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Measure of Modified Slope Rotatability for Second Order Response Surface Designs Using Balanced Incomplete Block Designs

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Abstract

Box and Hunter (1957) introduced the very important concept of rotatability for response surface designs. Das and Narasimham (1962) developed rotatable designs using balanced incomplete block designs (BIBD). As an analogue to Box and Hunter (1957) rotatability, Hader and Park (1978) introduced slope rotatability for second order response surface designs and developed slope rotatable central composite designs (SRCCD). Victorbabu and Narasimham (1991) extended the work of Hader and Park (1978) and constructed second order slope rotatable designs (SOSRD) using BIBD. Measure of slope rotatability that enable us to assess the degree of slope rotatability for a given response surface designs have been introduced by Park and Kim (1992). Modified slope rotatability for second order response surface designs was suggested by Victorbabu (2005, 2006). In this paper, measure of modified slope rotatability for second order response surface designs using BIBD is suggested for $3 \le v \le 16$ which enables us to assess the degree of modified slope rotatability for a given second order response surface design and variance of the estimated responses are also obtained.

Keywords: Experimental designs, estimation of slope, degree of slope rotatability, incomplete block designs.

1. Introduction

A design is said to be rotatable if the variance of the response estimate is a function only of the distance of the point from the design centre. The study of rotatable designs is mainly emphasized on the estimation of absolute response. Estimation of differences in response at two different points in the factor space will often be of great importance. If differences at two points close together, estimation of local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc. (Park 1987).

Hader and Park (1978) constructed SRCCD. Victorbabu and Narasimham (1991) studied in detail the conditions to be satisfied by a general second order slope rotatable designs (SOSRD) and constructed SOSRD using BIBD. Victorbabu (2005, 2006) studied modified SRCCD and modified

SOSRD using BIBD respectively. Victorbabu (2007) suggested a review on SOSRD. Park and Kim (1992) suggested measure of slope rotatability for second order response surface designs. Victorbabu and Surekha (2012, 2013 and 2016) suggested different measures of second order response surface designs. Recently, Victorbabu and Jyostna (2021) suggested measure of modified slope rotatability for second order response surface designs.

2. Conditions for Second Order Slope Rotatable Designs

Suppose we want to use the second order response surface design $D=(x_{iu})$ to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v}\sum_{j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u,$$

where x_{iu} denotes the level of the i^{th} factor $(i=1,2,...,v)$ in the u^{th} run $(u=1,2,\ldots,N)$ of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2 is said to be SOSRD if the variance of the estimate of first order partial derivative of $Y_u(x_1,x_2,\ \ldots,x_v)$ with respect to each of independent variables (x_i) is only a function of the distance $(d^2=\sum_{i=1}^{v} x_i^2)$ of the point $(x_1,x_2,\ \ldots,x_v)$ from the origin of the design.

Following Box and Hunter (1957), Hader and Park (1978) and Victorbabu and Narasimham (1991) the general conditions for second order slope rotatability can be obtained as follows. To simplify the fit of the second order polynomial from design points '*D*' through the method of least squares, we impose the following simple symmetry conditions on D to facilitate easy solutions of the normal equations:

1. $\sum x_{iu}=0,\ \sum x_{iu}x_{ju}=0,\ \sum x_{iu}x_{ju}^2=0,\ \sum x_{iu}x_{ju}x_{ku}=0,\ \sum x_{iu}^3=0,\ \sum x_{iu}x_{ju}^3=0,$
   $\sum x_{iu}x_{ju}x_{ku}^2=0,\ \sum x_{iu}x_{ju}x_{ku}x_{lu}=0;\ \text{for } i\neq j\neq k\neq l,$
2. (i) $\sum x_{iu}^2=\text{constant}=N\lambda_2,$
   (ii) $\sum x_{iu}^4=\text{constant}=cN\lambda_4;\ \text{for all } i,$
3. $\sum x_{iu}^2 x_{ju}^2=\text{constant}\ =N\lambda_4;\ \text{for } i\neq j,$ (1)

where c, λ_2 and λ_4 are constants. The variances and covariances of the estimated parameters are

$$V(\hat{b}_0)=\frac{\lambda_4(c+v-1)\sigma^2}{N\left[\lambda_4(c+v-1)-v\lambda_2^2\right]},\quad V(\hat{b}_i)=\frac{\sigma^2}{N\lambda_2},\quad V(\hat{b}_{ij})=\frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii})=\frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right],\quad Cov(\hat{b}_0,\hat{b}_{ii})=\frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj})\ =\ \frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]},\tag{2}$$

and other covariances vanish.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a non-singular second order design is

4. $\dfrac{\lambda_4}{\lambda_2^2}>\dfrac{v}{(c+v-1)}.$ (3)

For the second order model

$$\frac{\partial \hat{Y}}{\partial x_i} = \hat{b}_i + 2\hat{b}_{ii} + \sum_{j \neq i} \hat{b}_{ij} x_{ju},$$

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = V(\hat{b}_i) + 4x_{iu}^2 V(\hat{b}_{ii}) + \sum_{j \neq i} x_{ju}^2 V(\hat{b}_{ij}). \tag{4}$$

The condition for right hand side of the (4) to be a function of $d^2 = \sum_{i=1}^{v} x_i^2$ alone (for slope rotatability) is

$$4V(\hat{b}_{ii}) = V(\hat{b}_{ij}). \tag{5}$$

On simplification of (5), we get,

5. $\left[v(5-c)-(c-3)^2\right]\lambda_4 + \left[v(c-5)+4\right]\lambda_2^2 = 0.$ (6)

Therefore 1, 2 and 3 of (1), (3) and (6) give a set of conditions for slope rotatability in any general second order response surface design (Hader and Park 1978, Victorbabu and Narasimham 1991).

