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On the Identified Power Topp-Leone Distribution: Properties, Simulation and Applications

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Abstract

Generally, the identification problem is a serious problem facing many researchers in their empirical models and causes wrong interpretations leading to wrong decisions. In this paper, the identification problem impact is investigated on a new proposed life time model so called the Power Topp-Leone (PTL) distribution, some serious effects of the identification problem are illustrated and solved in PTL distribution. Some mathematical properties are obtained. Parameters estimation of the PTL distribution using maximum likelihood method (MLE) is performed. A simulation study is used to show the impact of ignoring the identification problem and study estimators' behavior, two real data sets are applied to illustrate the distribution flexibility.

Keywords: Identification problem, Topp-Leone distribution, moments, orders statistics, maximum likelihood estimation.

1. Introduction

Lifetime distributions, basically, are used to model the life of an item to study its properties so that generalizing lifetime distributions and increasing its flexibility may provide more useful information resulting more effective conclusions and decisions. The bounded Topp-Leone (TL) distribution, presented by Topp and Leone (1955), for empirical data with J-shaped histogram as powered band tool and automatic calculating machine failures. Many authors have studied the Topp-Leone distribution as Nadarajah and Kotz (2003), Ghitany et al. (2005), Van Dorp and Kotz (2006), Zhou et al. (2006), Kotz and Seier (2007), Nadarajah (2009) and Genç (2012).

The cumulative distribution function (CDF) and probability density function (PDF) of the classical TL distribution (Nadarajah and Kotz 2003) are

$$\underset{TL}{F}(y) = \left[y(2-y) \right]^{\alpha}; 0 < y < 1; \alpha > 0, \tag{1}$$

and

$$\underset{TL}{f}(y) = 2\alpha y^{\alpha-1} (2-y)^{\alpha-1} (1-y). \tag{2}$$

When parameter values cannot be determined or known completely, even if the true distribution $f(x;\cdot)$ is known, this problem is called the identification problem and this distribution is known as a non-identified distribution. Also, any nested distribution by a non-identified distribution is non-identified. Cleary, a parametric distribution is said to be identified if all its parameters values are identified. Imposing constraints on the parameters can solve some problems, that constraints are said to be identifying.

This paper goals to study the effect of the parameters identification problem on the new power Topp-Leone (PTL) distribution, also it aims to find constraints on the parameters to solve these problems. The rest of this paper is organized as follows: In Section 2, the PTL distribution is presented, its special cases are shown and its asymptotes are given. In Section 3, some properties are obtained. In Section 4, the Hazard function is given. In Section 5, the Rényi entropy is obtained. In Section 6, the stress strength model is proposed. In Section 7, order statistics are studied. In Section 8, the MLE method is used in order to estimate the distribution parameters. In Section 9, a simulation study is illustrated. Finally, in Section 10, some applications are used to clarify the flexibility of the identified distribution.

2. The New PTL Distribution

In this section, the PTL distribution is presented, for the first time, as follows: setting $x = y^{\frac{1}{\beta}}$ and substituting it into (1) gives

$$F_{PTL}(x) = x^{\alpha\beta}\left(2 - x^{\beta}\right)^{\alpha};0 < x < 1;\alpha > 0,\beta > 0, \tag{3}$$

one can see that when $\alpha\beta = 1$ the PTL distribution in (3) be non-identified. Basically, to avoid identification problem, in PTL distribution, the join product of $\alpha\beta$ must be constrained as follows

$$F_{PTL}(x) = x^{\alpha\beta}\left(2 - x^{\beta}\right)^{\alpha};0 < x < 1;\alpha > 0,\beta > 0;\alpha\beta \neq 1, \tag{4}$$

differentiating (3) with respect to x yields,

$$f_{PTL}(x) = 2\alpha\beta x^{\alpha\beta-1}\left(1 - x^{\beta}\right)\left(2 - x^{\beta}\right)^{\alpha-1}, \tag{5}$$

when $\beta = 1$, the PTL distribution reduces to TL distribution (Topp and Leone 1955). Some shapes of the density function for the identified PTL distribution are illustrated in Figure 1.

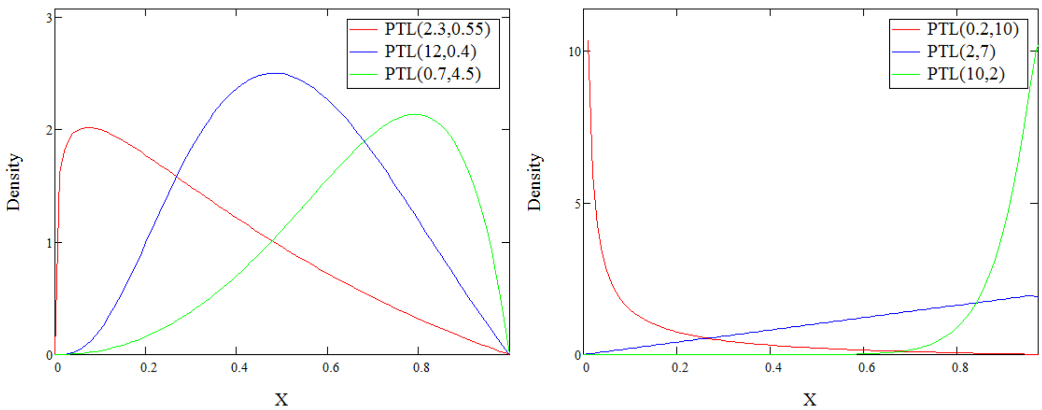

Figure 1 The identified PTL density functions

2.1. Expansions for the CDF and PDF

In this section, expansions for the CDF and PDF of the PTL distribution are given as follows:

