}{\gamma_{s}\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{l}E\left[B^{t}\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}\right], \tag{25}$$

where $B^{t}(x,y)=\dfrac{x^{i}y^{j-1}(\bar{H}(y))^{t}}{\lambda_{2}(y)}$ *and* $\lambda_{2}(y)=\dfrac{h(y)}{H(y)}.$

Proof: From Athar and Islam (2004), we have

$$E[\xi\{X(r,s,n,\tilde{m},k)\}]-E[\xi\{X(r,s-1,n,\tilde{m},k)\}]$$
$$=C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{\partial}{\partial y}\xi'(x,y)\left[\sum_{c=r+1}^{s}a_{i}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_{i}}\right]\left[\sum_{i=1}^{r}a_{i}(r)\left(\bar{F}(x)\right)^{\gamma_{i}}\right]\frac{f(x)}{\bar{F}(x)}dydx.$$

Let $\xi(x,y)=x^{i}y^{j}$, then

$$E[X^{i}(r,n,\tilde{m},k)X^{j}(s,n,\tilde{m},k)]-E[X^{i}(r,n,\tilde{m},k)X^{j}(s-1,n,\tilde{m},k)]$$
$$=jC_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{i}y^{j-1}\left[\sum_{c=r+1}^{s}a_{i}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_{i}}\right]\left[\sum_{i=1}^{r}a_{i}(r)\left(\bar{F}(x)\right)^{\gamma_{i}}\right]\frac{f(x)}{\bar{F}(x)}dydx.$$

Thus, from (7), we have

$$E[X^{i}(r,n,\tilde{m},k)X^{j}(s,n,\tilde{m},k)]-E[X^{i}(r,n,\tilde{m},k)X^{j}(s-1,n,\tilde{m},k)]$$

$$=\frac{j\,C_{s-1}}{\gamma_{s}\alpha\beta}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{i}y^{j-1}\left[\sum_{c=r+1}^{s}a_{i}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_{i}}\right]\left[\sum_{i=1}^{r}a_{i}(r)\left(\bar{F}(x)\right)^{\gamma_{i}}\right]\times\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}\frac{(\bar{H}(y))^{t}}{\lambda_{2}(y)}dydx.$$
$$=\frac{j}{\gamma_{s}\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty}B^{t}\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}$$
$$\times\left[\sum_{c=r+1}^{s}a_{i}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_{i}}\right]\left[\sum_{i=1}^{r}a_{i}(r)\left(\bar{F}(x)\right)^{\gamma_{i}}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx.$$

This leads to (25).

Corollary 4 *When* $\beta=1,$ *in Theorem 2, we get the relation between product moments of gos for the Lehman type I distributions as*

$$E[X^{i}(r,n,\tilde{m},k)X^{j}(s,n,\tilde{m},k)]-E[X^{i}(r,n,\tilde{m},k)X^{j}(s-1,n,\tilde{m},k)]$$
$$=\frac{j}{\gamma_{s}\alpha}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}E\left[B^{t}\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}\right]. \tag{26}$$

Corollary 5 *When* $\alpha=1$, *in Theorem 2*, *we get the relation between product moments of gos for the Lehman type II distributions as*

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$=\frac{j}{\gamma_s\beta}E\left[\phi\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}\right], \tag{27}$$

where $\phi(x,y)=\dfrac{\bar{H}(y)}{h(y)}x^i y^{j-1}$.

Corollary 6 *Let* $m_i=m$, $i=1,2,...,n-1$, *in* (25), *then the recurrence relation for product moments of gos for Case II is given by*

$$E[X^i(r,n,m,k)X^j(s,n,m,k)]-E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

$$=\frac{j}{\gamma_s\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}E\left[B^t\{X(r,n,m,k).X(s,n,m,k)\}\right]. \tag{28}$$

Proof: Khan et al. (2006) have shown that when $\gamma_i\neq\gamma_j$ but $m_i=m;i,j=1,2,\ldots,n-1$, then

$$a_i^{(r)}(s)=\frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}.$$

Hence the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ given in (23) reduces to (24). Therefore, the relation (30) can be obtained by replacing $\tilde{m}$ with m in (25).