3. Conditions for Modified Second Order Slope Rotatable Designs

Following Das et al. (1999), Hader and Park (1978), Victorbabu and Narasimham (1991), equations 1, 2 and 3 of (1), (2), (3) and (6) give the necessary and sufficient conditions for modified SOSRD (Victorbabu 2005, 2006).

The usual method of construction of SOSRD is to take combinations with unknown constants, associate a 2^v factorial combinations or a suitable fraction of it with factors each at ± 1 levels to make the level codes equidistant. All such combinations form a design. Generally, SOSRD need at least five levels (suitably coded) at $0, \pm 1, \pm a$ for all factors $((0,0,\ldots,0)$ – chosen center of the design, unknown level 'a' to be chosen suitably to satisfy slope rotatability). Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively, by putting some restrictions indicating some relation among $\sum x_{iu}^2, \sum x_{iu}^4$ and $\sum x_{iu}^2 x_{ju}^2$ some equations involving the unknowns are obtained and their solution gives the unknown levels. In SOSRD, the restrictions used is seems, not exploited well. We shall investigate the restriction $\left(\sum x_{iu}^2\right)^2 = N \sum x_{iu}^2 x_{ju}^2$ i.e., $\left(N\lambda_2\right)^2 = N(N\lambda_4)$ and $\lambda_2^2 = \lambda_4$ to get modified SOSRD. By applying the new restriction in (6), we get $c = 1$ or $c = 5$. The non-singularity condition (3) leads to $c = 5$. It may be noted $\lambda_2^2 = \lambda_4$ and $c = 5$ are equivalent conditions. The variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{(v+4)\sigma^2}{4N}, \quad V(\hat{b}_i) = \frac{\sigma^2}{N\sqrt{\lambda_4}}, \quad V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}, \quad V(\hat{b}_{ii}) = \frac{\sigma^2}{4N\lambda_4},$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\sigma^2}{4N\sqrt{\lambda_4}} \text{ and other covariances are zero, } V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = \left[\frac{\sqrt{\lambda_4} + d^2}{N\lambda_4}\right]\sigma^2. \tag{7}$$

4. Conditions of Measure of Slope Rotatability for Second Order Response Surface Designs

Following Hader and Park (1978), Victorbabu and Narasimham (1991), Park and Kim (1992), Equations (1), (2), (3) and (6) give the necessary and sufficient conditions for a measure of slope rotatability for any general second order response surface designs. Further we have, $V(b_i)$ are the same for all i, $V(b_{ii})$ are the same for all i, $V(b_{ij})$ are the same for all i, j where $i \neq j$, $Cov(b_i, b_{ii})$ = $Cov(b_i, b_{ij})$ = $Cov(b_{ii}, b_{ij})$ = $Cov(b_{ij}, b_{il})$ for all $i \neq j \neq l$.

The measure of slope rotatability for second order response surface design can be obtained by using the following equation (Park and Kim 1992, p.398):

$$Q_v(D) = \frac{1}{2(v-1)\sigma^4}\left\{(v+2)(v+4)\sum_{i=1}^{v}\left[\left(V(b_i) - \frac{1}{v}\sum_{i=1}^{v}V(b_i)\right) + \frac{\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right) - \frac{1}{v}\sum_{i=1}^{v}\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right)}{v+2}\right]^2\right.$$

$$+\frac{4}{v(v+2)}\sum_{i=1}^{v}\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right) - \frac{1}{v}\sum_{i=1}^{v}\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right)^2 + 2\sum_{i=1}^{v}\left[\left(4V(b_{ii}) - \frac{\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right)}{v}\right)^2 + \sum_{\substack{j=1\\ j\neq i}}^{v}\left(V(b_{ij}) - \frac{\left(4V(b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}V(b_{ij})\right)}{v}\right)^2\right]$$

$$\left. +4(v+4)\left[4Cov^2(b_i, b_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{v}Cov^2(b_i, b_{ij})\right] + 4\sum_{i=1}^{v}\left[4\sum_{\substack{j=1\\ j\neq i}}^{v}Cov^2(b_{ii}, b_{ij}) + \sum_{\substack{j<l\\ j,l\neq i}}^{v}\sum Cov^2(b_{ij}, b_{il})\right]\right\},$$

where $Q_v(D)$ is the measure of slope-rotatability. It can be verified that $Q_v(D)$ is zero if and only if a design D is slope-rotatable. $Q_v(D)$ becomes larger as D deviates from a slope-rotatable design. Further, $Q_v(D)$ is greatly simplified to $Q_v(D) = \frac{1}{\sigma^4}\left[4V(b_{ii}) - V(b_{ij})\right]^2$.

