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Expectation Identities of Generalized Order Statistics from the Bass Diffusion Model

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Abstract

In this paper, we have studied the Bass diffusion model to establish some recurrence relations for the single and product moments under generalized order statistics framework. These relations are further reduced in the special cases of generalized order statistics like order statistics and record values. We have also obtained the characterizing results by using conditional expectation and recurrence relation for the single moments of generalized order statistics from the Bass diffusion model.

Keywords: Order statistics, record values, generalized order statistics, Bass diffusion model, recurrence relation and characterization.

1. Introduction

The Bass diffusion model was developed by Bass (1969). It describes the behaviour of a new product or good that moves in the market as an interaction between the present customer and potential customer. This model provides a good platform for forecasting the long term sales behaviour of any product or technology that has launched in the market. Let X is a random variable (rv) defining the time at which the first purchase of new innovative product takes place, follows the Bass diffusion model with parameters α and β. Then the probability density function (pdf) of X is given by

$$f(x)=\frac{((\alpha+\beta)^2/\alpha)e^{-(\alpha+\beta)x}}{[1+(\beta/\alpha)e^{-(\alpha+\beta)x}]^2}, \quad x>0. \tag{1}$$

The parameters α and β determine the shape of the diffusion process and are interpreted as the coefficients of innovation (external influence) and imitation (internal influence), respectively. For practical purposes usually, the values of α and β are considered between 0 and 1, i.e. $0<\alpha\le 1$ and $0<\beta\le 1$. Moreover, this model satisfies the following Riccati differential equation with constant coefficients.

$$\frac{f(x)}{1-F(x)} = \alpha + \beta F(x), \quad x > 0, \tag{2}$$

where $F(x)$ denotes the cumulative distribution function (cdf) of X. After reparametrization, the pdf given in (1) can be written as

$$f(x) = \frac{p(1+q)e^{-px}}{(1+qe^{-px})^2}, \quad x > 0, \tag{3}$$

with $p = \alpha + \beta$ and $q = \beta/\alpha$. The corresponding cdf is given by

$$F(x) = \frac{1-e^{-px}}{1+qe^{-px}}, \quad x > 0,\ p > 0,\ q > -1, \tag{4}$$

and the survival function can be written as

$$\bar{F}(x) = \frac{(1+q)e^{-px}}{1+qe^{-px}}, \quad x > 0,\ p > 0,\ q > -1. \tag{5}$$

Here it may be noted that the function $f(x)$ in (3) is a well-defined pdf $f(x)$, when the parameters take values $p > 0, q > -1$. For $q = 0$, the pdf has given in (3) reduces to the pdf of the exponential distribution.

From (3) and (5), we have obtained the characterizing differential equation for the Bass diffusion model given as

$$f(x) = p[1-F(x)] - \frac{pq}{(1+q)}[1-F(x)]^2. \tag{6}$$

The concept of generalized order statistics (gos) was first introduced by Kamps (1995) which encompasses several models of ascending ordered random variable such as order statistics, record values, the k^{th} record values, Pfeifer record model etc. These models are very useful in many statistical applications and are extensively used in statistical modelling and inference. The models have applications in reliability analysis, the goodness of fit test, detection of outlier's, robust estimates, for detection of the strength of materials, flood frequency analysis etc.

Let $X_1, X_2, \ldots, X_n$ be the continuous and independently identically distributed random variables (rv's) with distribution function $F(x)$ and pdf $f(x)$. The r^{th} gos corresponding to the rv's denoted by $X(r,n,m,k)$. Suppose $X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ \ldots,X(n,n,\tilde{m},k)$ be n gos with parameters $n \in N$, $k \geq 1$, $\tilde{m} = \{m_1, m_2, \ldots, m_{n-1}\} \in \Re$, $M_r = \sum_{j=r}^{n-1} m_j, 1 \leq r \leq n-1$, such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1,2,\ldots,n-1\}$, $n \geq 2$. Their joint density function is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]f(x_n). \tag{7}$$

On the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots < F^{-1}(1)$. The definition of gos has considered two cases.

Case I. $\gamma_i \neq \gamma_j$ and $i,j = 1,2,\ldots,n-1$.

Case II. $m_i = m_j = m$ and $i,j = 1,2,\ldots,n-1$.

