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Marshall-Olkin Sujatha Distribution and Its Applications

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Abstract

In this paper, we introduced a two-parameter heavy-tailed, monotone non-increasing hazard rate distribution, and its regression model called the Marshall-Olkin Sujatha (MOS) distribution for life processes. This study extends the Sujatha distribution using the Marshall-Olkin method and offers a more flexible model for survival data. Some of its useful statistical properties such as the survival rate function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, probability generating function, moment generating function, characteristic function, stochastic ordering, Shannon, and Rényi entropies, heavy-tail property, and order statistics are derived. The study adopted the method of maximum likelihood estimation to estimate the parameters of the proposed model. Simulation studies are carried out to examine the flexibility behavior of the proposed model. The numerical applications and usefulness of the proposed lifetime model are investigated using two real-life data sets. The results obtained show that the proposed model yields the best goodness of fit to all the data sets.

Keywords: Marshall-Olkin family of distributions, MOS regression model, order statistics, quantile function, entropies.

1. Introduction

Survival and reliability analysis are very important areas in statistics. The survival or failure behavior of a system can be considered as a stochastic process or a random variable due to the changes from one system to another as a result of the nature of the system. Recently, to investigate the survival or failure rate of a system, numerous one parameter classical survival models have been proposed in the literature. This includes the Lindley (1958) one parameter Lindley distribution. Shanker (2015a, 2016c, 2016d) proposed one parameter Akash distribution, Aradhana distribution, and a thin-tailed non-decreasing one parameter Sujatha distribution. Shanker and Shukla (2017) proposed one parameter Ishita distribution. Shukla (2018) proposed one parameter Pranav distribution. Odom and Ijomah (2019) proposed one parameter Odoma distribution among others. Unfortunately, these distributions may not provide a good fit to some real-life situations with high kurtosis, non-increasing rate, and heavy tail. Thus, extending such distributions yield better flexibility to real-life situations.

In statistical literature, the extension of the classical or baseline one parameter survival models has been introduced using a various approach such as Marshall and Olkin (1997) introduced an approach based on the cumulative and probability densities function of a random variable. Gupta and Kundu (1999) introduced the exponentiation approach. Patil and Rao (1978) introduced sized-biased approach, and Mahdavi and Kundu (2017) proposed a method for generating new distributions called the alpha power transformation among others. Adopting many of these approaches to introduce more flexible distributions, researchers such as Pogány et al. (2015) proposed Marshall–Olkin exponential Weibull distribution, George and Thobias (2017) proposed Marshall-Olkin Kumaraswamy distribution, Ikechukwu et al. (2020) proposed A three parameter shifted exponential distribution, Eghwerido et al. (2020) proposed the Gompertz alpha power inverted exponential distribution, Eghwerido et al. (2021) proposed the alpha power Gompertz distribution, Agu, and Onwukwe (2019) proposed modified Laplace distribution, AbuJarad et al. (2020) investigated Bayesian reliability analysis of the Marshall and Olkin models, Agu and Runyi (2018) studied the goodness of fit test for normal distribution, Khaleel et al. (2020) proposed Marshall-Olkin exponential Gompertz distribution among others. All these researchers derived some useful statistical properties of their models and investigated the flexibility using real-life data sets. Their results have shown that their proposed models outperformed the baseline distributions.

Though, despite the easy in parameter estimation of the one-parameter distributions, the distributions may lack flexibility on its property to model real life situations that may not follow any classical distribution. Hence, the demand for more flexible distributions is increasing and to respond to this demand, it is very important to derive a flexible statistical model that can offer more flexibility for the failure or survival behavior of a system. Thus, the motivation behind this study is to propose a heavy-tailed, monotone non-increasing hazard rate, high kurtosis distribution that can offer more flexibility for failure or survival behavior of a system.

Therefore, this paper adopted the Marshall-Olkin approach to propose a two-parameter heavy-tailed, monotone non-decreasing distribution called the Marshall-Olkin Sujatha (MOS) distribution as an extension of the Sujatha distribution and introduced its regression model.

The rest of this paper is structured as follows: Section 2 presents the Marshall-Olkin family of distributions, Section 2.1 presents the Sujatha distribution, Section 2.2 presents the proposed Marshall-Olkin Sujatha distribution, section 2.3 presents the linear representation of the proposed model, Section 3 presents the statistical properties of the proposed model, section 4 presents the maximum likelihood estimation of the proposed model, Section 4.1 presents the Quantile function of the proposed model, Section 4.2 presents the simulation studies, Section 5 presents the regression model of the proposed distribution, section 6 presents the numerical applications of the proposed model, and finally, Section 7 presents the conclusions.

2. The Marshall-Olkin Family of Distributions

Marshall and Olkin (1997) proposed a method for improving the flexibility of a family of distributions. According to Marshall and Olkin, for a given cumulative density function (cdf) and probability density function (pdf) of a baseline survival model defined the cdf of the Marshall-Olkin family of distributions as

$$Q(x,\beta)=\frac{F(x)}{1-(1-\beta)F(x)}, \tag{1}$$

where $-\infty < x < \infty, \beta > 0.$ $F(x)$ and $f(x)$ are the cdf and pdf of the baseline distribution. The corresponding pdf is defined as

$$q(x,\beta)=\frac{\beta f(x)}{[1-(1-\beta)F(x)]^{2}}, \tag{2}$$

where $-\infty<x<\infty,\ \beta>0.$

2.1. The Sujatha distribution

Shanker (2016d) defined the cdf of the one parameter Sujatha distribution as

$$F(x,\theta)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^{2}+\theta+2}\right]e^{-\theta x}, \tag{3}$$

where $x>0,\ \theta>0.$ The corresponding pdf is given as

$$f(x,\theta)=\frac{\theta^{3}}{\theta^{2}+\theta+2}\left(1+x+x^{2}\right)e^{-\theta x}. \tag{4}$$