5. Modified Second Order Slope Rotatable Designs Using BIBD (Victorbabu 2006)

A balanced incomplete block design (BIBD) denoted by (v, b, r, k, λ) is an arrangement of v treatments in b blocks each containing $k(< v)$ treatments, if (i) every treatment occurs at most once in a block, (ii) every treatment occurs in exactly r blocks and (iii) every pair of treatments occurs together in λ blocks.

Let (v, b, r, k, λ) be a BIBD, $2^{t(k)}$ denotes a fractional replicate of 2^k with $+1$ or -1 levels in which no interaction with less than five factors is confounded. $[1-(v, b, r, k, \lambda)]$ denote the design points generated from the transpose of the incidence matrix of BIBD. $[1-(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (Raghavarao 1971). Let n_0 be the number of central points in modified SOSRD and $\cup$ denotes combination of the design points generated from different sets of points.

Let $(a, 0, 0, \ldots, 0)2^1$ denote the design points generated from $(a, 0, 0, \ldots, 0)$ point set. Repeat this set of additional design points, say n_a times when $r < 5\lambda$. Consider the design points, $[1-(v, b, r, k, \lambda)]2^{t(k)} \cup n_a(a, 0, 0, \ldots 0)2^1 \cup n_0$ will give a v dimensional modified SOSRD in

$N=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ design points if, $a^4=\frac{(5\lambda-r)2^{t(k)-1}}{n_a}$, $n_0=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}-(b2^{t(k)}+2n_a v)$ and n_0 turns out to be an integer.

6. Measure of Slope Rotatability for Second Order Response Surface Designs Using BIBD

The result of measure of slope rotatability for second order response surface designs using BIBD is suggested here (Victorbabu and Surekha 2012). Let (v,b,r,k,λ) denote a BIBD. Then the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)}\cup n_a(a,0,0,...,0)2^1\cup(n_0)$ will give a measure of slope rotatability for second order response surface designs using BIBD in $N=b2^{t(k)}+2vn_a+n_0$ design points, as follows:

$$Q_v(D)=\left[\frac{\sum x_{iu}^2}{N}\right]^4\left[4e-V(b_{ij})\right]^2,$$

where

$$e=V(b_{ii})=\frac{\begin{array}{c}(v-1)\left[2^{t(k)}\lambda n_0+2^{t(k)+1}\lambda vn_a-2^{t(k)+2}rn_a a^2-r^2 2^{2t(k)}+b\lambda 2^{2t(k)}\right]+\\ \left[2^{t(k)+1}bn_a+2n_a n_0+4n_a^2\right]a^4+(r-\lambda)\left[2^{t(k)+1}vn_a+2^{t(k)}n_0+b2^{2t(k)}\right]\end{array}}{\left[2^{t(k)}(r-\lambda)+2n_a a^4\right]\left[\begin{array}{l}v2^{t(k)}\left(\lambda n_0-4rn_a a^2\right)+2^{t(k)+1}n_a\left(v^2\lambda+ba^4\right)+2n_a n_0 a^4\\ +(r-\lambda)\left(b2^{2t(k)}+2^{t(k)+1}vn_a+2^{t(k)}n_0\right)+2^{2t(k)}v(b\lambda-r^2)\end{array}\right]}$$

If $Q_v(D)$ is zero, if and only if, a design '*D*' is slope-rotatable. $Q_v(D)$ becomes larger as '*D*' deviates from a slope rotatable design (Park and Kim 1992; Victorbabu and Surekha 2012).

7. Measure of Modified Slope Rotatability for Second Order Response Surface Designs Using BIBD

The proposed measure of modified slope rotatability for second order response surface designs using BIBD when $r<5\lambda$ is suggested here. Let (v,b,r,k,λ) denote a BIBD. Then the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)}\cup n_a(a,0,0,...0)2^1\cup n_0$ generated from BIBD in $N=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ design points, will give a measure of modified slope rotatability for second order response surface designs using BIBD with $a^4=\frac{(5\lambda-r)2^{t(k)-1}}{n_a}$, $n_0=\frac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}-(b2^{t(k)}+2n_a v)$ and n_0 turns out to be an integer. (Alternatively, N may be obtained directly as $N=b2^{t(k)}+2vn_a+n_0$ design points)

For the above design points the simple symmetry conditions 1, 2, 3 of (1) are true. Condition 1 of (1) is true obviously. Conditions 2 and 3 of (1) are true as follows:

1. (i) $\sum x_{iu}^2=r2^{t(k)}+2n_a a^2=N\lambda_2$, (8)

   (ii) $\sum x_{iu}^4=r2^{t(k)}+2n_a a^4=5N\lambda_4$, (9)

2. $\sum x_{iu}^2 x_{ju}^2=\lambda 2^{t(k)}=N\lambda_4$. (10)

