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Monte Carlo Simulation of Stress-Strength Model and Reliability Estimation for Extension of the Exponential Distribution

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Abstract

In this paper, parameter estimation and reliability estimation for the extension of the exponential distribution are discussed. Point estimations of the stress-strength model deliberated. The maximum likelihood, maximum product spacing and Bayesian estimation are obtained. The results of Bayesian estimation are computed under the squared error loss (SEL) based on Markov chain Monte Carlo (MCMC). Monte Carlo simulation introduced to explicate and comparison the precision of the obtained estimators. An empirical study by using real data sets were conducted to support the theoretical aspect.

Keywords: Maximum likelihood, maximum product spacing, Bayesian estimation, reliability stress-strength.

1. Introduction

The extension of the exponential (ExEx) distribution was introduced by Nadarajah and Haghighi (2011). We are interested in the inferential analysis for the ExEx distribution by using different estimation methods. The cumulative distribution function (cdf) and the probability density function (pdf) of the ExEx distribution are respectively as follows

$$F(x;\alpha,\lambda)=1-e^{\left[1-(1+\lambda x)^{\alpha}\right]};\ x\geq 0,\ \alpha,\lambda\succ 0, \tag{1}$$

and

$$f(x;\alpha,\lambda)=\lambda\alpha(1+\lambda x)^{\alpha-1}e^{\left[1-(1+\lambda x)^{\alpha}\right]}. \tag{2}$$

Figure 1 displays the pdf of the ExEx distribution for some values of parameters. The model can be considered as another useful two-parameter of the ExEx distribution. It is noted that the ExEx distribution is reduced to the exponential distribution for $\alpha=1$. Kumar et al. (2017) established recurrence relations for the single and product moments of order statistics from the EE distribution.

El-Damcese and Ramadan (2015) introduced modified ExEx distribution with three parameters. El-Din et al. (2016) discussed constant-stress accelerated life test is assumed when the lifetime of test units follows an ExEx distribution. In literature, estimation of parameters for ExEx distribution is discussed extensively, but no one has performed comparison of maximum likelihood estimation and maximum product spacing.

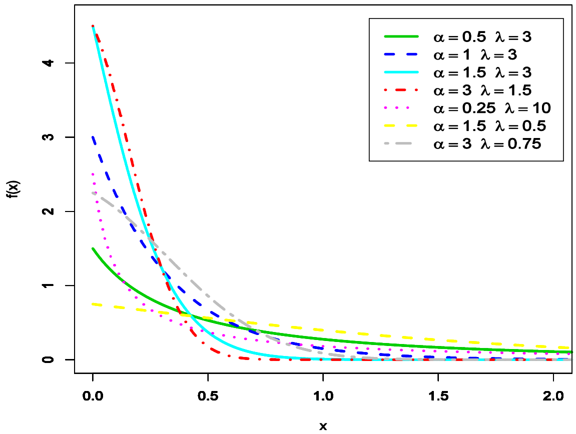

Figure 1 Plot of the ExEx PDF for some values of parameters

The statistical inferential analysis of lifetime data used follows a particular statistical distribution. Many of researchers uses traditional estimation methods such as the method of moments, maximum likelihood estimation (MLE) and maximum product spacing estimation (MPS). Each of them having their own advantages and limitations but the most popular method of estimation is MLE method.

Ekström (2006) discussed MPS an alternative to the MLE method. In many situations, the MPS method works better than the MLE method and attractive properties such as consistency and asymptotic efficiency of the MPS estimator closely parallel those of the MLE when the latter works well. For more information of the methods (see Ranneby 1984), (see for example, Singh et al. 2014b, Almetwally et al. 2018, Ahmad and Almetwally 2020, and El-Sherpieny et al. 2020).