2.2. The Marshall-Olkin Sujatha distribution

This section presents the proposed two parameters class of Sujatha distribution. Let X be a random variable of the Marshall-Olkin Sujatha (MOS) distribution. Substituting (3) into (1), we obtained the cdf of the proposed MOS model given as

$$Q(x;\beta,\theta)=\frac{(\theta^{2}+\theta+2)-\left[(\theta^{2}+\theta+2)+\theta x(\theta x+\theta+2)\right]e^{-\theta x}}{(\theta^{2}+\theta+2)-(1-\beta)\left[(\theta^{2}+\theta+2)+\theta x(\theta x+\theta+2)\right]e^{-\theta x}}, \tag{5}$$

where $-\infty<x<\infty,\ \beta>0,\ \theta>0.$ The corresponding pdf of the proposed MOS model is given as

$$q(x;\beta,\theta)=\frac{\theta^{3}\beta e^{-\theta x}(\theta^{2}+\theta+2)(1+x+x^{2})}{\left[(\theta^{2}+\theta+2)-(1-\beta)\left[(\theta^{2}+\theta+2)+\theta x(\theta x+\theta+2)\right]e^{-\theta x}\right]^{2}}, \tag{6}$$

where $-\infty<x<\infty,\ \beta>0,\ \theta>0.$ The asymptotic behavior of the MOS distribution can be investigated using (5) for $x\to 0,\ Q(x;\beta,\theta)=0$ and $x\to\infty, Q(x;\beta,\theta)=1.$ Hence, the MOS distribution has a valid pdf. Figures 1 and 2 presents the cdf and pdf plots of the MOS distribution for different parameters values.

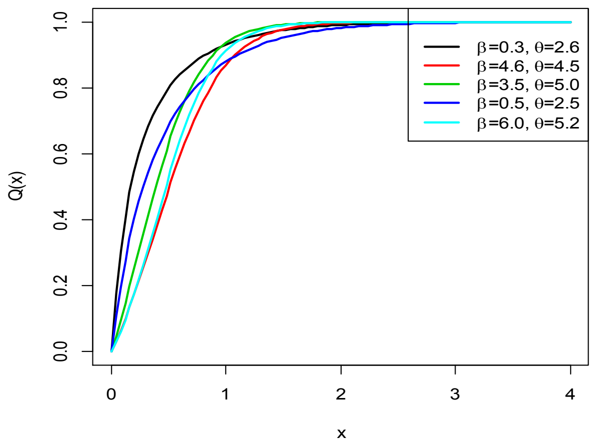

Figure 1 The cdf plots of the MOS distribution for different parameters values

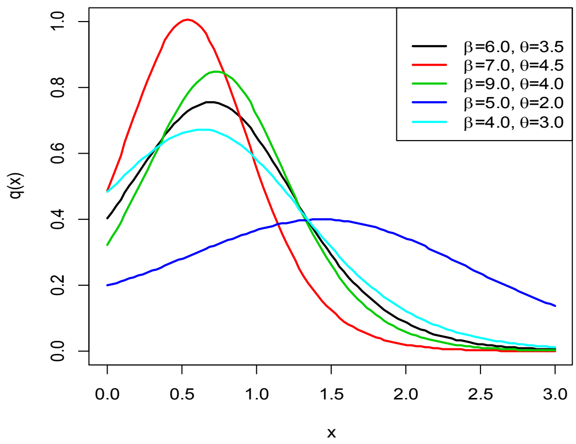

Figure 2 The pdf plots of the MOS distribution for different parameters values

2.3. The linear representation

The linear representation of the MOS distribution is examined in this section to facilitate and to make tractable the properties of the MOS distribution. Let the binomial expansion of $\left(a-x\right)^{-n}=\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k a^{-(n+k)}$. Then,

$$\left[\left(\theta^2+\theta+2\right)-\left(1-\beta\right)\left(\left(\theta^2+\theta+2\right)+\theta x\left(\theta x+\theta+2\right)\right)e^{-\theta x}\right]^{-2}=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}p_{kij}x^{i+j}e^{-\theta kx},$$

where $p_{kij}=\binom{k}{i}\binom{i}{j}\binom{k+1}{k}(-1)^k\left(\theta+2\right)^{i-j}\left(1-\beta\right)^k\theta^{i+j}\left(\theta^2+\theta+2\right)^{-(i+2)}$. Then, the pdf of the MOS density can be expressed as

$$q(x;\beta,\theta)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)x^{i+j}e^{-\theta x(k+1)}\left(x^2+x+1\right), \tag{7}$$

Equation (7) is the linear representation of (6).

3. Some Statistical Properties

This section derived some statistical properties of the MOS distribution such as the survival rate function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, generating function, moment generating function, characteristic function, stochastic ordering, Shannon and Rényi entropies, heavy-tail property, and order statistics.

3.1. The survival rate function

This section presents the survival rate function of the proposed MOS distribution. Let X be a random variable with MOS cdf and pdf defined in (5) and (6) respectively, and then the survival rate function of the MOS distribution is given as

$$S(x;\beta,\theta)=\frac{\beta e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}{(\theta^2+\theta+2)-(1-\beta)e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}.$$

3.2. The hazard rate function

The corresponding hazard rate function of a random variable X with cdf and pdf of the MOS distribution is given as

$$H(x;\beta,\theta)=\frac{\theta^3 e^{-\theta x}(1+x+x^2)}{(\theta^2+\theta+2)-(1-\beta)e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}.$$

Figures 3 and 4 present the survival and hazard rate functions of the proposed MOS distribution for different parameter values. The hazard plots indicated that the MOS model has a monotone non-increasing function.

3.3. The reversed hazard rate function

The reversed rate hazard function of a random variable X with the cdf and pdf of the MOS distribution is given as

$$Rv_f(x;\beta,\theta)=\frac{\theta^3\beta e^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{\left\{(\theta^2+\theta+2)-e^{-\theta x}\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}\right\}\times\left\{(\theta^2+\theta+2)-(1-\beta)e^{-\theta x}\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}\right\}}.$$

3.4. The cumulative hazard rate function

The cumulative hazard rate function of a random variable X with the cdf and pdf of the MOS is given as

$$ch_f(x;\beta,\theta)=-\ln\left[\frac{\beta e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}{(\theta^2+\theta+2)-(1-\beta)e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}\right].$$

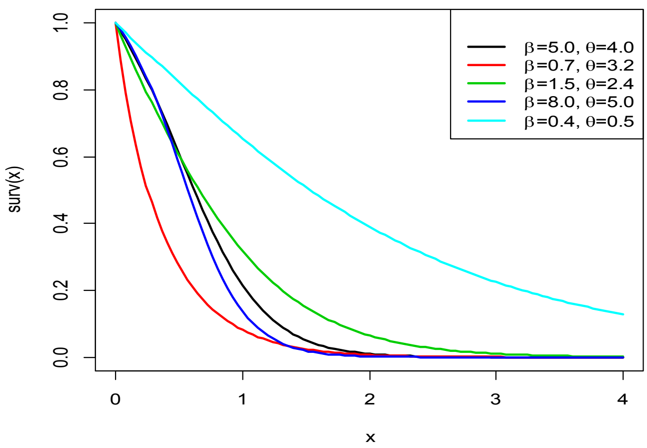

Figure 3 The survival rate plot of the MOS distribution for different parameters values