From (8), (9) and (10), and on simplification, we get $a^4 = \frac{(5\lambda - r)2^{t(k)-1}}{n_a}$, $\lambda_2 = \frac{r2^{t(k)} + 2n_a a^2}{N}$, and $\lambda_4 = \frac{\lambda 2^{t(k)}}{N}$. To obtain measure of modified slope rotatability for second order response surface designs using BIBD we investigate the restriction $(\sum x_{iu}^2)^2 = N\sum x_{iu}^2 x_{ju}^2$ i.e., $(N\lambda_2)^2 = N(N\lambda_4)$ and $\lambda_2^2 = \lambda_4$ (Victorbabu 2005; 2006) and on simplification, we get,

$$n_0 = \frac{(r2^{t(k)} + 2n_a a^2)^2}{\lambda 2^{t(k)}} - (b2^{t(k)} + 2n_a v) \text{ and } Q_v(D) = \left[\frac{\sum x_{iu}^2}{N}\right]^4 \left[4e - V(\hat{b}_{ij})\right]^2,$$

where $e = V(\hat{b}_{ii}) = \frac{b2^{t(k)} + 2vn_a + n_0}{4\left(r^2 2^{2t(k)} + 4r2^{t(k)} n_a a^2 + 4n_a^2 a^4\right)}$ (since $\lambda_2^2 = \lambda_4$).

Measure of modified slope rotatability for second order response surface designs using BIBD is

$$Q_v(D) = \left[\frac{r2^{t(k)} + 2n_a a^2}{N}\right]^4 \left[4e - \frac{1}{\lambda 2^{t(k)}}\right]^2.$$

Example: We illustrate the measure of modified slope rotatability for second order response surface designs for $v = 7$ factors with the help of a BIBD. The design points, $[1-(v=7, b=7, r=3, k=3, \lambda=1)]2^3 \cup n_a(a,0,0,\ldots 0)2^1 \cup n_0$ will give a measure of modified slope rotatability for second order response surface designs in $N = 128$ design points. We have from (8), (9) and (10), we get

$$\sum x_{iu}^2 = 24 + 2n_a a^2 = N\lambda_2, \quad (11)$$

$$\sum x_{iu}^4 = 24 + 2n_a a^4 = 5N\lambda_4, \quad (12)$$

$$\sum x_{iu}^2 x_{ju}^2 = 8 = N\lambda_4. \quad (13)$$

Equations (12) and (13) leads to $n_a a^4 = 8$, which implies $a^2 = 2$ for $n_a = 2$. From (11), Equation (13) using the modified condition $(\lambda_2^2 = \lambda_4)$ with $a^2 = 2$ and $n_a = 2$, we get $N = 128$, $n_0 = 44$. At $a = 1.4142$, we get $e = 0.03125$ then $Q_v(D)$ is zero. Then the design is modified slope rotatable. Variance of the estimated response for measure of modified slope rotatability for second order response surface designs using BIBD is

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = 0.03125\sigma^2 + 0.125d^2\sigma^2.$$

Suppose if we take $a = 2.5$ instead of taking $a = 1.4142$ for 7 factors we get $e = 0.01333$ then $Q_v(D) = 1.10369 \times 10^{-4}$. Here $Q_v(D)$ becomes larger it deviates from modified slope rotatability. Variance of the estimated response for measure of modified slope rotatability for second order response surface designs using BIBD is

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = 0.0204\sigma^2 + 0.0533d^2\sigma^2.$$

We may point out here that this measure of modified slope rotatability for second order response surface designs using BIBD for 7-factors has only 128 design points, whereas the corresponding measure of modified slope rotatability for second order response surface designs using CCD obtained by Victorbabu and Jyostna (2021) needs 144 design points. Thus, the new method leads to a 7-factor measure of modified slope rotatability for second order response surface designs using BIBD in less number of design points than the corresponding measure of modified slope rotatability for second order response surface designs using CCD and same is the case in some other cases also (please see the table for $v = 9,13$).

Table 1 gives the values of measure of modified slope rotatability $(Q_v(D))$ for second order response surface designs using BIBD, at different values of 'a' for $3 \le v \le 16$. It can be verified that $Q_v(D)$ is zero, if and only if a design 'D' is modified second order slope rotatable. $Q_v(D)$ becomes larger as 'D' deviates from a modified SOSRD. Variance of the estimated responses for measure of modified slope rotatability for second order response surface designs using BIBD for different values of 'a' are also included in the Table 1.

Table 1 Values of measure of modified slope rotatability for second order response surface designs using BIBD

$(3,3,2,2,1), N=100,\ n_a=6,\ a=1.0$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
*1.0	0.0000	$0.0500\,\sigma^2 + 0.2500\,d^2\sigma^2$
1.3	3.9999×10^{-4}	$0.0354\,\sigma^2 + 0.1250\,d^2\sigma^2$
1.6	7.5520×10^{-4}	$0.0258\,\sigma^2 + 0.0667\,d^2\sigma^2$
1.9	3.1185×10^{-3}	$0.0195\,\sigma^2 + 0.0380\,d^2\sigma^2$
2.2	9.8335×10^{-3}	$0.0151\,\sigma^2 + 0.0229\,d^2\sigma^2$
2.5	0.0263	$0.0120\,\sigma^2 + 0.0145\,d^2\sigma^2$
2.8	0.0116	$0.0098\,\sigma^2 + 0.0096\,d^2\sigma^2$
3.1	0.1370	$0.0081\,\sigma^2 + 0.0066\,d^2\sigma^2$
3.4	0.2790	$0.0068\,\sigma^2 + 0.0046\,d^2\sigma^2$
3.7	0.5358	$0.0058\,\sigma^2 + 0.0034\,d^2\sigma^2$
4.0	0.9801	$0.0050\,\sigma^2 + 0.0025\,d^2\sigma^2$