2.1.1. An expansion for the CDF

Since,

$$\left(2-z\right)^{c}=\sum_{j=0}^{\infty}(-1)^{j}\,2^{c-j}\binom{c}{j}z^{j}, \tag{1}$$

then, using (6) into (4) gives

$$\underset{PTL}{F}(x)=\sum_{j=0}^{\infty}\left(-1\right)^{j}\,2^{\alpha-j}\binom{\alpha}{j}x^{(\alpha+j)\beta},$$

hence,

$$\underset{PTL}{F}(x)=\sum_{j=0}^{\infty}m_{j}\,x^{(\alpha+j)\beta}, \tag{2}$$

where,

$$m_{j}=\left(-1\right)^{j}\,2^{\alpha-j}\binom{\alpha}{j}.$$

2.1.2. An expansion for the PDF

Using (6) into (5) gives

$$\underset{PTL}{f}(x)=2\alpha\beta\sum_{i=0}^{\infty}w_{i}\,x^{(\alpha+i)\beta-1}\left(1-x^{\beta}\right), \tag{3}$$

where,

$$w_{i}=\left(-1\right)^{i}\,2^{\alpha-1-i}\binom{\alpha-1}{i}.$$

Condition of the expansion for the PDF, since

$$2\alpha\sum_{i=0}^{\infty}w_{i}\int_{0}^{1}\beta x^{(\alpha+i)\beta-1}\left(1-x^{\beta}\right)dx=1,$$

then

$$2\alpha\sum_{i=0}^{\infty}w_{i}B(\alpha+i,2)=1, \tag{4}$$

where $B(\cdot,\cdot)$ is the beta function.

2.2. The asymptotes of the CDF and PDF

In this section, the asymptotes of the CDF and PDF of the PTL distribution are obtained.

2.2.1. The asymptotes of the CDF

First: as x converges to zero, since,

$$\lim_{x\to 0}\left(2-x^{\beta}\right)^{\alpha}=2^{\alpha},$$

then

$$\underset{PTL}{F}(x)\sim 2^{\alpha}\,x^{\alpha\beta}.$$

Second: as x converges to 1, using only first and second terms of binominal expansion leads to

$$\underset{PTL}{F}(x)\sim x^{\alpha\beta}\left(2-\alpha x^{\beta}\right).$$

2.2.2. The asymptotes of the PDF

First: as x converges to zero, since

$$\lim_{x\to 0}\left(1-x^{\beta}\right)=1,\ \lim_{x\to 0}\ \left(2-x^{\beta}\right)^{\alpha-1}=2^{\alpha-1},$$

then

$$\underset{PTL}{f}(x)\sim \alpha\beta\,2^{\alpha}\,x^{\alpha\,\beta-1}.$$

Second: as x converges to 1, using only first and second terms of binominal expansion gives

$$\underset{PTL}{f}(x)\sim 2\alpha\beta x^{\alpha\beta-1}\left(1-x^{\beta}\right)\left(2-(\alpha-1)x^{\beta}\right).$$

3. Some Properties of the PTL Distribution

In this section some properties of the PTL distribution is considered as follows:

3.1. The r^{th} moment

Generally, the r^{th} moment of a continuous random variable X is given by (Johnson et al. 1995) $E(X^r)=\int_x x^r f(x)\,dx.$ Substituting (8) into the last equation yields

$$E(X^r)=2\alpha\sum_{i=0}^{\infty} w_i\int_0^1 \beta x^{(\alpha+i)\beta+r-1}\left(1-x^{\beta}\right)dx,$$

then

$$E(X^r)=2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+r}{\beta},2\right).$$

It can see that, setting $r=0$ leads to

$$E(X^0)=2\alpha\sum_{i=0}^{\infty} w_i B\left((\alpha+i),2\right),$$

substituting (9) into the last equation gives

$$E(X^0)=1.$$

Mean, variance, coefficient of variation (CV), coefficient of skewness (S), and coefficient of kurtosis (K) of the PTL distribution can be given as follows

$$E\left(X\right)=2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+1}{\beta},2\right),$$

$$V(X)=2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+2}{\beta},2\right)-\left[2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+1}{\beta},2\right)\right]^2,$$

$$CV=\frac{\sqrt{2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+2}{\beta},2\right)-\left[2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+1}{\beta},2\right)\right]^2}}{2\alpha\sum_{i=0}^{\infty} w_i B\left(\frac{(\alpha+i)\beta+1}{\beta},2\right)},$$

in the same way, S and K can be given by substituting into the following equations

$$S=\frac{E(X^3)-3E(X)V(X)-\left[E(X)\right]^3}{\left[\sqrt{V(X)}\right]^3},$$

and

$$K=\frac{E(X^4)-4E(X^3)E(X)+6E(X^2)\left[E(X)\right]^2-3\left[E(X)\right]^4}{\left[V(X)\right]^2}.$$

Mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis of the PTL distribution can be calculated, numerically, for different values of α and β in Table 1 for the non-identified case and in Table 2 for the identified case.

Table 1 Mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis of PTL distribution for the non-identified case

Measure	$\beta=0.1$, $\alpha=10$	$\beta=0.3$, $\alpha=3.33$	$\beta=0.5$, $\alpha=2$	$\beta=0.75$, $\alpha=1.33$	$\beta=0.9$, $\alpha=1.11$	$\beta=1.5$, $\alpha=0.66$	$\beta=2.0$, $\alpha=0.5$
Mean	10.618	6.430	5.145	4.469	3.984	2.900	2.600
Variance	352.366	73.058	40.824	28.368	20.264	9.123	7.079
CV	1.767	1.329	1.241	1.191	1.129	1.040	1.023
Skewness	5.826	2.995	1.246	0.936	0.652	0.452	0.235
Kurtosis	36.164	15.612	5.789	3.861	1.696	0.892	0.518

Table 2 Mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis of PTL distribution for the identified case