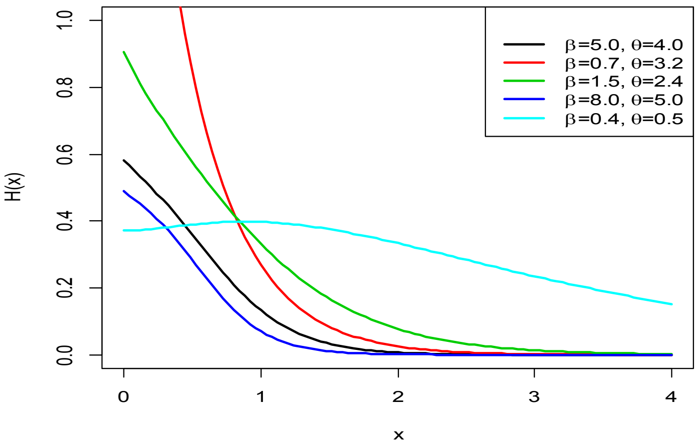

Figure 4 The hazard rate function plot of the MOS distribution for different parameters values

3.5. Generating functions

The probability generating function (pgf) of the MOS density is defined as

$$M(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\frac{(\log t)^p}{p!}\int_0^{\infty} x^{i+j+p}\left(x^2+x+1\right)e^{-\theta x(k+1)}dx. \tag{8}$$

On simplifying (8), we have the pgf as

$$M(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\frac{(\log t)^p}{p!}\begin{pmatrix}\left(\theta(k+1)\right)^{-(p+i+j+2)}\Gamma(p+i+j+3)\\ +\left(\theta(k+1)\right)^{-(p+i+j+1)}\Gamma(p+i+j+2)\\ +\left(\theta(k+1)\right)^{-(p+i+j)}\Gamma(p+i+j+1)\end{pmatrix}.$$

3.6. Moment generating functions (mgf)

The MOS mgf for a random variable X is defined as

$$M_X(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\int_0^{\infty} x^{i+j}\left(x^2+x+1\right)e^{-\theta x(k+1)+tx}\,dx.$$

On simplifying, we have

$$M_X(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\begin{pmatrix}\left(\theta(k+1)-t\right)^{-(i+j+2)}\Gamma(i+j+3)\\ +\left(\theta(k+1)-t\right)^{-(i+j+1)}\Gamma(i+j+2)\\ +\left(\theta(k+1)-t\right)^{-(i+j)}\Gamma(i+j+1)\end{pmatrix}.$$

The r^{th} moment of the MOS distribution is defined as

$$M_X(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\begin{pmatrix}\left(\theta(k+1)+r\right)^{-(i+j+2)}\Gamma(i+r+j+3)\\ +\left(\theta(k+1)+r\right)^{-(i+j+1)}\Gamma(i+r+j+2)\\ +\left(\theta(k+1)+r\right)^{-(i+j)}\Gamma(i+r+j+1)\end{pmatrix}.$$

3.7. The characteristic function

Let X be a random variable with the cdf and pdf of the MOS distribution. The characteristic function of X is defined as

$$\phi_X(t) = E(e^{ztx}) = \sum_{r=0}^{\infty} \frac{(zt)^r}{r!} E(X^r),$$

where $E(X^r)$ is the r^{th} of the MOS distribution. Thus, the characteristic function of the MOS distribution is defined as

$$\phi_X(t) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} \frac{p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)(zt)^r}{r!} \begin{pmatrix} \left(\theta(k+1)+r\right)^{-(i+j+2)}\Gamma(i+r+j+3) \\ +\left(\theta(k+1)+r\right)^{-(i+j+1)}\Gamma(i+r+j+2) \\ +\left(\theta(k+1)+r\right)^{-(i+j)}\Gamma(i+r+j+1) \end{pmatrix}.$$

3.8. Stochastic orderings

The comparative study of the behavior of continuous random variables can be investigated using stochastic ordering. Consider X and Y to be random variables. X is said to be smaller than Y in Shanthikumar (1994) if the following conditions hold.

$$X \le_L Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_m Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

Stochastic order $(X \le_{st} Y)$ if $Q_X(x) \ge Q_y(Y)$ for all x, hazard rate order $(X \le_{hr} Y)$ if $q_x(x) \ge q_y(y)$ for all x, mean residual life order $(X \le_m Y)$ if $m_x(x) \le m_y(y)$ for all x, likelihood ratio order $(X \le_L Y)$ if $\dfrac{q_x(x)}{q_x(x)}$ decreases in x.

Theorem 1. *Let X and $Y \sim$ Agu-F-D with θ_1, θ_2 and β_1, β_2 respectively. If $\theta_1 \ge \theta_2$ and $\beta_1 \ge \beta_2$, then $X \le_L Y$. Hence, $X \le_{hr} Y$, $X \le_m Y$ and $X \le_{st} Y$.*

Proof: The MOS distribution will be ordered based on the strongest likelihood ordering as established in Shanthikumar (1994).

$$\frac{q(x,\theta_1,\beta_1)}{q(x,\theta_2,\beta_2)} = \frac{\dfrac{\theta_1^3\beta_1 e^{-\theta_1 x}(\theta_1^2+\theta_1+2)(1+x+x^2)}{\left[(\theta_1^2+\theta_1+2)-(1-\beta_1)\left[(\theta_1^2+\theta_1+2)+\theta_1 x(\theta_1 x+\theta_1+2)\right]e^{-\theta_1 x}\right]^2}}{\dfrac{\theta_2^3\beta_2 e^{-\theta_2 x}(\theta_2^2+\theta_2+2)(1+x+x^2)}{\left[(\theta_2^2+\theta_2+2)-(1-\beta_2)\left[(\theta_2^2+\theta_2+2)+\theta_2 x(\theta_2 x+\theta_2+2)\right]e^{-\theta_2 x}\right]^2}}.$$

$$\frac{d}{dx}\ln\frac{q_x(x,\theta_1,\beta_1)}{q_y(x,\theta_2,\beta_2)} = \frac{2(1-\beta_1)\left[\theta_1^2+2x(\theta_1^2+\theta_1)\right]}{\left[(\theta_1^2+\theta_2+2)-(1-\beta_1)\left[(\theta_1^2+\theta_1+2)+\theta_1 x(\theta_1 x+\theta_1+2)\right]e^{-\theta_1 x}\right]^2}$$
$$-\frac{2(1-\beta_2)\left[\theta_2^2+2x(\theta_2^2+\theta_2)\right]}{\left[(\theta_2^2+\theta_2+2)-(1-\beta_2)\left[(\theta_2^2+\theta_2+2)+\theta_2 x(\theta_2 x+\theta_2+2)\right]e^{-\theta_2 x}\right]^2}+\theta_1-\theta_2.$$