Table 1 (Continued)

$(4,6,3,2,1)$, $N=64$, $n_a=1$, $a=1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	1.3411×10^{-5}	$0.0714\,\sigma^2+0.3265\,d^2\sigma^2$
1.3	1.4101×10^{-6}	$0.0650\,\sigma^2+0.2706\,d^2\sigma^2$
*1.4142	0.0000	$0.0625\,\sigma^2+0.2500\,d^2\sigma^2$
1.6	5.1260×10^{-6}	$0.0584\,\sigma^2+0.2184\,d^2\sigma^2$
1.9	4.7913×10^{-5}	$0.0520\,\sigma^2+0.1732\,d^2\sigma^2$
2.2	1.7064×10^{-4}	$0.0461\,\sigma^2+0.1362\,d^2\sigma^2$
2.5	4.4148×10^{-4}	$0.0408\,\sigma^2+0.1362\,d^2\sigma^2$
2.8	9.6964×10^{-4}	$0.0361\,\sigma^2+0.0835\,d^2\sigma^2$
3.1	1.9242×10^{-3}	$0.0320\,\sigma^2+0.0657\,d^2\sigma^2$
3.4	3.5589×10^{-3}	$0.0285\,\sigma^2+0.0519\,d^2\sigma^2$
3.7	6.2453×10^{-3}	$0.0254\,\sigma^2+0.0413\,d^2\sigma^2$
4.0	0.0123×10^{-3}	$0.0223\,\sigma^2+0.0156\,d^2\sigma^2$

$(5,10,6,3,3)$, $N=150$, $n_a=1$, $a=2.4495$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	4.1495×10^{-6}	$0.0200\,\sigma^2+0.0600\,d^2\sigma^2$
1.3	3.1611×10^{-6}	$0.0195\,\sigma^2+0.0568\,d^2\sigma^2$
1.6	2.0772×10^{-6}	$0.0188\,\sigma^2+0.0532\,d^2\sigma^2$
1.9	1.0402×10^{-6}	$0.0181\,\sigma^2+0.0492\,d^2\sigma^2$
2.2	2.5562×10^{-7}	$0.0173\,\sigma^2+0.0451\,d^2\sigma^2$
*2.4495	0.0000	$0.0167\,\sigma^2+0.0417\,d^2\sigma^2$
2.5	1.2449×10^{-8}	$0.0165\,\sigma^2+0.0410\,d^2\sigma^2$
2.8	7.1041×10^{-7}	$0.0157\,\sigma^2+0.0367\,d^2\sigma^2$
3.1	2.8934×10^{-6}	$0.0149\,\sigma^2+0.0073\,d^2\sigma^2$
3.4	7.2905×10^{-6}	$0.0141\,\sigma^2+0.0297\,d^2\sigma^2$
3.7	1.4867×10^{-5}	$0.0134\,\sigma^2+0.0264\,d^2\sigma^2$
4.0	2.6886×10^{-5}	$0.0125\,\sigma^2+0.0234\,d^2\sigma^2$

Table 1 (Continued)

(6,6,5,5,4), $N = 529$, $n_a = 30$, $a = 1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	6.3360×10^{-7}	$0.0071\,\sigma^2 + 0.0270\,d^2\sigma^2$
1.3	2.8139×10^{-9}	$0.0055\,\sigma^2 + 0.0161\,d^2\sigma^2$
*1.4142	0.0000	$0.0050\,\sigma^2 + 0.0132\,d^2\sigma^2$
1.6	1.3375×10^{-6}	$0.0043\,\sigma^2 + 0.0097\,d^2\sigma^2$
1.9	9.1298×10^{-6}	$0.0034\,\sigma^2 + 0.0060\,d^2\sigma^2$
2.2	3.3293×10^{-5}	$0.0027\,\sigma^2 + 0.0039\,d^2\sigma^2$
2.5	9.3488×10^{-5}	$0.0022\,\sigma^2 + 0.0026\,d^2\sigma^2$
2.8	2.2573×10^{-4}	$0.0018\,\sigma^2 + 0.0018\,d^2\sigma^2$
3.1	5.9124×10^{-3}	$0.0015\,\sigma^2 + 0.0012\,d^2\sigma^2$
3.4	1.0110×10^{-3}	$0.0013\,\sigma^2 + 0.0001\,d^2\sigma^2$
3.7	1.8903×10^{-3}	$0.0011\,\sigma^2 + 0.0007\,d^2\sigma^2$
4.0	3.4224×10^{-3}	$0.0010\,\sigma^2 + 0.0005\,d^2\sigma^2$