Measure	$\beta=3$, $\alpha=10$	$\beta=3$, $\alpha=3.33$	$\beta=3$, $\alpha=2$	$\beta=3$, $\alpha=1.33$	$\beta=3$, $\alpha=1.11$	$\beta=3$, $\alpha=0.66$	$\beta=3$, $\alpha=0.5$
Mean	2.596	1.839	1.548	1.462	1.324	1.205	1.152
Variance	2.874	0.889	0.429	0.349	0.190	0.101	0.067
CV	0.653	0.512	0.423	0.404	0.329	0.263	0.224
Skewness	0.414	0.034	−0.311	−0.580	−0.782	−1.224	−1.621
Kurtosis	0.979	0.727	0.597	0.501	0.476	0.340	0.117

From last tables, as β increases and α is fixed, mean, variance, skewness and kurtosis decrease. On the other hand, as α decreases and β is fixed, mean, variance, skewness and kurtosis decrease. An impact of identification problem appears in last tables, as it can be seen that, when α is fixed, the coefficient of variation of the identified distribution is smaller than the coefficient of variation of the non-identified distribution.

3.2. Moment generating function

Basically, the moment generating function (MGF) of a continuous random variable X is given by

$$M_x(t)=E(e^{tx})=\int_x e^{tx}f(x)dx.$$

A first representation can be obtained by substituting (8) into the last equation, yields

$$M_x(t)=2\alpha\beta\sum_{i=0}^{\infty}w_i\int_0^1 e^{tx}x^{(\alpha+i)\beta-1}\left(1-x^{\beta}\right)dx,$$

using binomial expansion for the last equation gives

$$M_x(t) = 2\alpha\beta\sum_{i=0}^{\infty} w_i^* \int_0^1 e^{tx} x^{(\alpha+i+j)\beta-1} dx,$$

where

$$w_i^* = w_i \sum_{j=0}^{1} (-1)^j \binom{1}{j}.$$

Then the following integration (Gradshteyn and Ryzhik 2000) is used

$$\frac{{}_1F_1\left(a; a+1; t\right)}{a} = \int_0^1 e^{tz} z^{a-1} dz, \tag{5}$$

using (10) yields

$$M_x(t) = 2\alpha\sum_{i=0}^{\infty} w_i^* \frac{{}_1F_1\left(\left(\alpha+i+j\right)\beta; \left(\alpha+i+j\right)\beta+1; t\right)}{\left(\alpha+i+j\right)}.$$

Furthermore, the following expansion (Gradshteyn and Ryzhik 2000) is applied

$${}_1F_1(a; b; z) = \sum_{u=0}^{\infty} \frac{\Gamma(a+u)}{\Gamma(b+u)} \frac{z^u}{u!}, \tag{6}$$

using (11) gives

$$M_x(t) = \sum_{i=0}^{\infty} \frac{2\alpha}{\alpha+i+j} w_i^* \sum_{k=0}^{\infty} \frac{\Gamma\left((\alpha+i+j)\beta+k\right)}{\Gamma\left((\alpha+i+j)\beta+1+k\right)} \frac{t^k}{k!},$$

hence,

$$M_x(t) = \sum_{i=0}^{\infty} \frac{2\alpha}{\alpha+i+j} w_i^* \sum_{k=0}^{\infty} \frac{t^k}{k!\left(\left(\alpha+i+j\right)\beta+k\right)}.$$

A second representation for MGF based on the exponential expansion can be obtained as follows: Since $M_x(t) = E(e^{tx})$, then using exponential expansion in the last equation gives

$$M_x(t) = E\left(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!}\right),$$

then

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left(x^k\right).$$

3.3. The quantile function and the median

The definition of the $100u^{\text{th}}$ is

$$u = P(X \le x_u) = F(x_u)\ ;\ x_u > 0,\ 0 < u < 1,$$

substituting (4) into u gives

$$u = x^{\alpha\beta}\left(2 - x^{\beta}\right)^{\alpha}.$$

One can see that the last equation is a nonlinear quantile function needing a numerical solution, with respect to x, to be solved.

3.4. The mean deviation

Basically, the mean deviation about the mean and about the median for a random variable X can be given, respectively, by

$$S_1(x)=\int_x |x-\mu| f(x)dx \text{ and } S_2(x)=\int_x |x-M| f(x)dx,$$

easily, it can be given by, the proof is included in Appendix I of Ali Ahmed (2021),

$$S_1(x)=2\mu F(\mu)-2t(\mu) \text{ and } S_2(x)=\mu-2t(M),$$

where $T(q)=\int_{-\infty}^{q} x f(x)dx$ is the linear incomplete moment.

Substituting (8) into $T(\cdot)$ gives

$$T(q)=2\alpha\sum_{i=0}^{\infty} w_i \int_0^q \beta x^{(\alpha+i)\beta}\left(1-x^{\beta}\right)dx,$$

then,

$$T(q)=2\alpha\sum_{i=0}^{\infty} w_i B\left(q;(\alpha+i)\beta+1,2\right),$$

where $B(\cdot;\cdot,\cdot)$ is the incomplete beta function.

3.5. The mode

The natural logarithm of (5) is

$$\log \underset{PTL}{f}(x)=\log(2\alpha\beta)+(\alpha\beta-1)\log x+\log(1-x^{\beta})+(\alpha-1)\log(2-x^{\beta}),$$

differentiating the last equation, with respect to x, and equating it to zero gives

$$\frac{(\alpha\beta-1)}{x}-\frac{\beta x^{\beta-1}}{1-x^{\beta}}-\frac{(\alpha-1)\beta x^{\beta-1}}{2-x^{\beta}}=0.$$

The last equation is a nonlinear equation which does not have an analytic solution with respect to x, therefore it has to be solved numerically, if x_0 is a root for the last equation then it must be $f''\left[\log(x_0)\right]<0$.