Therefore, for $\theta_1 < \theta_2$ and $\beta_1, \beta_2 > 1$, $\frac{d}{dx}\ln\frac{q_x\left(x,\theta_1,\beta_1\right)}{q_y\left(x,\theta_2,\beta_2\right)} < 0 \Rightarrow X \leq_L Y$, for $\theta_1 > \theta_2$ and $\beta_1, \beta_2 < 1$, $\frac{d}{dx}\ln\frac{q_x\left(x,\theta_1,\beta_1\right)}{q_y\left(x,\theta_2,\beta_2\right)} > 0$, and if $\theta_1 = \theta_2$ and $\beta_1, \beta_2 = 1$, $\frac{d}{dx}\ln\frac{q_x\left(x,\theta_1,\beta_1\right)}{q_y\left(x,\theta_2,\beta_2\right)} = 0$. Thus, $X \leq_{hr} Y$, $X \leq_m Y$ and $X \leq_{st} Y$.

3.9. Entropies

Let X be a continuous random variable with the pdf defined in (7). The entropy of X deals with the measure of uncertainty or spread of X. Rényi (1961) defined the entropy of X as

$$R_E = \frac{1}{1-a}\log\left\{\lim_{c\to\infty}\int_0^c \left[q(x;\beta,\theta)\right]^a dx\right\}, a > 0, a \neq 0. \tag{9}$$

Substituting (7) into (9), we have that

$$R_E = \frac{1}{1-a}\log\left[\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\right]^a \left\{\lim_{c\to\infty}\int_0^c \left[x^{i+j}\left(x^2+x+1\right)e^{-\theta x(k+1)+tx}\right]^a dx\right\}.$$

Thus, the Rényi entropy of the MOS distribution can be expressed as

$$R_E = \frac{1}{1-a}\log\left\{\left[\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\right]^a \begin{pmatrix} \left(\theta a\left(k+1\right)+r\right)^{-a(i+j+2)}\Gamma\left(i+r+j+3\right) \\ +\left(\theta a\left(k+1\right)+r\right)^{-a(i+j+1)}\Gamma\left(i+r+j+2\right) \\ +\left(\theta a\left(k+1\right)+r\right)^{-a(i+j)}\Gamma\left(i+r+j+1\right) \end{pmatrix}\right\}.$$

3.10. The Shannon entropy

Shannon (1948) defined the Shannon entropy of a continuous random variable X with a pdf $q(x;\beta,\theta)$ as

$$SH_\gamma = -E\left\{\log\left[q(x;\beta,\theta)\right]\right\}. \tag{10}$$

Thus, the Shannon entropy of the MOS distribution based on Equation (10) can be expressed as

$$SH_\gamma = -E\left\{\log\left[\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} p_{kij}\theta^3\beta\left(\theta^2+\theta+2\right)\begin{pmatrix} \left(\theta\left(k+1\right)+r\right)^{-(i+j+2)}\Gamma\left(i+r+j+3\right) \\ +\left(\theta\left(k+1\right)+r\right)^{-(i+j+1)}\Gamma\left(i+r+j+2\right) \\ +\left(\theta\left(k+1\right)+r\right)^{-(i+j)}\Gamma\left(i+r+j+1\right) \end{pmatrix}\right]\right\}.$$

3.11. The heavy-tailed property

Supposing X is a continuous random variable with a probability density function $q(x;\beta,\theta)$ defined in (6). By definition, $q(x;\beta,\theta)$ is said to be heavy-tailed if and only if

$$\lim_{x\to\infty}\sup\left[q(x;\beta,\theta)\right]e^{tx} = \infty; \quad \forall t > 0. \tag{11}$$

Substituting (6) into (11), we obtain

$$\lim_{x\to\infty}\sup\left[q(x;\beta,\theta)\right]e^{tx} = \lim_{x\to\infty}\sup\left[\frac{\theta^3\beta e^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{\left[(\theta^2+\theta+2)-(1-\beta)\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}e^{-\theta x}\right]^2}\right]e^{tx}.$$

Thus,

$$\lim_{x\to\infty}\sup\left[q(x;\beta,\theta)\right]e^{tx} = \lim_{x\to\infty}\sup\left[\frac{\theta^3\beta(\theta^2+\theta+2)(1+x+x^2)}{\left[(\theta^2+\theta+2)-(1-\beta)\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}\right]^2}\right]e^{x(\theta+t)} = \infty.$$

Therefore, the pdf defined in (6) is a heavy-tailed density function.

3.12. Order statistics

Let $x_1, x_2, ..., x_n$ be a random sample of size n of the MOS distribution with pdf $q(x;\beta,\theta)$ and cdf $Q(x;\beta,\theta)$. Then, the order statistics of $X_1 \le X_2 \le ... \le X_n$ for X random variable is defined as

$$\mho_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}q(x)\left[Q(x)\right]^{j-1}\left[1-Q(x)\right]^{n-j}. \tag{12}$$

The pdf of the minimum order statistic of the MOS distribution is obtained by setting $j=1$ in (12) such that

$$\mho_{1:n}(x) = nq(x)\left[1-Q(x)\right]^{n-1}.$$

Thus, the minimum order statistic of the MOS distribution can be expressed as

$$\mho_{1:n}(x;\beta,\theta) = \frac{n\theta^3\beta e^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{\left[(\theta^2+\theta+2)-(1-\beta)\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}e^{-\theta x}\right]^2} \times\left[\frac{\beta e^{-\theta x}\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]}{(\theta^2+\theta+2)-(1-\beta)e^{-\theta x}\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}}\right]^{n-1}. \tag{13}$$