(7,7,3,3,1), $N = 128$, $n_a = 2$, $a = 1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	3.0691×10^{-6}	$0.0357\,\sigma^2 + 0.1616\,d^2\sigma^2$
1.3	1.057×10^{-4}	$0.0325\,\sigma^2 + 0.1353\,d^2\sigma^2$
*1.4142	0.0000	$0.0313\,\sigma^2 + 0.125\,d^2\sigma^2$
1.6	1.2815×10^{-6}	$0.0292\,\sigma^2 + 0.1092\,d^2\sigma^2$
1.9	1.1978×10^{-5}	$0.026\,\sigma^2 + 0.0866\,d^2\sigma^2$
2.2	4.2660×10^{-5}	$0.0231\,\sigma^2 + 0.0681\,d^2\sigma^2$
2.5	1.1037×10^{-4}	$0.0204\,\sigma^2 + 0.0533\,d^2\sigma^2$
2.8	2.4241×10^{-4}	$0.0181\,\sigma^2 + 0.0418\,d^2\sigma^2$
3.1	4.8104×10^{-4}	$0.016\,\sigma^2 + 0.0328\,d^2\sigma^2$
3.4	6.4861×10^{-4}	$0.0142\,\sigma^2 + 0.0259\,d^2\sigma^2$
3.7	5.1587×10^{-3}	$0.0127\,\sigma^2 + 0.0206\,d^2\sigma^2$
4.0	4.1322×10^{-3}	$0.0114\,\sigma^2 + 0.0165\,d^2\sigma^2$

Table 1 (Continued)

$(8,14,7,4,3)$, $N=432$, $n_a=4$, $a=2$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	5.0028×10^{-7}	$0.0083\,\sigma^2+0.0300\,d^2\sigma^2$
1.3	3.0915×10^{-7}	$0.0071\,\sigma^2+0.0274\,d^2\sigma^2$
1.6	1.2642×10^{-7}	$0.0076\,\sigma^2+0.0246\,d^2\sigma^2$
1.9	9.8451×10^{-9}	$0.0071\,\sigma^2+0.0217\,d^2\sigma^2$
*2.0	0.0000	$0.0069\,\sigma^2+0.0208\,d^2\sigma^2$
2.2	4.8881×10^{-8}	$0.0066\,\sigma^2+0.0190\,d^2\sigma^2$
2.5	3.7807×10^{-7}	$0.0062\,\sigma^2+0.0165\,d^2\sigma^2$
2.8	1.1947×10^{-6}	$0.0057\,\sigma^2+0.0142\,d^2\sigma^2$
3.1	2.7814×10^{-6}	$0.0053\,\sigma^2+0.0121\,d^2\sigma^2$
3.4	5.5356×10^{-6}	$0.0049\,\sigma^2+0.0103\,d^2\sigma^2$
3.7	5.9220×10^{-6}	$0.0045\,\sigma^2+0.0116\,d^2\sigma^2$
4.0	1.6935×10^{-5}	$0.0042\,\sigma^2+0.0075\,d^2\sigma^2$

$(9,12,4,3,1)$, $N=162$, $n_a=1$, $a=1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	4.4465×10^{-7}	$0.0294\,\sigma^2+0.1401\,d^2\sigma^2$
1.3	4.4432×10^{-8}	$0.0283\,\sigma^2+0.1294\,d^2\sigma^2$
*1.4142	0.0000	$0.0278\,\sigma^2+0.1250\,d^2\sigma^2$
1.6	1.5215×10^{-7}	$0.0269\,\sigma^2+0.1176\,d^2\sigma^2$
1.9	1.3309×10^{-6}	$0.0255\,\sigma^2+0.1053\,d^2\sigma^2$
2.2	4.4165×10^{-6}	$0.0241\,\sigma^2+0.0818\,d^2\sigma^2$
2.5	1.0622×10^{-5}	$0.0225\,\sigma^2+0.0818\,d^2\sigma^2$
2.8	2.1672×10^{-5}	$0.0211\,\sigma^2+0.0713\,d^2\sigma^2$
3.1	3.9978×10^{-5}	$0.0210\,\sigma^2+0.0713\,d^2\sigma^2$
3.4	3.2676×10^{-5}	$0.0181\,\sigma^2+0.0533\,d^2\sigma^2$
3.7	1.1281×10^{-4}	$0.0168\,\sigma^2+0.0459\,d^2\sigma^2$
4.0	1.7786×10^{-4}	$0.0156\,\sigma^2+0.0396\,d^2\sigma^2$

Table 1 (Continued)

$(10,15,6,4,2),\ N=392,\ n_a=2,\ a=2$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	2.6766×10^{-7}	$0.0100\,\sigma^2+0.0392\,d^2\sigma^2$
1.3	2.2851×10^{-7}	$0.0097\,\sigma^2+0.0382\,d^2\sigma^2$
1.6	6.5354×10^{-8}	$0.0094\,\sigma^2+0.0347\,d^2\sigma^2$
1.9	4.9800×10^{-9}	$0.0091\,\sigma^2+0.0321\,d^2\sigma^2$
*2.0	0.0000	$0.0089\,\sigma^2+0.0313\,d^2\sigma^2$
2.2	2.4136×10^{-8}	$0.0087\,\sigma^2+0.0295\,d^2\sigma^2$
2.5	1.8187×10^{-7}	$0.0083\,\sigma^2+0.0267\,d^2\sigma^2$
2.8	5.5904×10^{-7}	$0.0079\,\sigma^2+0.0240\,d^2\sigma^2$
3.1	1.2648×10^{-6}	$0.0074\,\sigma^2+0.0217\,d^2\sigma^2$
3.4	2.4446×10^{-6}	$0.0070\,\sigma^2+0.0194\,d^2\sigma^2$
3.7	4.2898×10^{-6}	$0.0066\,\sigma^2+0.0172\,d^2\sigma^2$
4.0	7.0498×10^{-6}	$0.0063\,\sigma^2+0.0153\,d^2\sigma^2$