4. The Hazard Function of the PTL Distribution

Basically, the survival function of a random variable X (Meeker and Escobar 1998) can be given by

$$S(x)=1-F(x),$$

substituting (4) into the last equation yields

$$\underset{PTL}{S}(x)=1-x^{\alpha\beta}\left(2-x^{\beta}\right)^{\alpha};0<x<1;\alpha>0,\beta>0;\alpha\beta\neq 1.$$

(12)

On the other hand, the hazard function (Meeker and Escobar 1998) can be given by

$$H(x)=\frac{f(x)}{S(x)},$$

substituting (5) and (12) into last equation yields

$$\underset{PTL}{H}(x)=\frac{2\alpha\beta x^{\alpha\beta-1}\left(1-x^{\beta}\right)\left(2-x^{\beta}\right)^{\alpha-1}}{1-x^{\alpha\beta}\left(2-x^{\beta}\right)^{\alpha}};0<x<1;\alpha>0,\beta>0;\alpha\beta\neq 1.$$

Some shapes of the hazard function for the identified PTL distribution are illustrated in Figure 2.

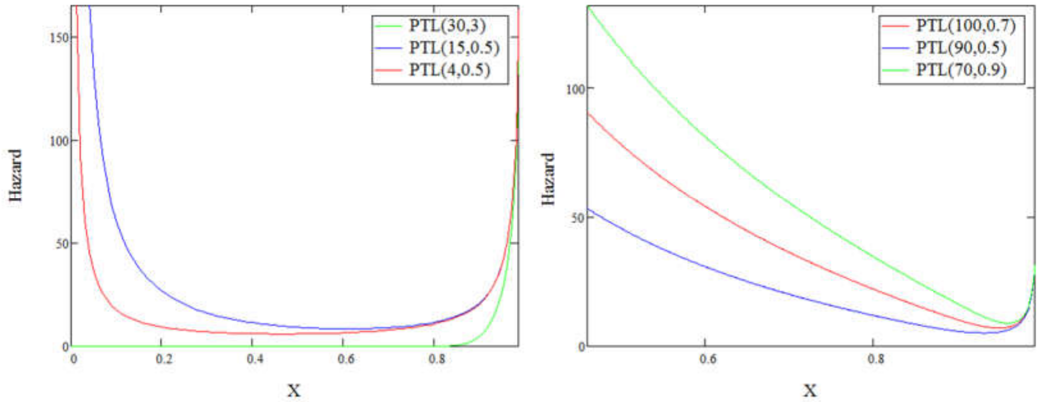

Figure 2 The identified PTL hazard functions

One can see in Figure 2, three types of hazard functions curves of the PTL distribution are described as follows: A decreasing then constant hazard curve, a constant then increasing hazard curve and a decreasing then constant then increasing (bathtub) hazard curve.

5. The Rényi Entropy of the PTL Distribution

The Rényi entropy of a random variable X (Meeker and Escobar 1998) is given by

$$e_R(\rho) = \frac{1}{1-\rho}\log\left[\int_x [f(x)]^\rho \, dx\right],$$

substituting (8) into the last equation gives

$$e_{R_{PTL}}(\rho) = \frac{1}{1-\rho}\log\left\{(2\alpha\beta)^\rho \int_0^1 x^{(\alpha\beta-1)\rho}\left(1-x^\beta\right)^\rho \left\{\sum_{i=0}^{\infty} w_i \left(x^\beta\right)^i\right\}^\rho dx\right\},$$

since

$$\left\{\sum_{i=0}^{\infty} w_i \left(x^\beta\right)^i\right\}^\rho = \sum_{i=0}^{\infty} n_i \left(x^\beta\right)^i, \text{ (Gradshteyn and Ryzhik 2000)}$$

where $n_0 = w_0^\rho$, $n_t = \frac{1}{t\,w_0}\sum_{i=1}^{t}\left(i\rho - t + i\right) w_i\, n_{t-i}; t \ge 1$, then

$$e_{R_{PTL}}(\rho) = \frac{1}{1-\rho}\log\left\{(2\alpha)^\rho \beta^{\rho-1}\sum_{i=0}^{\infty} n_i \int_0^1 \beta x^{(\alpha\beta-1)\rho+i\beta}\left(1-x^\beta\right)^\rho dx\right\},$$

hence,

$$e_{R_{PTL}}(\rho) = \frac{1}{1-\rho}\log\left\{(2\alpha)^\rho \beta^{\rho-1}\sum_{i=0}^{\infty} n_i\, B\left(\frac{(\alpha\beta-1)\rho+i\beta+1}{\beta}, \rho+1\right)\right\}.$$

6. Reliability: The Stress Strength Model of the PTL Distribution

Basically, the stress strength model of a random variable X (Meeker and Escobar 1998) can be given by

$$R = \int_x f_1(x;\lambda_1)F_2(x;\lambda_2)dx,$$

substituting (7) and (8) into the last equation, β is a common parameter, gives

$$R = 2\alpha_1 \sum_{i,j=0}^{\infty} w_i\, m_j \int_0^1 \beta x^{(\alpha_1+i+\alpha_2+j)\beta-1}\left(1-x^{\beta}\right)dx,$$

hence,

$$R = 2\alpha_1 \sum_{i,j=0}^{\infty} w_i\, m_j\, B\left(\alpha_1+i+\alpha_2+j,2\right).$$