Here we have considered only Case I with the following results. The marginal pdf of $X(r,n,m,k)$ is given by

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}[\bar{F}(x))]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)),\quad -\infty<x<\infty. \tag{8}$$

The joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\le r<s\le n$ is

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=&\frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^{m}f(x)g_m^{r-1}(F(x))\\ &\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y),\quad y>x, \end{aligned} \tag{9}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\ \gamma_r=k+(n-r)(m+1),\ h_m(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1}, & m\ne -1\\ -\ln\ (1-x), & m=-1,\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0),\quad x\in[0,1).$$

By choosing the appropriate values of parameters of gos, we obtain the models of gos such as order statistics ($k=1,\ m=0$ i.e. $\gamma_r=n-r+1$), the k^{th} record values ($m=-1,\ \gamma_r=k$), sequential order statistics ($\gamma_r=(n-r+1)\tau_r;\ \tau_1,\tau_2,....,\tau_n>0$), order statistics with non-integral sample size ($\gamma_r=\alpha+r-1;\alpha>0$), Pfeifer's record values ($\gamma_r=\beta_r,\ \beta_1,\beta_2,...,\beta_n>0$).

The literature reveals that the concept of gos have been used by various authors such as Pawlas and Szynal (2001), Ahmad and Fawzy (2003), Bieniek and Szynal (2003), AL-Hussaini (2005), Khan et al. (2007), Ahmad (2007, 2008), Abdul-Moniem (2012), Anwar and Khan (2018), Gupta et al. (2018), Singh et al. (2018), Khan (2018) and Gupta and Anwar (2019).

Recurrence relations are very useful in reducing the amount of time and steps involved in the computation of moments. The results have obtained in this paper are the generalization of Pushkarna et al. (2013).

The rest of the paper is organized as follows. In Section 2, we have obtained the recurrence relations for the single moments of gos from the Bass diffusion model and these relations also discussed for order statistics and record values. In Section 3, recurrence relations are presented for product moments of gos from the Bass diffusion model. In the final section, the characterizing results are discussed.

2. Recurrence Relations for Single Moments

Theorem 2.1 *Let X be a continuous rv follows the Bass diffusion model as given in (3). For* $1\le r<n,\ j=1,2,...,$ *the following recurrence relation is satisfied*

$$\begin{aligned} E\left[X^{j}(r,n,m,k)\right]=&\frac{p\gamma_r}{(j+1)}\left\{E\left[X^{j+1}(r,n,m,k)\right]-E\left[X^{j+1}(r-1,n,m,k)\right]\right\}\\ &-\frac{pqK}{(1+q)(j+1)}\left\{E\left[X^{j+1}(r,n,m,k+1)\right]-E\left[X^{j+1}(r-1,n,m,k+1)\right]\right\}, \end{aligned} \tag{10}$$

where $K=\dfrac{(\gamma_r+1)C_{r-1}}{C_{r-1}^{(k+1,m)}}.$

Proof: From (8), we have

$$E\left[X^j(r,n,m,k)\right]=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{11}$$

Using (6) and (11), we get

$$E\left[X^j(r,n,m,k)\right]=\frac{pC_{r-1}}{(r-1)!}I_1(x)-\frac{pq}{(1+q)}\frac{C_{r-1}}{(r-1)!}I_2(x), \tag{12}$$

where

$$I_1(x)=\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r}\,g_m^{r-1}(F(x))\,dx \text{ and } I_2(x)=\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r+1}\,g_m^{r-1}(F(x))\,dx.$$

Simplifying the above integrations on integrating by parts treating x^j for integration and rest of the integrand for differentiation, we get

$$I_1(x)=\frac{\gamma_r}{(j+1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))\,dx-\frac{(r-1)}{(j+1)}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))\,dx, \tag{13}$$

$$I_2(x)=\frac{\gamma_r+1}{(j+1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r}f(x)g_m^{r-1}(F(x))\,dx-\frac{(r-1)}{(j+1)}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m+1}f(x)g_m^{r-2}(F(x))\,dx. \tag{14}$$

Now substituting (13) and (14) in (12), we get

$$\begin{aligned}E\left[X^j(r,n,m,k)\right]&=\frac{p\gamma_r}{(j+1)}\left\{E\left[X^{j+1}(r,n,m,k)\right]-E\left[X^{j+1}(r-1,n,m,k)\right]\right\}\\&-\frac{pq(\gamma_r+1)}{(1+q)(j+1)}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r}f(x)g_m^{r-1}(F(x))\,dx\\&+\frac{pq}{(1+q)(j+1)}\frac{C_{r-1}}{(r-2)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m+1}f(x)g_m^{r-2}(F(x))\,dx.\end{aligned} \tag{15}$$

After simplification (15), we obtain result given in (10).