$(11,11,5,5,2),\ N=450,\ n_a=10,\ a=1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	4.6106×10^{-7}	$0.0100\,\sigma^2+0.0450\,d^2\sigma^2$
1.3	5.0041×10^{-8}	$0.0088\,\sigma^2+0.0347\,d^2\sigma^2$
*1.4142	0.0000	$0.0083\,\sigma^2+0.0313\,d^2\sigma^2$
1.6	1.8851×10^{-7}	$0.0076\,\sigma^2+0.0261\,d^2\sigma^2$
1.9	1.8295×10^{-6}	$0.0066\,\sigma^2+0.0194\,d^2\sigma^2$
2.2	6.7682×10^{-6}	$0.0057\,\sigma^2+0.0144\,d^2\sigma^2$
2.5	1.8174×10^{-5}	$0.0049\,\sigma^2+0.0107\,d^2\sigma^2$
2.8	4.1360×10^{-5}	$0.0042\,\sigma^2+0.0080\,d^2\sigma^2$
3.1	8.4858×10^{-5}	$0.0037\,\sigma^2+0.0061\,d^2\sigma^2$
3.4	1.6188×10^{-4}	$0.0032\,\sigma^2+0.0046\,d^2\sigma^2$
3.7	2.9223×10^{-4}	$0.0028\,\sigma^2+0.0036\,d^2\sigma^2$
4.0	5.0486×10^{-4}	$0.0025\,\sigma^2+0.0028\,d^2\sigma^2$

Table 1 (Continued)

(12,33,11,4,3), $N=768$, $n_a=2$, $a=2$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	2.4861×10^{-8}	$0.0056\,\sigma^2+0.0237\,d^2\sigma^2$
1.3	1.4960×10^{-8}	$0.0055\,\sigma^2+0.0231\,d^2\sigma^2$
1.6	5.9218×10^{-9}	$0.0064\,\sigma^2+0.0221\,d^2\sigma^2$
1.9	1.9316×10^{-8}	$5.2510\,\sigma^2+0.0186\,d^2\sigma^2$
*2.0	0.0000	$0.0052\,\sigma^2+0.0208\,d^2\sigma^2$
2.2	2.1134×10^{-9}	$0.0051\,\sigma^2+0.0201\,d^2\sigma^2$
2.5	1.5608×10^{-8}	$0.0050\,\sigma^2+0.0190\,d^2\sigma^2$
2.8	4.6944×10^{-8}	$0.0048\,\sigma^2+0.0179\,d^2\sigma^2$
3.1	1.0378×10^{-7}	$0.0047\,\sigma^2+0.0167\,d^2\sigma^2$
3.4	1.9577×10^{-7}	$0.0045\,\sigma^2+0.0155\,d^2\sigma^2$
3.7	3.3499×10^{-7}	$0.0043\,\sigma^2+0.0144\,d^2\sigma^2$
4.0	5.3644×10^{-7}	$0.0042\,\sigma^2+0.0133\,d^2\sigma^2$

(13,13,4,4,1), $N=324$, $n_a=2$, $a=1.4142$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	1.1116×10^{-7}	$0.0147\,\sigma^2+0.0701\,d^2\sigma^2$
1.3	1.1108×10^{-8}	$0.0141\,\sigma^2+0.0647\,d^2\sigma^2$
*1.4142	0.0000	$0.0139\,\sigma^2+0.0625\,d^2\sigma^2$
1.6	3.8037×10^{-8}	$0.0135\,\sigma^2+0.0588\,d^2\sigma^2$
1.9	2.7418×10^{-7}	$0.0127\,\sigma^2+0.0536\,d^2\sigma^2$
2.2	1.1041×10^{-6}	$0.0120\,\sigma^2+0.0466\,d^2\sigma^2$
2.5	2.6554×10^{-6}	$0.0112\,\sigma^2+0.0409\,d^2\sigma^2$
2.8	5.4179×10^{-6}	$0.0105\,\sigma^2+0.0356\,d^2\sigma^2$
3.1	9.9945×10^{-6}	$0.0097\,\sigma^2+0.0309\,d^2\sigma^2$
3.4	1.7215×10^{-5}	$0.0091\,\sigma^2+0.0267\,d^2\sigma^2$
3.7	2.8204×10^{-5}	$0.0084\,\sigma^2+0.0230\,d^2\sigma^2$
4.0	4.4465×10^{-5}	$0.0078\,\sigma^2+0.0210\,d^2\sigma^2$

Table 1 (Continued)