Remark 3 Let $m_i=0;i=1,2,...,n-1$ and $k=1$ in (25), then recurrence relation for the product moments of order statistics is

$$E[X_{r,s:n}^{(i,j)}]-E[X_{r,s-1:n}^{(i,j)}]=\frac{j}{(n-s+1)\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}E\left[B^t(X_{r,s:n})\right]. \tag{29}$$

Remark 4 Let $m_i\to-1;i=1,2,...,n-1$ in (25), then recurrence relation for the product moments of k^{th} upper record values is

$$E[(X_{u(r)}^{(k)})^i(X_{u(s)}^{(k)})^j]-E[(X_{u(r)}^{(k)})^i(X_{u(s-1)}^{(k)})^j]=\frac{j}{k\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}E\left[B^t(X_{u(r,s)}^{(k)})\right]. \tag{30}$$

3.1. Examples

3.3.1. Exponentiated generalized power function distribution

For the given cdf in (14), we have

$$B^t(x,y)=\frac{x^i y^{j-1}(1-\nu^{-p}y^p)^t}{(p/y)}=\frac{1}{p}\sum_{u=0}^{t}(-1)^u\binom{t}{u}\nu^{-pu}x^i y^{j+pu}.$$

Thus, from (25), we get

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$=\frac{j}{\gamma_s p\alpha\beta}\sum_{t=1}^{\infty}\sum_{u=0}^{t}(-1)^t\binom{[\alpha+t-1]}{t}\binom{t}{u}\nu^{-pu}E\left[X^i(r,n,\tilde{m},k)X^{j+pu}(s,n,\tilde{m},k)\right], \tag{31}$$

Now at $\alpha = 1$, we have

$$\sum_{t=1}^{\infty} B^t(x,y) = \phi(x,y) = \frac{1-H(y)}{h(y)} x^i y^{j-1} = \frac{x^i}{p}[\nu^p y^{j-p} - y^j].$$

Thus, from (27), we get the relation for product moments of gos from exponentiated power function distribution as

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$= \frac{j}{p\beta\gamma_s}\Big[\nu^p E[X^i(r,n,\tilde{m},k)X^{j-p}(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]\Big]. \quad (32)$$

Futher, at $\beta = 1$ with $j = p$, we have the relation for product moments of gos from power function distribution as

$$E[X^i(r,n,\tilde{m},k)X^p(s,n,\tilde{m},k)] = \frac{\gamma_s}{\gamma_s+1} E[X^i(r,n,\tilde{m},k)X^p(s-1,n,\tilde{m},k)] + \frac{\nu^p}{\gamma_s+1} E[X^i(r,n,\tilde{m},k)]. \quad (33)$$

3.3.2. Exponentiated generalized Pareto distribution

For the given CDF in (18), we have

$$B^t(x,y) = \frac{x^i y^{j-1}(\lambda^p y^{-p})^t (1-\lambda^p y^{-p})}{p\lambda^p y^{-(p+1)}} = \frac{1}{p}\Big[\lambda^{p(t-1)} x^i y^{j+p(1-t)} - \lambda^{tp} x^i y^{j-tp}\Big].$$

Thus in view of (25), we get

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$= \frac{j}{\gamma_s p\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}\lambda^{tp}\Big[\frac{1}{\lambda^p}E[X^i(r,n,\tilde{m},k)X^{j+(1-t)p}(s,n,\tilde{m},k)]$$

$$-E[X^i(r,n,\tilde{m},k)X^{j-tp}(s,n,\tilde{m},k)]\Big]. \quad (34)$$

Further, at $\alpha = 1$

$$\sum_{t=1}^{\infty} B^t(x,y) = \phi(x,y) = \frac{1-H(y)}{h(y)} x^i y^{j-1} = \frac{x^i y^j}{p}.$$

Thus using (27) we get,

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$= \frac{j}{p\beta\gamma_s} E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]$$

or
$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] = \frac{p\beta\gamma_s}{p\beta\gamma_s - j} E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]. \quad (35)$$

The relation (35) is the relation for product moments of gos from exponentiated Pareto distribution. At $\beta = 1$ in (35), we get the relation for product moments of gos from Pareto distribution as

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] = \frac{p\gamma_s}{p\gamma_s - j} E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]. \quad (36)$$

The above relation (36) is obtained by Athar et al. (2012).