The pdf of the maximum order statistic of the MOS distribution is obtained by setting $j=n$ in (12) such that

$$\mho_{n:n}(x) = nq(x)\left[Q(x)\right]^{n-1}. \tag{14}$$

Hence, using (14), the maximum order statistic of the MOS distribution can be expressed as

$$\mho_{n:n}(x;\beta,\theta) = \frac{n\theta^3\beta e^{-\theta x}(\theta^2+\theta+2)(1+x+x^2)}{\left[(\theta^2+\theta+2)-(1-\beta)\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}e^{-\theta x}\right]^2}\left[\frac{(\theta^2+\theta+2)-\left[(\theta^2+\theta+2)+\theta x(\theta x+\theta+2)\right]e^{-\theta x}}{(\theta^2+\theta+2)-(1-\beta)\begin{bmatrix}(\theta^2+\theta+2)\\+\theta x(\theta x+\theta+2)\end{bmatrix}e^{-\theta x}}\right]^{n-1}.$$

4. Maximum Likelihood Estimation

In the literature, numerous approaches have been used for parameter estimation. In this paper, we used the maximum likelihood estimation method to obtain the parameters of the MOS distribution.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution function with pdf $q(x;\beta,\theta)$. Then, the log-likelihood function can be defined as

$$\ln L(x_1, x_2, ..., x_n; \beta, \theta) = \sum_{j=1}^{n} \ln q(x_1, x_2, ..., x_n; \beta, \theta).$$

Hence, the log-likelihood function of the MOS distribution can be derived as

$$\ln L(x_1, x_2, ..., x_n; \beta, \theta) = \sum_{j=1}^{n} \ln q(x_1, x_2, ..., x_n; \beta, \theta)$$

$$= \sum_{j=1}^{n} \ln \left[\frac{\theta^3 \beta e^{-\theta x_j} (\theta^2 + \theta + 2)(1 + x_j + x_j{}^2)}{\left[(\theta^2 + \theta + 2) - (1-\beta) \begin{bmatrix} (\theta^2 + \theta + 2) \\ +\theta x_j (\theta x_j + \theta + 2) \end{bmatrix} e^{-\theta x_j} \right]^2} \right],$$

where $j = 1, 2, 3, ...n$.

Let $D_j = (\theta^2 + \theta + 2) - (1-\beta)\left[(\theta^2 + \theta + 2) + \theta x_j (\theta x_j + \theta + 2)\right]$ and $Z = \theta^2 + \theta + 2$ such that

$$\ln L(x_1, x_2, ..., x_n; \beta, \theta) = \sum_{j=1}^{n} \ln Z \theta^3 \beta (1 + x_j + x_j^2) e^{\theta x_j} D_j^{-2}.$$

This implies that

$$\ln L(x_1, x_2, ..., x_n; \beta, \theta) = 3n \ln\theta + n \ln\beta + n \ln Z + \sum_{j=1}^{n} (1 + x_j + x_j^2) + \theta \sum_{j=1}^{n} x_j - 2 \sum_{j=1}^{n} \ln D_j = 0. \quad (15)$$

However, taking the partial derivative of (15) with respect to each of the parameters and equating to zero, we have

$$\frac{d \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta} = \frac{n}{\beta} - 2 \sum_{j=1}^{n} \frac{D_j'}{D_j} = 0, \quad (16)$$

$$\frac{d \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\theta} = \frac{3n}{\theta} + \sum_{j=i}^{n} x_j + \frac{n(2\theta + 1)}{(\theta^2 + \theta + 2)} - 2 \sum_{j=1}^{n} \frac{D_{j\theta}'}{D_{j_\theta}} = 0, \quad (17)$$

where $D_{j\beta}' = (\theta^2 + \theta + 2) + \theta x_j (\theta x_j + \theta + 2)$ and

$D_{j\theta}' = (2\theta + 1) - (1-\beta)\left[(2\theta + 1) + 2x_j (\theta x_j + \theta + 1)\right]$, respectively.

Equations (16) and (17) cannot be expressed in closed form therefore solving it directly will be very cumbersome. However, these equations can be solved iteratively using Fisher's scoring method. We therefore have

$$\frac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta^2} = -\frac{n}{\beta^2} + 2 \sum_{j=1}^{n} \frac{\left[(\theta^2 + \theta + 2) + \theta x_j (\theta x_j + \theta + 2)\right]^2}{D_{j_\beta}^2}.$$

Let $P_j = (\theta^2 + \theta + 2) + \theta x_j (\theta x_j + \theta + 2)$ such that

$$\frac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta^2} = -\frac{n}{\beta^2} + 2\sum_{j=1}^{n} \frac{P_j^2}{D_{j_\beta}^2}.$$

From (17), let $K_j = v_j u_j' - u_j v_j'$ such that

$$K_j = 2(\theta^2+\theta+2) - (2\theta+1)^2 + 2(2\theta+1)(1-\beta)\left[(2\theta+1) + 2x_j(\theta x_j + \theta + 1)\right]$$
$$-2(1-\beta)\left[(1+x_j^2+x_j)(\theta^2+\theta+2) + (\theta^2+\theta+2) + \theta x_j(\theta x_j+\theta+2)\right] +$$
$$(1-\beta)^2\left[2(1+x_j^2+x_j)(\theta^2+\theta+2+\theta x_j(\theta x_j+\theta+2) - \left[2\theta+1+2x_j(\theta x_j+\theta+1)\right]^2\right].$$

Then,

$$\frac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\theta^2} = -\frac{3n}{\theta^2} + \frac{n\left[3-2\theta(\theta+1)\right]}{(\theta^2+\theta+2)^2} - 2\sum_{j=1}^{n} K_j D_j^{-2}.$$

Recall that

$$P_j = (\theta^2+\theta+2) + \theta x_j(\theta x_j + \theta + 2) \text{ such that } \frac{d^2 P_j}{d\beta d\theta} = P_j'.$$

Similarly,

$$D_j = (\theta^2+\theta+2) - (1-\beta)\left[(\theta^2+\theta+2) + \theta x_j(\theta x_j+\theta+2)\right].$$

Then, $\dfrac{d^2 D_j}{d\beta d\theta} = D_j'$. Thus, we have

$$\frac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta d\theta} = -2\sum_{j=1}^{n}\left(D_j P_j' - P_j D_j'\right) D_j^{-2}.$$

The maximum likelihood estimates $(\hat{\beta}, \hat{\theta})$ of (β, θ) of the MOS distribution are the solution of the following equations

$$\begin{bmatrix} \dfrac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta^2} & \dfrac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta d\theta} \\ \dfrac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta d\theta} & \dfrac{d^2 \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\theta^2} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{d \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\beta} \\ \dfrac{d \ln L(x_1, x_2, ..., x_n; \beta, \theta)}{d\theta} \end{bmatrix},$$

where $\hat{\beta}$ and $\hat{\theta}$ are the initial estimate values of β and θ, respectively. The R package maxLik function was used to obtain the estimated values of the $\hat{\beta}$ and $\hat{\theta}$, respectively.