(15,15,7,7,3), $N=1,200$, $n_a=1$, $a=4$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	1.0397×10^{-8}	$0.0022\,\sigma^2+0.0059\,d^2\sigma^2$
1.3	9.2953×10^{-9}	$0.0021\,\sigma^2+0.0059\,d^2\sigma^2$
1.6	8.1153×10^{-9}	$0.0020\,\sigma^2+0.0058\,d^2\sigma^2$
1.9	7.0259×10^{-9}	$0.0022\,\sigma^2+0.0058\,d^2\sigma^2$
2.2	5.7302×10^{-9}	$0.0022\,\sigma^2+0.0057\,d^2\sigma^2$
2.5	4.4001×10^{-9}	$0.0022\,\sigma^2+0.0057\,d^2\sigma^2$
2.8	3.1029×10^{-9}	$0.0022\,\sigma^2+0.0056\,d^2\sigma^2$
3.1	1.9171×10^{-9}	$0.0021\,\sigma^2+0.0055\,d^2\sigma^2$
3.4	3.4887×10^{-10}	$0.0021\,\sigma^2+0.0053\,d^2\sigma^2$
3.7	2.5486×10^{-10}	$0.0021\,\sigma^2+0.0053\,d^2\sigma^2$
*4.0	0.0000	$0.0021\,\sigma^2+0.0052\,d^2\sigma^2$

(16,16,6,6,2), $N=676$, $n_a=1$, $a=2.8284$		
a	$Q_v(D)$	$V\left(\partial\hat{Y}/\partial x_i\right)$
1.0	3.7031×10^{-8}	$0.0052\,\sigma^2+0.0180\,d^2\sigma^2$
1.3	3.0297×10^{-8}	$0.0051\,\sigma^2+0.0177\,d^2\sigma^2$
1.6	2.2713×10^{-8}	$0.0051\,\sigma^2+0.0174\,d^2\sigma^2$
1.9	1.4945×10^{-8}	$0.0050\,\sigma^2+0.0170\,d^2\sigma^2$
2.2	7.8375×10^{-9}	$0.0049\,\sigma^2+0.0166\,d^2\sigma^2$
2.5	2.4369×10^{-9}	$0.0049\,\sigma^2+0.0162\,d^2\sigma^2$
2.8	2.0686×10^{-11}	$0.0048\,\sigma^2+0.0157\,d^2\sigma^2$
*2.82843	0.0000	$0.0048\,\sigma^2+0.0156\,d^2\sigma^2$
3.1	2.1303×10^{-9}	$0.0047\,\sigma^2+0.0151\,d^2\sigma^2$
3.4	1.0611×10^{-8}	$0.0046\,\sigma^2+0.0146\,d^2\sigma^2$
3.7	2.7655×10^{-8}	$0.0046\,\sigma^2+0.0140\,d^2\sigma^2$
4.0	5.5855×10^{-8}	$0.0045\,\sigma^2+0.0135\,d^2\sigma^2$

Note: * denotes the exact modified slope rotatability value using BIBD

8. Conclusions

In this paper, measure of modified slope rotatability for second order response surface designs using BIBD has been proposed which enables us to assess the degree of slope rotatability for a given response surface design. This measure of modified slope rotatability for second order response surface designs using BIBD, $Q_v(D)$ has the value zero, if and only if, the design 'D' is modified slope rotatable design, and becomes larger as 'D' deviates from a modified slope rotatable design.

Variances of the estimated response for measure of modified slope rotatability for second order response surface designs using BIBD are also obtained.

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References

Box GEP, Hunter JS. Multifactor experimental designs for exploring response surfaces. Ann Math Stat. 1957; 28: 195-241.

Das MN, Narasimham VL. Construction of rotatable designs through balanced incomplete block designs. Ann Math Stat. 1962; 33: 1421-1439.

Das MN, Parsad R, Manocha VP. Response surface designs, symmetrical and asymmetrical, rotatable and modified. Stat Appl. 1999; 1: 17-34.

Hader RJ, Park SH. Slope-rotatable central composite designs. Technometrics. 1978; 20(4): 413-417.

Park SH. A class of multi-factor designs for estimating the slope of response surface. Technometrics. 1987; 29(4): 449-453.

Park SH, Kim HJ. A measure of slope-rotatability for second order response surface experimental designs. J Appl Stat. 1992; 19: 391-404.

Raghavarao D. Constructions and combinatorial problems in design of experiments. New York: John Wiley & Sons; 1971.

Victorbabu BRe. Modified slope rotatable central composite designs. J Korean Stat Soc. 2005; 34: 153-160.

Victorbabu BRe. Modified second order slope rotatable designs using BIBD. J Korean Stat Soc. 2006; 35(2): 179-192.

Victorbabu BRe. On second order slope rotatable designs-A review. J Korean Stat Soc. 2007; 33: 373-386.

Victorbabu BRe, Narasimham VL. Construction of second order slope rotatable designs through balanced incomplete block designs. Commun Stat-Theory Method. 1991; 20: 2467-2478.

Victorbabu BRe, Jyostna P. Measure of modified slope rotatability for second order response surface designs. Thail Stat. 2021; 19(1): 195-207.

Victorbabu BRe, Surekha ChVVS. Construction of measure of second order slope rotatable designs using balanced incomplete block designs. J Stat. 2012; 19: 1-10.

Victorbabu BRe, Surekha ChVVS. On measure of second order slope rotatable designs using partially balanced incomplete block type designs. Thail Stat. 2013; 11(2): 159-171.

Victorbabu BRe, Surekha ChVVS. A note on measure of slope rotatability for second order response surface designs using a pair of balanced incomplete block designs. Thail Stat. 2016; 14(1): 25-33.