Special Cases

(1) Putting $k=1$ and $m=0$ in (10), the recurrence relation given in (10) reduces to the recurrence relation for single moments of order statistics from the Bass diffusion model

$$\begin{aligned}E\left[X_{r:n}^j\right]&=\frac{p(n-r+1)}{(j+1)}\left\{E\left[X_{r:n}^{j+1}\right]-E\left[X_{r-1:n}^{j+1}\right]\right\}\\&-\frac{pq(n-r+1)(n-r+2)}{(1+q)(j+1)(n+1)}\left\{E\left[X_{r:n+1}^{j+1}\right]-E\left[X_{r-1:n+1}^{j+1}\right]\right\}.\end{aligned}$$

The above relation obtained by Pushkarna et al. (2013).

(2) Putting $m=-1$ in (10), the recurrence relation given in (10) reduced to the recurrence relation for single moments of the k^{th} record values from the Bass diffusion model

$$\begin{aligned}E[(X_{U(r)}^{(k)})^j]&=\frac{pk}{(j+1)}\{E[(X_{U(r)}^{(k)})^{j+1}]-E[(X_{U(r-1)}^{(k)})^{j+1}]\}\\&-\frac{pqk^r}{(1+q)(j+1)(k+1)^{r-1}}\{E[(X_{U(r)}^{(k+1)})^{j+1}]-E[(X_{U(r-1)}^{(k+1)})^{j+1}]\}.\end{aligned}$$

3. Recurrence Relation for Product Moment

Theorem 3.1 *Let* X *is a continuous rv follows the Bass diffusion model as given in (3). For* $1 \le r < s \le n-1$, *and* $i, j = 1,2,....,$ *the following recurrence relation satisfied*

$$E[X^i(r,n,m,k)X^j(s,n,m,k)]$$
$$= \frac{p\gamma_s}{(j+1)}\{E[X^i(r,n,m,k)X^{j+1}(s,n,m,k)] - E[X^i(r,n,m,k)X^{j+1}(s-1,n,m,k)]\}$$
$$-\frac{pqK^*}{(1+q)(j+1)}\{E[X^i(r,n,m,k)X^{j+1}(s,n,m,k+1)] - E[X^i(r,n,m,k)X^{j+1}(s-1,n,m,k+1)]\}, \quad (16)$$

where $K^* = \frac{(\gamma_s+1)C_{s-1}}{C_{s-1}^{(k+1,m)}}.$

Proof: From (9), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^j [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y). \quad (17)$$

Using (6) and (17), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{pC_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i [F(x)]^m f(x) g_m^{r-1}(F(x)) I_1(x)dx - \frac{pqC_{s-1}}{(1+q)(r-1)!(s-r-1)!}\int_0^\infty x^i [F(x)]^m f(x) g_m^{r-1}(F(x)) I_2(x)dx, \quad (18)$$

where

$$I_1(x) = \int_x^\infty y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s} dy,$$

and

$$I_2(x) = \int_x^\infty y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s+1} dy.$$

To simplifying the above integrations on integrating by parts treating x^j for integration and rest of the integrand for differentiation, we get

$$I_1(x) = \frac{\gamma_s}{(j+1)}\int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(y)dy - \frac{(s-r-1)}{(j+1)}\int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_s+m} f(y)dy, \quad (19)$$

$$I_2(x) = \frac{(\gamma_s+1)}{(j+1)}\int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s} f(y)dy - \frac{(s-r-1)}{(j+1)}\int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_s+m+1} f(y)dy. \quad (20)$$

Substituting (19) and (20) in (18), we get

$$E[X^i(r,n,m,k)X^j(s,n,m,k)]$$
$$=\frac{p\gamma_s}{(j+1)}\{E[X^i(r,n,m,k)X^{j+1}(s,n,m,k)]-E[X^i(r,n,m,k)X^{j+1}(s-1,n,m,k)]\}$$
$$-\frac{pq(\gamma_s+1)C_{s-1}}{(1+q)(j+1)(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j+1}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s^{(k+1,m)}-1}f(y)dydx \tag{21}$$
$$+\frac{pqC_{s-1}}{(1+q)(j+1)(r-1)!(s-r-2)!}\int_0^\infty\int_x^\infty x^i y^{j+1}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_{s-1}^{(k+1,m)}}f(y)dydx.$$

After simplification (21), we obtain the required result.