4.1. Quantile function

This section presents the quantile function of the propose MOS distribution. The quantile function of the MOS distribution can be derived as follows:

Recall

$$Q(x; \beta, \theta) = \frac{(\theta^2+\theta+2) - \left[(\theta^2+\theta+2) + \theta x(\theta x+\theta+2)\right]e^{-\theta x}}{(\theta^2+\theta+2) - (1-\beta)\left[(\theta^2+\theta+2) + \theta x(\theta x+\theta+2)\right]e^{-\theta x}}.$$

Let $\theta^2 + \theta + 2 = m$ and $1 - \beta = n$, such that

$$Q(x;\beta,\theta)=\frac{m-\left[m+\theta^2x^2+\theta^2x+2\theta x\right]e^{-\theta x}}{m-n\left[m+\theta^2x^2+\theta^2x+2\theta x\right]e^{-\theta x}},$$

and

$$\theta^2x^2+\theta^2x+2\theta x=\theta^2x^2+x(\theta^2+2\theta).$$

Similarly, let $u=\frac{m-\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}}{m-n\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}}$, where $u\in[0,1]$. On simplifying, we have

$$m-\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}=u\left[m-n\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}\right]$$
$$=um-un\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}.$$

$$un\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}-\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}=um-m=m(u-1).$$

This implies that

$$\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}(un-1)=m(u-1).$$

Also, let $\left[m+\theta^2x^2+x(\theta^2+2\theta)\right]e^{-\theta x}=\frac{m(u-1)}{(un-1)}=k.$ Thus,

$me^{-\theta x}+\theta^2x^2e^{-\theta x}+x(\theta^2+2\theta)e^{-\theta x}=k,$ and let $w=\theta^2+2\theta.$ Thus, we have $\theta^2x^2e^{-\theta x}+xwe^{-\theta x}+me^{-\theta x}=k.$

Hence,

$$xe^{-\theta x}=-\frac{(w+m)\pm\sqrt{(w+m)^2+4\theta^2k}}{2\theta^2}.$$

However, let $xe^{-\theta x}=\alpha$ then, $\frac{x}{e^{\theta x}}=\alpha\Rightarrow x=\alpha e^{\theta x}$

$e^{-\theta x^j}=\sum_{j=1}^{n}\frac{(-1)^j\theta^jx^{j+1}}{j!}$, $(j=1,2,3...,n)$. Also $xe^{-\theta x}=\alpha.$ We have $\sum_{j=1}^{n}\frac{(-1)^j\theta^jx^{j+1}}{j!}=\alpha$ and

$x^{i+1}=\sum_{j=1}^{n}j!\alpha(-1)^{-j}\theta^{-j}.$ The MOS distribution quantile function can be expressed as

$$x=\left[\sum_{j=1}^{n}j!\alpha(-1)^{-j}\theta^{-j}\right]^{\left(\frac{1}{j+1}\right)}. \tag{18}$$

4.2. The simulation study

In this section, we present a simulation study to examine the flexibility, and performance of the MOS distribution. We examined the mean estimates (ME), variance (V), biases (B) and root mean square errors (RMSEs) of the MLEs. The bias is calculated as

$$\hat{B}_s=\frac{\sum_{k=1}^{600}\left(\hat{T}_j-T\right)}{600},\text{ for } T=\theta,\beta.$$

Similarly, the MSEs is obtained as

$$\widehat{MSE}_s = \frac{\sum_{k=1}^{600}\left(\hat{T}_j - T\right)^2}{600}.$$

Tables 1 and 2 show the simulation for different parameter values for the MOS distribution. The simulation is performed as follows:

a. Data is generated using the derived quantile function.

b. The sample sizes are taken as n = 50, 100, 200, 300, 500 and 600.

c. The parameters values are set as $\theta = 0.5,\ \beta = 2.3$ and $\theta = 2.5,\ \beta = 0.8$.

d. Each sample size is replicated 600 times.

The results obtained in Tables 1 and 2 show that as the sample size increases, the RMSEs, biases, and variances of the MLEs of the parameters become smaller respectively. This result is in line with the first-order asymptotic theory.

Table 1 Simulation results for mean estimates (ME), variance (V), biases (B), and mean squared errors (MSE) of the $\hat{\theta}$ and $\hat{\beta}$, respectively for the MOS distribution

Parameter	n	ME	B	V	MSE
$\theta = 0.5$	50	2.1691	1.0826	0.6931	1.8652
$\beta = 2.3$		0.9750	1.7178	0.3820	3.3328
$\theta = 0.5$	100	2.1890	1.1122	0.6140	1.8509
$\beta = 2.3$		0.8339	1.6787	0.2565	3.0746
$\theta = 0.5$	200	2.1654	1.1237	0.4813	1.7441
$\beta = 2.3$		0.7041	1.6830	0.2018	3.0375
$\theta = 0.5$	300	2.0560	0.5599	0.3517	0.6652
$\beta = 2.3$		0.5841	0.9701	0.1381	1.0791
$\theta = 0.5$	500	1.9033	1.1177	0.2356	1.4849
$\beta = 2.3$		0.4496	0.5296	0.0844	0.3649
$\theta = 0.5$	600	1.8497	0.3576	0.1907	0.3186
$\beta = 2.3$		0.4099	0.2322	0.0687	0.1226

Table 2 Simulation results for mean estimates (ME), variance (V), biases (B), and mean squared errors (MSE) of the $\hat{\theta}$ and $\hat{\beta}$ respectively for the MOS distribution

Parameter	n	ME	B	V	MSE
$\theta = 2.5$	50	3.8114	1.1095	0.1283	1.3594
$\beta = 0.8$		0.4856	1.1268	0.0259	1.2956
$\theta = 2.5$	100	3.7409	1.0711	0.1218	1.2691
$\beta = 0.8$		1.2115	1.0970	0.0100	1.2156
$\theta = 2.5$	200	3.6035	1.0190	0.1204	1.1607
$\beta = 0.8$		0.3423	1.0668	0.0075	1.1455
$\theta = 2.5$	300	3.4877	0.9799	0.1103	1.0705
$\beta = 0.8$		0.3055	1.0495	0.0074	1.1089
$\theta = 2.5$	500	3.2638	0.2225	0.0054	0.0549
$\beta = 0.8$		0.2527	1.0245	0.0060	1.0555
$\theta = 2.5$	600	3.2043	0.1634	0.0032	0.0299
$\beta = 0.8$		0.2392	1.0130	0.0057	1.0319