7. Order Statistics of the PTL Distribution

The density function $f(x_{u:v})$ of the u^{th} order statistics for $u=1,2,\ldots,v$ from i,i.d. random variables $X_1, X_2,\ldots, X_v$ following the PTL distribution (Arnold et al. 1992) is given by

$$f(x_{u:v}) = \frac{f(x_u)}{B(u,v-u+1)} F(x_u)^{u-1}\left[1-F(x_u)\right]^{v-u},$$

using binomial expansion in the last equation gives

$$f(x_{u:v}) = \frac{f(x_u)}{B(u,v-u+1)} \sum_{k=o}^{v-u} (-1)^k \binom{v-u}{k} \left[F(x_u)\right]^{u+k-1}, \tag{13}$$

substituting (7) and (8) into (13) yields

$$f(x_{u:v}) = \frac{2\alpha\beta \sum_{k=o}^{v-u} \binom{v-u}{k} (-1)^k \sum_{i=0}^{\infty} w_i}{B(u,v-u+1)} x_u^{(\alpha+i)\beta-1+\beta\alpha(u+k-1)}\left(1-x_u^{\beta}\right)\left[\sum_{j=0}^{\infty} m_j x_u^{j\beta}\right]^{u+k-1},$$

since

$$\left[\sum_{j=0}^{\infty} m_j \left(x_u^{\beta}\right)^j\right]^{u+k-1} = \sum_{j=0}^{\infty} p_j \left(x_u^{\beta}\right)^j, \text{ (Gradshteyn and Ryzhik 2000)}$$

where $p_0 = m_0^{u+k-1}, p_n = \frac{1}{n\, m_0} \sum_{j=1}^{n} \left(j(u+k-1)-n+j\right) m_j p_{n-j}; n \geq 1,$ then

$$f(x_{u:v}) = \frac{2\alpha\beta}{B(u,v-u+1)} \sum_{k=o}^{v-u} s_k \sum_{i,j=0}^{\infty} w_i p_j\, x_u^{(\alpha+i+\alpha(u+k-1)+j)\beta-1}\left(1-x_u^{\beta}\right), \tag{7}$$

where $s_k = \binom{v-u}{k}(-1)^k.$

7.1. The r^{th} moment of order statistics

The r^{th} moment of order statistics of the PTL distribution can be got by $E\left(X_{u:v}^r\right) = \int_x x_u^r f(x_u) dx_u$, substituting (14) into the last equation yields

$$E\left(X_{u:v}^r\right) = \frac{2\alpha}{B(u,v-u+1)} \sum_{k=o}^{v-u} s_k \sum_{i,j=0}^{\infty} w_i p_j \int_0^1 \beta x_u^{(\alpha+i+\alpha(u+k-1)+j)\beta+r-1}\left(1-x_u^{\beta}\right)dx_u,$$

then,

$$E\left(X_{u:v}^r\right) = \frac{2\alpha}{B\left(u,v-u+1\right)} \sum_{k=o}^{v-u} s_k \sum_{i,j=0}^{\infty} w_i p_j B\left(\frac{\left(\alpha+i+\alpha\left(u+k-1\right)+j\right)\beta+r}{\beta},2\right).$$

8. Estimation of the PTL Distribution Parameters

Let $X_1, X_2, \ldots, X_n$ be the i.i.d. random variables from the $\mathrm{PTL}(x;\Lambda)$ distribution, where $\Lambda = (\alpha, \beta)$, then the likelihood function for the vector of parameter $\Lambda = (\alpha, \beta)$ (Garthwaite et al. 1995) can be given by

$$L = (2\alpha\beta)^n \prod_{i=1}^{n} x_i^{\alpha\beta - 1} \prod_{i=1}^{n} \left[1 - x_i^{\beta}\right] \prod_{i=1}^{n} \left[2 - x_i^{\beta}\right]^{\alpha - 1},$$

the log likelihood function is given by

$$\ell = n\log(2\alpha\beta) + (\alpha\beta - 1)\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\left(1 - x_i^{\beta}\right) + (\alpha - 1)\sum_{i=1}^{n} \log\left(2 - x_i^{\beta}\right).$$

The score functions for the parameters α and β are given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \beta\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\left(2 - x_i^{\beta}\right), \tag{8}$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \alpha\sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \frac{x_i^{\beta}\left(\log x_i\right)}{1 - x_i^{\beta}} - \sum_{i=1}^{n} \frac{x_i^{\beta}\left(\log x_i\right)}{2 - x_i^{\beta}}. \tag{9}$$

The maximum likelihood estimators (MLEs) of the distribution parameters are obtained by solving the nonlinear likelihood (15) and (16), numerically. Estimating the parameters needs an iterative technique such as a Newton-Raphson algorithm.

Let Λ be the vector of the distribution parameter (α, β), then any element of the 2×2 information matrix $I(\alpha, \beta)$ can be obtained by

$$I_{ij}(\hat{\Lambda}) = E\left[-\frac{\partial^2 \ell(\Lambda)}{\partial \Lambda_i\, \partial \Lambda_j}\Big|_{\Lambda = \hat{\Lambda}}\right],$$

where $I_{ij}^{-1}(\hat{\Lambda})$ is the variance covariance matrix of the unknown parameters, the asymptotic distributions of the PTL parameters is

$$\sqrt{n}\,(\hat{\Lambda}_i - \Lambda_i) \approx N_2(0, I^{-1}(\hat{\Lambda}_i)),\ i = 1, 2,$$

the approximation $100(1-\gamma)\%$ confidence intervals of the unknown parameters using the asymptotic distribution of the $PTL(\alpha, \beta)$ distribution are derived, respectively, as

$$\hat{\Lambda}_i \pm Z_{\gamma/2}\sqrt{I^{-1}(\hat{\Lambda}_i)},\ i = 1, 2,$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)^{\text{th}}$ percentile of a standard normal distribution.