Special Cases

(1) Putting $k=1$ and $m=0$ in (16), the recurrence relation given in (16) reduces to the recurrence relations for product moments of order statistics from the Bass diffusion model

$$E[X_{r:n}^i X_{s:n}^j]=\frac{p\gamma_s}{(j+1)}\{E[X_{r:n}^i X_{s:n}^{j+1}]-E[X_{r:n}^i X_{s-1}^{j+1}]\}-\frac{pq(n-s+1)(n-s+2)}{(1+q)(j+1)(n+1)}$$
$$\times\{E[X_{r:n}^i X_{s:n+1}^j]-E[X_{r:n}^i X_{s-1:n+1}^{j+1}]\}.$$

As obtained by Pushkarna et al. (2013).

(2) Putting $m=-1$ in (16), the recurrence relation given in (16) reduces to the recurrence relations for product moments of $k-$th record values from the Bass diffusion model

$$E\left[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j\right]=\frac{pk}{(j+1)}\left\{E\left[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^{j+1}\right]-E\left[(X_{U(r)}^{(k)})^i(X_{U(s-1)}^{(k)})^{j+1}\right]\right\}$$
$$-\frac{pqk^s}{(1+q)(j+1)(k+1)^{s-1}}\left\{E\left[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k+1)})^{j+1}\right]-E\left[(X_{U(r)}^{(k)})^i(X_{U(s-1)}^{(k+1)})^{j+1}\right]\right\}.$$

4. Characterization

In this section, we have discussed the characterization results of the Bass diffusion model using conditional expectation and recurrence relations for single moments of gos.

The conditional pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\le r<s\le n$ gos is given by

$$f_{X(s,n,m,k)|\ X(r,n,m,k}(y/x)$$
$$=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}\frac{[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}}f(y),\ x<y,\ m\neq-1. \tag{22}$$

Theorem 4.1 *Let X be a continuous rv with cdf $F(x)$ and pdf $f(x)$ as given in (3) and (4), then*

$$E[e^{pX(s,n,m,k)}/X(r,n,m,k)]=e^{px}A_{s/r}(x)+B_{s/r}(x), \tag{23}$$

where

$$A_{s/r}(x)=\prod_{l=1}^{s-r}\frac{\gamma_{r+l}}{(\gamma_{r+l}-1)}\ \ and\ \ B_{s/r}(x)=q(A_{s/r}-1).$$

If and only if

$$F(x)=\frac{1-e^{-px}}{1+qe^{-px}},\quad x>0,p>0,q>0.$$

Proof: From (22), we have

$$E[e^{pX(s,n,m,k)}/X(r,n,m,k)]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \times\int_x^\infty e^{py}\left[1-\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{24}$$

Substituting $z=\frac{\bar{F}(y)}{\bar{F}(x)}$ in (24), we get

$$E[e^{pX(s,n,m,k)}/X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^1(e^{px}+q)z^{\gamma_s-2}(1-z^{m+1})^{s-r-1}dz -q\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^1 z^{\gamma_s-1}(1-z^{m+1})^{s-r-1}dz. \tag{25}$$

Substituting $z^{m+1}=t$ in (25), we get

$$E[e^{pX(s,n,m,k)}/X(r,n,m,k)=x]=\frac{C_{s-1}(e^{px}+p)}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\frac{\Gamma\left(\frac{k-1}{(m+1)}+n-s\right)\Gamma(s-r)}{\Gamma\left(\frac{k-1}{(m+1)}+n-r\right)} -q\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\frac{\Gamma\left(\frac{k}{(m+1)}+n-s\right)\Gamma(s-r)}{\Gamma\left(\frac{k}{(m+1)}+n-r\right)}.$$

$$E[e^{pX(s,n,m,k)}/X(r,n,m,k)=x]=\frac{C_{s-1}(e^{px}+q)}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\int_0^1 t^{\frac{k-1}{(m+1)}+n-s-1}(1-t)^{s-r-1}dt -q\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\int_0^1 t^{\frac{k}{(m+1)}+n-s-1}(1-t)^{s-r-1}dt, \tag{26}$$

After simplification (26), we get the necessary part. For sufficient part, from (23) and (24) we have