5. The MOS Regression Model

This section introduced the regression model for the proposed MOS distribution. Let P be a random variable with MOS probability density function and $\theta = \varphi$ as the parameter; a function of $X = \left(1, x_1, x_2, ..., x_n\right)^T$ such that

$$\varphi = \eta + \phi_1 x_1 + \phi_2 x_2 + ... + \phi_n x_n. \tag{19}$$

Then, the probability function is written as

$$q_{reg.}(p \mid \varphi) = \frac{\varphi^3 \beta e^{-\varphi p}(\varphi^2 + \varphi + 2)(1 + p + p^2)}{\left[(\varphi^2 + \varphi + 2) - (1 - \beta)\left[(\varphi^2 + \varphi + 2) + \varphi p(\varphi p + \varphi + 2)\right] e^{-\varphi p}\right]^2},$$

where $\beta > 0$, and φ is the regression model defined in (19) above. The survival rate function of the MOS regression model is given by

$$S(p \mid \varphi) = \frac{\beta e^{-\varphi p}\left[(\varphi^2 + \varphi + 2) + \varphi p(\varphi p + \varphi + 2)\right]}{(\varphi^2 + \varphi + 2) - (1 - \beta) e^{-\varphi p}\left[(\varphi^2 + \varphi + 2) + \varphi p(\varphi p + \varphi + 2)\right]}.$$

The corresponding hazard rate function for MOS regression model is given as

$$H(p \mid \varphi) = \frac{\varphi^3 e^{-\varphi p}(1 + p + p^2)}{(\varphi^2 + \varphi + 2) - (1 - \beta) e^{-\varphi p}\left[(\varphi^2 + \varphi + 2) + \varphi p(\varphi p + \varphi + 2)\right]}.$$

6. Numerical Application and Goodness-of-fit Test

The numerical applications of the MOS distribution are demonstrated using two real lifetime data sets and its fitness performance is investigated using Akaike information criteria (AIC), Bayesian information criteria (BIC), Hannan and Quinn information criteria (HQIC), Standard Errors (S.E), Kolmogorov-Smirnov (K-S), Anderson-Darling (A), and Cramér-von Misses (W) tests. The distribution with the least AIC, BIC, HQIC values is considered the best model to the data sets.

Data set I consist of the survival times in days of 72 guinea pigs infected with virulent tubercle bacilli reported in Bjerkedal (1960). The data set is presented in Table 3. The fitness of the MOS distribution to the data set I is compared with the Power Ishita distribution (PID) (Shukla and Shanker 2018), Sujatha distribution (Shanker 2016d), and Quasi-Akash distribution (A Quasi) (Shanker et al. 2018), Marshall-Olkin extended Lindley distribution (MOELD) of Ghitany et al. (2012), and Marshall-Olkin exponential distribution (MOED) of Marshall and Olkin (1997).

Data set II consists of the number of vehicle fatalities for 39 cities in South Carolina for 2012 collected by the National Highway Traffic Safety Administration (www-fars.nhtsa.dot.gov/States). The data set is presented in Table 4. The fitness performance of the MOS distribution is compared with Sujatha distribution (Shanker 2016d), Quasi-Akash distribution (A Quasi) (Shanker et al. 2018), Marshall-Olkin extended Lindley distribution (MOELD) of Ghitany et al. (2012), and Marshall-Olkin exponential distribution (MOED) of Marshall and Olkin (1997).

Firstly, we examined the descriptive statistics for the data sets and are presented in Tables 5 and 6. Tables 7, 8, 9, and 10 provide the values of the test statistics, parameter estimates, standard errors, and the maximum likelihood estimates (MLE) values for the fitted models to the data sets I and II respectively. The results obtained in Tables 7, 8, 9 and 10 show that the proposed MOS distribution gave the least values of the AIC, BIC, HQIC, A, W, and K-S and the highest p-values to all the data sets. Therefore, the MOS distribution is considered the best model for the data sets under consideration and could be an alternative to the models compared to modeling real-lifetime situations.

Table 3 Data set I

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555

Table 4 Data set II

4, 48, 9, 9, 31, 22, 26, 17,20, 12, 6, 5, 14, 9, 16, 27, 3, 33, 9, 20, 68, 13, 51, 13, 48, 23, 12, 13, 10, 15, 8, 1, 2, 4, 17, 16, 6, 52, 50

Table 5 Descriptive statistics for the data set I

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
10.00	108.00	149.50	176.80	224.00	555.00	1.34	4.99

Table 6 Descriptive statistics for the data set II

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
1.00	9.00	14.00	19.54	24.50	68.00	1.29	3.79

Table 7 The parameters estimates of $\hat{\theta}$, $\hat{\beta}$, S.E., K-S, and p-values of the fitted model using data set I

Model	Parameters Estimates		K-S	p-value
	$\hat{\theta}$ (S.E.)	$\hat{\beta}$ (S.E.)		
Sujatha	0.2031 (0.041)	-	0.2327	0.0660
MOS	**0.0048 (0.030)**	**1.8100 (0.136)**	**0.0314**	**0.8101**
PID	0.2284 (0.341)	0.0096 (0.473)	0.1510	0.0590
A Quasi	0.0046 (0.552)	0.5060 (0.506)	0.1542	0.0299
MOELD	1.5565 (0.804)	0.0169 (0.617)	0.7931	0.0000
MOED	0.0157 (0.602)	11.6995 (0.561)	0.7148	0.0001

Table 8 The MLE, AIC, BIC, HQIC, A, and W values of the fitted models using data set I

Model	MLE	AIC	BIC	HQIC	A	W
Sujatha	14.0409	34.0819	40.5113	36.6106	0.5969	0.0972
MOS	**14.0277**	**32.1353**	**35.0647**	**33.1640**	**0.5652**	**0.0908**
PID	14.1885	38.3769	46.9495	41.7486	0.7043	0.4011
A Quasi	14.5027	37.0055	45.5780	40.3777	0.6857	0.3288
MOELD	567.0000	1138.0000	1142.5600	1139.8200	0.9937	0.7402
MOED	428.1300	860.2600	864.8100	862.0700	0.8005	0.5558