3.3.3. Exponentiated generalized inverse Weibull distribution

For the cdf as given in (21)

$$B^t(x,y)=\frac{x^i y^{j-1}(1-e^{-\lambda y^{-p}})^t}{\lambda p y^{-(p+1)}}=\frac{1}{p}\sum_{v=0}^{t}\sum_{d=0}^{\infty}(-1)^{v+d}\binom{t}{v}\frac{v^d}{d!}\lambda^{d-1}x^{j+p(1-d)}.$$

Therefore from (25), we get

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$=\frac{j}{p\alpha\beta\gamma_s}\sum_{t=1}^{\infty}\sum_{v=0}^{t}\sum_{d=0}^{\infty}(-1)^{v+d}\binom{[\alpha+t-1]}{t}\binom{t}{v}\frac{v^d}{d!}\lambda^{d-1}E[X^i(r,n,\tilde{m},k)X^{j+p(1-d)}(s,n,\tilde{m},k)]. \quad (37)$$

At $\alpha=1$, we have

$$\sum_{t=1}^{\infty}B^t(x,y)=\phi(x,y)=\frac{1-H(y)}{h(y)}x^i y^{j-1}=\frac{1}{p}\sum_{b=1}^{\infty}\frac{\lambda^{b-1}}{b!}x^i y^{p(1-b)+1}.$$

Now by using (27), we get

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$=\frac{j}{p\beta\gamma_s}\sum_{b=1}^{\infty}\frac{\lambda^{b-1}}{b!}E[X^i(r,n,\tilde{m},k)X^{j+p(1-b)+1}(s,n,\tilde{m},k)], \quad (38)$$

which is the relation between moments of gos for exponentiated inverse Weibull distribution. Also at $\beta=1$ in (37), we get the relation for inverse Weibull distribution.

4. Characterization

In this section characterization of exponentiated generalized class of distributions as considered in (3) is presented through recurrence relations for single and product moments of gos as well as through conditional expectation.

Theorem 3 *Fix a positive integer* k *and let* j *be a non-negative integer. A necessary and sufficient condition for a random variable* X *to be distributed with PDF given by* (4) *is that*

$$E[X^j(r,n,\tilde{m},k)]-E[X^j(r-1,n,\tilde{m},k)]=\frac{j}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}E\left[A^l\{X(r,n,\tilde{m},k)\}\right], \quad (39)$$

where $A^l(x)=\dfrac{x^{j-1}(\bar{H}(x))^l}{\lambda_1(x)}$ and $\lambda_1(x)=\dfrac{h(x)}{H(x)}$.

Proof: The necessary part follows from (8). On the other hand, suppose the relation in (39) is satisfied, then on using Athar and Islam (2004) for $\xi(x)=x^j$, we have

$$jC_{r-2}\int_{-\infty}^{\infty}x^{j-1}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}dx=\frac{jC_{r-1}}{\gamma_r\alpha\beta}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}\int_{-\infty}^{\infty}\frac{x^{j-1}}{\lambda_1(x)}[\bar{H}(x)]^l\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}f(x)dx.$$

This implies

$$\frac{j}{\gamma_r\alpha\beta}C_{r-1}\int_{-\infty}^{\infty}x^{j-1}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}\left\{\alpha\beta\bar{F}(x)-\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}\frac{[\bar{H}(x)]^l}{\lambda_1(x)}f(x)\right\}dx=0. \quad (40)$$

Applying the extension of Müntz-Szász theorem (see, for example, Hwang and Lin (1984)) to (40), we get

$$\frac{\bar{F}(x)}{f(x)}=\frac{1}{\alpha\beta\lambda_1(x)}\sum_{l=1}^{\infty}\binom{[\alpha+l-1]}{l}[\bar{H}(x)]^l.$$

Thus, $f(x)$ has the PDF as given in (4). Hence Theorem 3 holds.

Theorem 4 *Suppose $X(r,n,m,k), r=1,2,...n$ be the the r^{th} gos based on CDF and $E(X)$ exists. Then for two consecutive values r and $r+1$, such that $1\le r<r+1\le n$,*

$$E[H^{\alpha}\{X(r+1,n,m,k)\}\,|\,X(r,n,m,k)=x]=\frac{\beta\gamma_{r+1}H^{\alpha}(x)+1}{\beta\gamma_{r+1}+1}. \tag{41}$$

if and only if

$$\bar{F}(x)=[1-H^{\alpha}(x)]^{\beta},-\infty<x<\infty;\alpha,\beta>0. \tag{42}$$

Proof: Khan and Alzaid (2004) have shown that for *gos*

$$E[h(X(s,n,m,k))\,|\,X(r,n,m,k)=x]=a^{*}h(x)+b^{*} \tag{43}$$

if and only if

$$\bar{F}(x)=[ah(x)+b]^{c}, \tag{44}$$

with $a^{*}=\prod_{j=r+1}^{s}\left(\frac{c\gamma_j}{1+c\gamma_j}\right)$ and $b^{*}=-\frac{b}{a}(1-a^{*})$.