The derivatives in the observed information matrix $I(\alpha, \beta)$ for the unknown parameters are obtained as follows

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2}, \quad \frac{\partial^2 \ell}{\partial \beta^2} = \frac{-n}{\beta^2},$$

and

$$\frac{\partial^2 \ell}{\partial \alpha\, \partial \beta} = \frac{-n}{\beta^2} - \sum_{i=1}^{n} \frac{x_i^{\beta}\left(\log x_i\right)^2}{1 - x_i^{\beta}}\left(1 + \frac{x_i^{\beta}}{1 - x_i^{\beta}}\right) - \sum_{i=1}^{n} \frac{x_i^{\beta}\left(\log x_i\right)^2}{2 - x_i^{\beta}}\left(1 + \frac{x_i^{\beta}}{2 - x_i^{\beta}}\right).$$

9. A Simulation Study

Obtaining MLEs of parameters of the PTL distribution using random numbers is the goal of this section to study the finite sample behavior of the MLEs. The algorithm of obtaining parameters estimates is discussed in the following steps:

Step (1) Generating a random sample $X_1, X_2, \ldots, X_n$ of sizes $n = (10, 20, 30, 50, 100, 300)$ by using the PTL distribution.

Step (2) Two different sets values of the parameters are selected as: Set(1): $(\alpha = 1.5, \beta = 0.5)$, Set(2): $(\alpha = 2, \beta = 0.5)$

Step (3) Solving (15) and (16) by iteration to get MLEs, biases, the root of mean squared error (RMSE) and the Pearson type of parameters estimators, Pearson (1895), of the PTL distribution.

Step (4) Repeating steps, from 1 to 3, 10,000 times.

Random numbers samples are generated using Mathcad package V14.0 where the conjugate gradient iteration method is performed. All results are indicated in tables and included in Appendix II.

From results of the study, in Appendix II, it is clear that, as sample size increases, estimators, biases and RMSEs decrease, as expected. Moreover, the sampling distribution of $\hat{\beta}$ can be the Pearson type IV distribution in all times, the sampling distribution of $\hat{\alpha}$ differs according to sample size. An impact of identification problem appears here where $\hat{\alpha}$ and $\hat{\beta}$, in the identified distribution (Set (1)), can be consistent, specially, when sample size increases, but in the non-identified distribution (Set (2)) they cannot be consistent.

10. Applications

In this application, two real data sets are given to investigate the flexibility of the identified PTL distribution, practically, via Mathematica package version 10. In this examples, different distributions are used as: the PTL distribution, the TL distribution, the Kumaraswamy distribution, Kumaraswamy (1980), the beta distribution, and the Weibull distribution, the following data sets are given from the UK National Physical Laboratory, more details are available at: http://www.npl.co.uk/.

Example 1. The following data represents the lifetime (Hours) of classical lamps for 50 devices: 0.913, 0.786, 0.860, 0.904, 0.971, 0.616, 0.961, 0.789, 0.817, 0.722, 0.956, 0.835, 0.853, 0.692, 0.850, 0.677, 0.898, 0.965, 0.820, 0.964, 0.865, 0.947, 0.798, 0.746, 0.926, 0.709, 0.615, 0.747, 0.931, 0.913, 0.895, 0.745, 0.839, 0.766, 0.690, 0.531, 0.838, 0.846, 0.876, 0.817, 0.719, 0.907, 0.915, 0.879, 0.890, 0.865, 0.869, 0.772, 0.933, 0.875

The results of some goodness of fit measures are in Table 3, the results of likelihood ratio tests are in Table 4, Figure 3 illustrates probability density functions for different distributions having similar skewness and kurtosis, Figure 4 illustrates probability density functions for nested distribution.

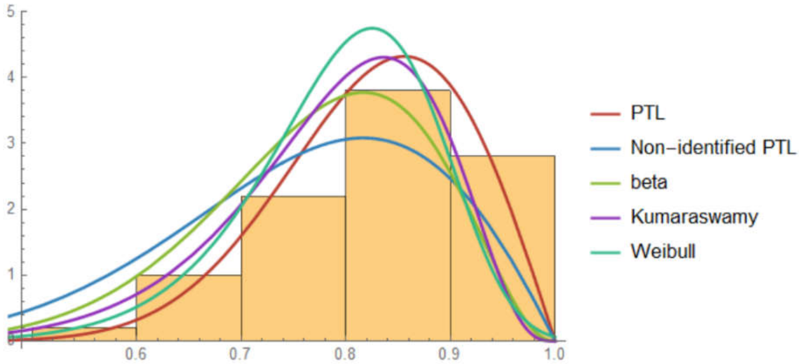

Figure 3 Probability density functions for different distributions having similar skewness and kurtosis

Table 3 The MLE of the parameters and the associated AIC, BIC and CAIC values

Distributions	MLE parameters		Skewness	Kurtosis	KS	p-value	Log Likelihood	AIC	BIC	CAIC
	α	β								
PTL	2.286 (1.054)	4.328 (0.008)	−0.625	3.703	0.110	0.537	48.078	−94.141	−92.229	−94.058
Non-dentified PTL	4.286 (3.845)	0.233 (0.618)	−0.909	2.823	0.470	0.005	26.985	−64.054	−60.230	−63.799
Beta	11.087 (2.269)	3.231 (0.423)	−0.823	3.203	0.287	0.011	39.488	−74.975	−71.151	−74.720
Kumaraswamy	8.666 (1.111)	4.288 (0.022)	−0.643	3.750	0.238	0.015	40.524	−77.049	−73.225	−76.793
Weibull	10.695 (1.238)	0.832 (0.032)	−0.765	3.943	0.246	0.013	39.993	−75.985	−72.161	−75.729

In Table 3, the MLEs of distributions parameters, parameters standard error (SEs), in parentheses, Kolmogorov-Smirnov (KS) test statistic, Akaike information criterion (AIC), the consistent Akaike information criterion (CAIC) and Bayesian information criterion (BIC), Merovcia and Puka (2014), are calculated for every distribution having similar skewness and kurtosis values. The null hypothesis that the data follow the PTL distribution, can be accepted at significance level $\alpha = 0.05$. It is clear that the PTL distribution has the smallest KS, AIC, CAIC, BIC, SEs and the largest log likelihood and p-value, so that, the PTL distribution can be the best fitted distribution to the data compared with other distributions having similar skewness and kurtosis. Cleary, the non-identified PTL distribution has the largest KS, AIC, CAIC, BIC, SEs and the smallest log likelihood and p-value, all of that explain several effects of the identification problem.