Table 9 The parameters estimates of $\hat{\theta}$, $\hat{\beta}$, S.E., K-S and p-values for the fitted model using data set II

Model	Parameters Estimates		K-S	p-value
	$\hat{\theta}$ (S.E.)	$\hat{\beta}$ (S.E.)		
Sujatha	0.1480 (0.014)	-	0.9396	0.0413
MOS	**0.0865 (0.002)**	**0.1618 (0.142)**	**0.9020**	**0.6421**
A Quasi	0.1154 (0.024)	10.3074 (11.565)	0.9289	0.2047
MOELD	0.3107 (0.054)	1.000 (0.432)	0.9532	0.0000
MOED	0.0718 (0.018)	2.0228 (1.040)	0.9121	0.3946

Table 10 The MLE, AIC, BIC, HQIC, A, and W values of the fitted models using data set II

Model	MLE	AIC	BIC	HQIC	A	W
Sujatha	158.2024	318.4047	320.0683	319.0016	39.7878	6.7294
MOS	**154.6959**	**313.3918**	**316.7190**	**314.5856**	**34.1918**	**6.4556**
A Quasi	154.9862	315.7322	318.0595	317.9160	39.0247	6.6136
MOELD	174.9122	345.8243	342.4972	344.6306	NA	NA
MOED	156.5340	314.2510	317.5322	316.2087	37.5374	6.6023

Note: NA means not available.

7. Conclusions

In this paper, we adopt the Marshall-Olkin approach of developing more flexible distributions to introduce a heavy-tailed, monotone non-increasing two parameters Marshall-Olkin Sujatha (MOS) distribution as an extension of the classical Sujatha distribution. Some of its useful statistical properties have been derived. The regression model of the proposed MOS distribution has been introduced. Simulation studies have been carried out to investigate the flexibility behavior of the proposed MOS distribution. The estimate of the parameters of the proposed MOS distribution was obtained by the method of maximum likelihood and its numerical applications have been studied using two real-life data sets. The results obtained indicated that the proposed MOS distribution provides a better goodness-of-fit than the Sujatha distribution (SD), Power Ishita distribution (PID), A Quasi-Akash (A Quasi) distribution, Marshall-Olkin extended Lindley distribution (MOELD), and Marshall-Olkin exponential (MOED) distribution for all the data sets under consideration. The results obtained are in line with the motivation of the study. Hence, the MOS distribution could be used as an alternative model to SD, PID, A Quasi, MOELD, and MOED in modeling real-lifetime scenarios. Further studies can be carried out to examine other statistical properties of the proposed model not covered in this study and its applications to other data sets as well as the application of the MOS regression model in modeling real life situations.

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References

AbuJarad MH, Khan AA, Khaleel MA, AbuJarad ES, AbuJarad AH, Oguntunde PE. Bayesian reliability analysis of Marshall and Olkin model. Ann Data Sci. 2020; 7(3): 461-489.

Agu FI, Onwukwe CE. Modified Laplace distribution, its statistical properties and applications. Asian J Prob Stat. 2019; 4(1): 1-14.

Ikechukwu AF, Jaspa OE, Emmanuel RF, Adeyinka O. A three parameter shifted exponential distribution: properties and applications. Thail Stat. 2020; 18(4): 470-480.

Agu FI, Runyi FE. Comparison of goodness of fit tests for normal distribution. Asian J Prob Stat. 2018; 1(2): 1-32.

Bjerkedal T. Acquisition of resistance in Guinea Pies infected with different doses of virulent tubercle bacilli. Am J Hyg. 1960; 72(1): 130-148.

Eghwerido JT, Zelibe SC, Efe-Eyefia E. Gompertz-Alpha power iverted exponential distribution: properties and applications. Thail Stat. 2020; 18(3): 319-332.

Eghwerido JT, Nzei LC, Agu FI. The Alpha power Gompertz distribution: characterization, properties, and applications. Sankhya A. 2021; 83(1): 449-475.

George R, Thobias S. Marshall-Olkin Kumaraswamy distribution. Int Math Forum. 2017; 12(2): 47-69.

Ghitany ME, Al-Mutairi DK, Al-Awadhi FA, Al-Burais MM. Marshall-Olkin extended Lindley distribution and its application. Int J Appl Math. 2012; 25(5): 709-721.

Gupta RD, Kundu D. Theory & methods: generalized exponential distributions. Aust N Z J Stat. 1999; 41(2): 173-188.

Khaleel MA, Al-Noor NH, Abdal-Hameed MK. Marshall Olkin exponential Gompertz distribution: properties and applications. Period. Eng. Nat. Sci. 2020; 8(1): 298-312.

Lindley DV. Fiducial distributions and Bayes' theorem. J R Stat Soc. 1958; 20(1): 102-107.

Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika. 1997; 84(3): 641-652.

Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution. Commun Stat - Theory Methods. 2017; 46(13): 6543-6557.

Odom CC, Ijomah MA. Odoma Distribution and Its Application. Asian J Prob Stat. 2019; 4(1): 1-11.

Patil GP, Rao CR. Weighted distributions and size-biased sampling with applications to wildlife populations and human families. Biometrics. 1978; 34(2): 179-189.

Pogány TK, Saboor A, Provost S. The marshall-olkin exponential Weibull distribution. Hacettepe J Math Stat. 2015; 44(6): 1579-1594.

Rényi A. On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. 1961; Volume 1: Contributions to the Theory of Statistics (pp. 547-561). University of California Press.

Shanthikumar JG. Stochastic orders and their applications. New York: Academic Press; 1994.

Shannon CE. A mathematical theory of communication. Bell Syst Tech J. 1948; 27(3): 379-423.

Shanker R. Aradhana distribution and its applications. Int J Stat Appl. 2016c; 6(1): 23-34.

Shanker R. Sujatha distribution and its applications. Stat Transit. 2016d; 17(3): 1-20.

Shanker R, Shukla KK. Ishita distribution and its applications. Biom Biostat Int J. 2017; 5(2): 39-46.

Shanker R, Shukla KK, Shanker R. A quasi Poisson Akash distribution and its applications to ecology. Int J Stat Appl Math. 2018; 3(3): 111-119.

Shukla KK. Pranav distribution with properties and its applications. Biom Biostat Int J. 2018; 7(3): 244-254.

Shukla KK, Shanker R. Power Ishita distribution and its application to model lifetime data. Stat Transit New. 2018; 19(1): 135-148.