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}}\int_x^\infty e^{py}[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy=[\bar{F}(x)]^{\gamma_{r+1}}D_{s/r}(x), \tag{27}$$

where $D_{s/r}(x)=e^{px}A_{s/r}(x)+B_{s/r}(x).$

Now differentiating (27) with respect to x, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}}\int_x^\infty e^{py}[h_m(F(y))-h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_s-1}f(y)dy =-\gamma_{r+1}[\bar{F}(x)]^{\gamma_{r+1}-1}f(x)D_{s/r}(x)+[\bar{F}(x)]^{\gamma_{r+1}}D'_{s/r}(x)$$

$$-\gamma_{r+1}[\bar{F}(x)]^{\gamma_{r+2}+m}f(x)D_{s/r+1}(x)=-\gamma_{r+1}[\bar{F}(x)]^{\gamma_{r+1}-1}f(x)D_{s/r}(x)+[\bar{F}(x)]^{\gamma_{r+1}}D'_{s/r}(x),$$

$$\frac{f(x)}{\bar{F}(x)}=-\frac{D'_{s/r}(x)}{\gamma_{r+1}[D_{s/r+1}-D_{s/r}]}. \tag{28}$$

After simplification (28), we get

$$f(x)=\frac{p\bar{F}(x)}{1+qe^{-px}} \text{ and } F(x)=\frac{1-e^{-px}}{1+qe^{-px}},\quad x>0, p>0, q>-1.$$

Hence the sufficient part is proved.

Remark: For $m=-1,$ we have obtained the characterizing result of the Bass diffusion model for $k-$th record values

$$E[e^{pX_{U(s)}^{(k)}}/X_{U(r)}^{(k)}]=e^{px}A_{s/r}(x)+B_{s/r}(x),\ k>1,$$

where

$$A_{s/r}(x)=\left(\frac{k}{k-1}\right)^{s-r} \text{ and } B_{s/r}(x)=q(A_{s/r}-1).$$

Theorem 4.2 *Let X be a continuous rv as given in (3) and (4), then the following relation is satisfied*

$$\begin{aligned}E\left[X^{j}(r,n,m,k)\right]&=\frac{p\gamma_r}{(j+1)}\left\{E[X^{j+1}(r,n,m,k)]-E[X^{j+1}(r-1,n,m,k)]\right\}\\&-\frac{pqK}{(1+q)(j+1)}\left\{E\left[X^{j+1}(r,n,m,k+1)\right]-E\left[X^{j+1}(r-1,n,m,k+1)\right]\right\},\end{aligned} \tag{29}$$

if and only if

$$F(x)=\frac{1-e^{-px}}{1+qe^{-px}},\quad x>0, p>0, q>-1.$$

Proof: The necessary part immediately follows from Theorem 2.1. Now, if the relation in (29) is satisfied, then using (8) in (29), we get

$$\begin{aligned}&\frac{(j+1)C_{r-1}}{(r-1)!}\int_0^\infty x^{j}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\&=\frac{p\gamma_r C_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\&-\frac{p\gamma_r C_{r-2}}{(r-2)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx\\&-\frac{pq}{(1+q)}\frac{(\gamma_r+1)C_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r}f(x)g_m^{r-1}(F(x))dx\\&+\frac{pq}{(1+q)}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m+1}f(x)g_m^{r-2}(F(x))dx.\end{aligned} \tag{30}$$

Equation (30) can be rewritten as

$$\frac{(j+1)C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$=\frac{pC_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}g_m^{r-1}(F(x))\{-\frac{d}{dx}[\bar{F}(x)]^{\gamma_r}\}dx$$
$$-\frac{p\gamma_r C_{r-2}}{(r-2)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx \tag{31}$$
$$-\frac{pq}{(1+q)}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+1}g_m^{r-1}(F(x))\{-\frac{d}{dx}[\bar{F}(x)]^{\gamma_r+1}\}dx$$
$$-\frac{pq}{(1+q)}\frac{C_{r-1}}{(r-2)!}\int_0^\infty x^{j+1}[\bar{F}(x)]^{\gamma_r+m+1}f(x)g_m^{r-1}(F(x))dx.$$

Now integrating the first and third terms of R.H.S of (31) and after simplification, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\{f(x)-p\bar{F}(x)+\frac{pq}{(1+q)}[\bar{F}(x)]^2\}=0. \tag{32}$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin 1984) in (32), we get

$$f(x)-p\bar{F}(x)+\frac{pq}{(1+q)}[\bar{F}(x)]^2=0 \text{ and } f(x)=p\bar{F}(x)-\frac{pq}{(1+q)}[\bar{F}(x)]^2.$$

This is the characterizing differential of the Bass diffusion model. Hence this proves that

$$F(x)=\frac{1-e^{-px}}{1+qe^{-px}}, \quad x>0, p>0, q>-1.$$

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