Comparing (42) with (44), we get

$$a=-1,\ \ b=1,\ \ c=\beta,\ \ h(x)=H^{\alpha}(x).$$

Thus, the theorem can be proved in view of (43).

Corollary 7 *For the r^{th} order statistics $X_{r:n}, r=1,2,...n$ and under the condition as stated under Theorem 4*

$$E[H^{\alpha}(X_{r+1:n})\,|\,X_{r:n}=x]=\frac{\beta(n-r)H^{\alpha}(x)+1}{\beta(n-r)+1}, \tag{45}$$

and consequently

$$E[H^{\alpha}(X_{n:n})\,|\,X_{n-1:n}=x]=E[H^{\alpha}(X)\,|\,X\ge x]=\frac{\beta}{\beta+1}H^{\alpha}(x)+\frac{1}{\beta+1} \tag{46}$$

if and only if

$$\bar{F}(x)=[1-H^{\alpha}(x)]^{\beta};-\infty<x<\infty,\alpha,\beta>0. \tag{47}$$

Remark 5 The characterization result for adjacent upper records is given as

$$E[H^{\alpha}(X_{u(n)})\,|\,X_{u(n-1)}=x]=E[H^{\alpha}(X)\,|\,X\ge x]=\frac{\beta}{\beta+1}H^{\alpha}(x)+\frac{1}{\beta+1}. \tag{48}$$

Theorem 5 *Fix a positive integer k and let i,j be non-negative integers. A necessary and sufficient condition for a random variables X,Y to be distributed with pdf given by* (4) *is that*

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]-E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$
$$=\frac{j}{\gamma_s\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}E\left[B^t\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}\right], \tag{49}$$

where $B^t(x,y)=\dfrac{x^i y^{j-1}(\bar{H}(y))^t}{\lambda_2(y)}$ *and* $\lambda_2(y)=\dfrac{h(y)}{H(y)}$.

Proof: The necessary part follows from (25). On the other hand, suppose the relation in (49) is satisfied, then on using Athar and Islam (2004) for $\xi(x,y)=x^i y^j$, we have

$$jC_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^i y^{j-1}\left[\sum_{c=r+1}^{s}a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r}a_i(r)\left(\bar{F}(x)\right)^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}dydx$$
$$=\frac{j}{\gamma_s\alpha\beta}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^i y^{j-1}\frac{(\bar{H}(y))^t}{\lambda(y)}\left[\sum_{c=r+1}^{s}a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]$$
$$\times\left[\sum_{i=1}^{r}a_i(r)\left(\bar{F}(x)\right)^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx.$$

This implies

$$\frac{j}{\gamma_s\alpha\beta}C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^i y^{j-1}\left[\sum_{i=1}^{r}a_i(r)\left(\bar{F}(x)\right)^{\gamma_i}\right]\left[\sum_{c=r+1}^{s}a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}$$
$$\times\left\{\alpha\beta-\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}\frac{(\bar{H}(y))^t}{\lambda(y)}\frac{f(y)}{\bar{F}(y)}\right\}dydx=0. \tag{50}$$

Applying the extension of Müntz-Szász theorem (see, for example, Hwang and Lin (1984)) to (50), we get

$$\frac{\bar{F}(y)}{f(y)}=\frac{1}{\alpha\beta\lambda(y)}\sum_{t=1}^{\infty}\binom{[\alpha+t-1]}{t}[\bar{H}(y)]^t.$$

Thus $f(y)$ has the PDF as given in (4). Hence Theorem 5 holds.

5. Conclusions

The moments of ordered random variables and recurrence relations between them have received great attention in the past few years in statistical literature. Recurrence relations reduce the amount of direct computations. The characterization results play an important role in the determination of probability distributions. The several well- known exponentiated distributions can be driven from the considered exponentiated generalized class. The moment properties of some known exponentiated distributions are studied by some authors in literature (See, for example, Khan et al. (2008), Khan and Kumar (2010), Kulshrestha and Kumar (2012), Kumar (2013), Aziz et al. (2013), among several others. The main purpose of this study to unify results based on moments of generalized order statistics for several exponentiated distributions. Since, generalized order statistics is unified approach for several ordered random variables, thus results obtained can be easily deduced for order statistics, record values, sequential order statistics etc.

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