Table 4 The log-likelihood function, the likelihood ratio tests statistic and p-value

Distribution	Parameters		ℓ (log-likelihood)	Λ (Likelihood ratio test statistics)	DF (Degrees of freedom)	p-value
	α	β				
TL	24.581 (2.61)	-	40.147	15.862	1	6.813×10^{-5}

*Note that the log likelihood of the PTL distribution = 48.078

In Table 4, based on the likelihood ratio test, the null hypothesis is the data follow the nested model and the alternative is the data follow the full model, where the TL distribution is nested by the PTL distribution, it is clear that, null hypothesis can be rejected at significance level $\alpha = 0.05$.

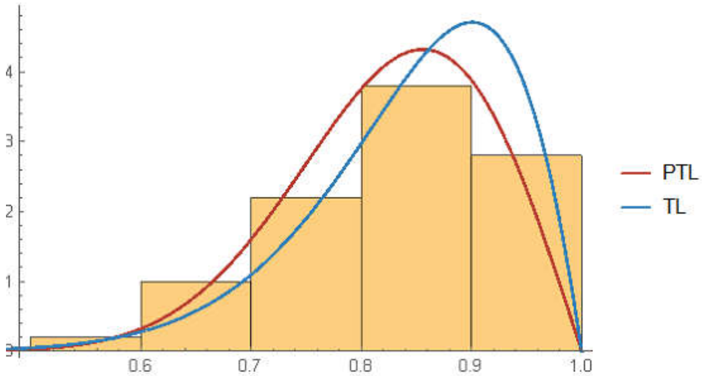

Figure 4 Probability density functions for nested distribution by PTL distribution

Example 2. The following data represents the lifetime (Hours) of classical lamps for 20 devices: 0.618, 0.711, 0.600, 0.553, 0.188, 0.313, 0.176, 0.300, 0.834, 0.004, 0.053, 0.614, 0.263, 0.751, 0.216, 0.416, 0.242, 0.232, 0.241, 0.039.

The results of some goodness of fit measures are in Table 5, the results of likelihood ratio tests are in Table 6, Figure 5 illustrates probability density functions for different distributions having similar skewness and kurtosis, Figure 6 illustrates probability density functions for nested distribution.

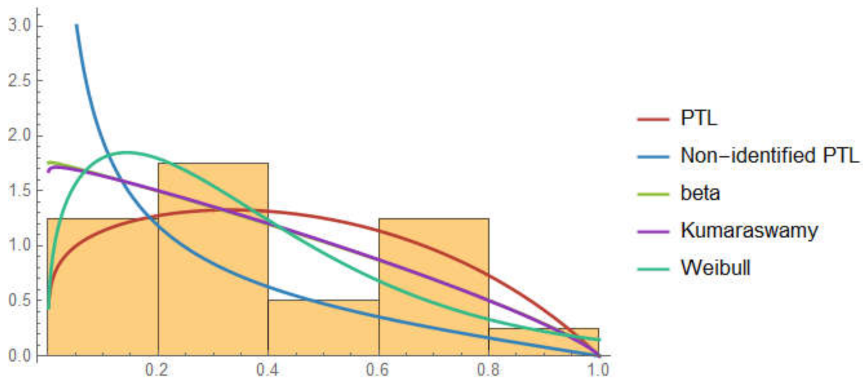

Figure 5 Probability density functions for different distributions having similar skewness and kurtosis

Table 5 The MLE of the parameters and the associated AIC, BIC and CAIC values

Distributions	MLE parameters α	MLE parameters β	Skewness	Kurtosis	KS	p-value	Log Likelihood	AIC	BIC	CAIC
PTL	0.8 (0.298)	2.000 (0.625)	0.754	12.514	0.238	0.104	−1.222	6.444	8.435	7.150
Non-dentified PTL	3.476 (1.249)	2.956 (0.863)	1.375	4.140	0.485	0.000	−5.102	11.029	12.025	11.252
Beta	1.003 (0.284)	1.787 (0.558)	−0.214	1.801	0.344	0.012	−2.434	8.868	10.859	9.573
Kumaraswamy	1.012 (0.261)	1.804 (0.601)	−0.202	1.802	0.342	0.013	−2.288	8.576	10.567	9.282
Weibull	1.341 (0.251)	0.397 (0.068)	3.183	9.692	0.371	0.010	−2.994	9.988	11.980	10.694

In Table 5, the MLEs of distributions parameters, SEs (in parentheses), KS test statistic, AIC, CAIC and BIC are calculated for every distribution having similar skewness and kurtosis values. The null hypothesis that the data follow the PTL distribution, can be accepted at significance level $\alpha = 0.05$. It is clear that the PTL distribution has the smallest KS, AIC, CAIC, BIC, SEs and the largest log likelihood and p-value, so that, the PTL distribution can be the best fitted distribution to the data compared with other distributions having similar skewness and kurtosis. The non-identified PTL distribution has the largest KS, AIC, CAIC, BIC, SEs and the smallest log likelihood and p-value, all of that explain several effects of the identification problem.

Table 6 The log-likelihood function, the likelihood ratio tests statistic and p-value

Distribution	Parameters α	Parameters β	ℓ (log-likelihood)	Λ (Likelihood ratio test statistics)	DF (Degrees of freedom)	p-value
TL	0.4 (0.15)	-	−4.514	6.584	1	0.01

*Note that the log likelihood of the PTL distribution = −1.222

In Table 6, based on the likelihood ratio test, the null hypothesis is the data follow the nested model and the alternative is the data follow the full model, where the TL distribution is nested by the PTL distribution, it is clear that, null hypothesis can be rejected at significance level $\alpha = 0.05$.

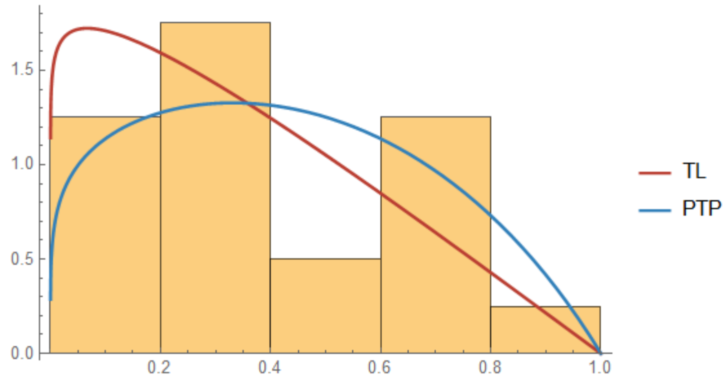

Figure 6 Probability density functions for nested distribution by PTL distribution

11. Conclusions

In empirical models, the impact of the identification problem leads models estimators to be inconsistent causing wrong interpretations making wrong decisions. The power Topp-Leone distribution is a useful lifetime distribution having flexible properties and wide applications but must be constrained to avoid the identification problem. The author encourages researchers, in future, to study more distributions suffering from identification problem and investigate its estimator's behavior in censored and complete samples.

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Appendices

Appendix (I)

The Mean Deviation about Mean and about Median

The mean deviation about mean and about median can be given by, respectively,

$$\delta_1(x) = \int_y |x-\mu| f(x)dx \ \text{ and } \ S_2(x) = \int_x |x-M|\, f(x)dx|x-M|,$$

easily, it can be given by

$$S_1(x) = 2\mu F(\mu) - 2t(\mu) \ \text{ and } \ S_2(x) = \mu - 2t(M),$$

where $T(q) = \int_{-\infty}^{q} x\, f(x)\, dx$ is the linear incomplete moment.

Proof:

First: mean deviation about mean:

Since

$$\delta_1(x) = \int_{-\infty}^{\infty} |x-\mu|\, f(x)dx,$$

then

$$\delta_1(x) = \int_{\mu}^{\infty} (x-\mu)\, f(x)\, dx + \int_{-\infty}^{\mu} (\mu - x)\, f(x)\, dx,$$

hence

$$\delta_1(x) = \int_{\mu}^{\infty} xf(x)dx - \mu \int_{\mu}^{\infty} f(x)dx \ + \int_{-\infty}^{\mu} \mu f(x)dx - \int_{-\infty}^{\mu} xf(x)dx,$$

so

$$\delta_1(x) = \int_{\mu}^{\infty} xf(x)dx - \mu + \mu F(\mu)\mu\mu + \mu F(\mu) - \int_{-\infty}^{\mu} xf(x)dx,$$

adding and subtracting to $\int_{-\infty}^{\mu} xf(x)dx$ gives

$$\delta_1(x) = \int_{\mu}^{\infty} xf(x)dx - \mu + 2\mu F(\mu) - \int_{-\infty}^{\mu} xf(x)dx + \int_{-\infty}^{\mu} xf(x)dx - \int_{-\infty}^{\mu} xf(x)dx,$$

then

$$\delta_1(x) = \int_{-\infty}^{\infty} xf(x)dx - \mu + 2\mu F(x) - 2\int_{-\infty}^{\mu} xf(x)dx,$$

hence

$$\delta_1(x) = 2\mu F(\mu) - 2T(\mu);\ T(\mu) = \int_{-\infty}^{\mu} xf(x)dx.$$

Similarly, the mean deviation about median can be given.

Appendix (II)

Set(1): $(\alpha = 1.5, \beta = 0.5)$

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 1.5$	32.396	30.896	30.924	68.819	68.848	−0.326	I
	$\beta = 0.5$	1.823	1.323		2.017		0.316	IV
20	$\alpha = 1.5$	20.805	19.305	19.313	52.748	52.770	1.639	VI
	$\beta = 0.5$	1.067	0.567		1.530		0.354	IV
30	$\alpha = 1.5$	13.497	11.997	12.001	46.993	47.005	0.318	IV
	$\beta = 0.5$	0.828	0.328		1.086		0.456	IV
50	$\alpha = 1.5$	4.112	2.612	2.614	27.150	27.151	0.432	IV
	$\beta = 0.5$	0.618	0.118		0.249		0.638	IV
100	$\alpha = 1.5$	2.907	1.407	1.407	5.117	5.117	0.360	IV
	$\beta = 0.5$	0.535	0.035		0.039		0.764	IV
300	$\alpha = 1.5$	1.527	0.027	0.027	0.832	0.832	1.019	VI
	$\beta = 0.5$	0.501	0.001		0.009		0.738	IV

Set(2): $(\alpha = 2, \beta = 0.5)$

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 2$	331.187	329.187	329.341	625.334	329.189	−0.383	I
	$\beta = 0.5$	10.582	10.082		5.427		0.305	IV
20	$\alpha = 2$	125.529	123.529	123.761	338.073	338.091	1.572	VI
	$\beta = 0.5$	8.077	7.577		3.442		0.338	VI
30	$\alpha = 2$	72.242	70.242	70.608	244.636	244.643	0.377	IV
	$\beta = 0.5$	7.682	7.182		1.783		0.575	IV
50	$\alpha = 2$	24.053	22.053	22.422	108.187	108.188	0.335	IV
	$\beta = 0.5$	4.553	4.053		0.535		0.207	IV
100	$\alpha = 2$	16.119	14.119	14.435	36.652	36.653	0.848	IV
	$\beta = 0.5$	3.507	3.007		0.282		0.325	IV
300	$\alpha = 2$	5.332	3.332	3.892	9.308	9.309	0.379	IV
	$\beta = 0.5$	2.512	2.012		0.153		0.238	IV