The stress-strength reliability involving two independent random variables X and Y, where X represents the stress variable and Y represents the strength variable is defined as $R = P(Y < X)$. The estimation of stress-strength reliability has been widely used in the fields of aeronautical, civil, mechanical and electronic engineering. Singh et al. (2014a) discussed the problem of the estimation of stressed system reliability under both classical and Bayesian paradigms. Mokhlis et al. (2017) presented characterizations, associated with the stress strength reliability, of distributions with some general exponential and general inverse exponential forms. Different methods to estimate Fréchet distribution parameter based on stress-strength model (R) as maximum likelihood estimator, moments estimator, regression estimator, percentile estimator, least square estimator and L-moments estimator are studied by Abid (2014). Sabry et al. (2021) discussed fuzzy stress strength reliability model for inverse Rayleigh distribution. Abu El Azm et al. (2021) discussed stress-strength reliability for exponentiated inverted Weibull distribution with application on breaking of jute fiber and carbon fibers. Yousef and Almetwally (2021) discussed multi stress-strength reliability based on progressive first failure for Kumaraswamy distribution.

In this paper, the MLE, MPS and Bayesian estimation under square error loss function are applied to estimate the parameters of the ExEx distribution. The purpose of this study is to discuss the best estimation method for parameters of ExEx distribution, which we think would be of deep interest to statisticians. Reliability Functions (survival and hazard) are used for comparing between methods. The expression of reliability form stress-strength model based on ExEx distribution has been derived. A simulation study is conducted to compare the preferences between estimation methods. Also, a real data set is used and analyzed to investigate the model.

2. Reliability Estimation

In this section, we propose the estimation of survival and hazard functions using MLE, MPS and Bayesian estimation under square error loss function for specified value of time say $\left(t = \text{mean}(x)\right)$ where $\text{mean}(x) = \frac{1}{\lambda}\left(e\,\text{Gamma}\left(1+\frac{1}{\alpha},1\right)-1\right)$. Cheng and Amin (1983) had mentioned in their paper that MPS also shows the invariance property just like MLE. So, on this basis using the invariance property we estimate the survival and hazard functions. The survival and hazard functions are given as follows

$$\hat{\text{s}}(x=t) = e^{\left[1-(1+\hat{\lambda}x)^{\hat{\alpha}}\right]}, \tag{3}$$

the hazard function defined by

$$\hat{H}(x=t) = \hat{\lambda}\hat{\alpha}(1+\hat{\lambda}x)^{\hat{\alpha}-1}. \tag{4}$$

Figure 2 displays the hazard functions of the ExEx distribution by (4) for some values of parameters as follows

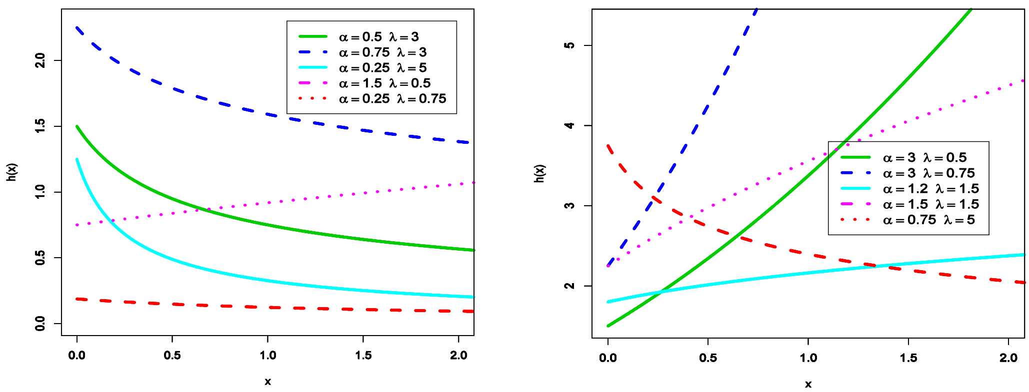

Figure 2 Plots of the ExEx hazard functions for some values of parameters

To obtained parameters estimation $\hat{\alpha}$ and $\hat{\lambda}$, we can use more estimation methods, but for the considered distribution, the MLE method, one which is not very common i.e. MPS method and Bayesian estimation by MCMC method will be used for estimating the parameters of an extension of the exponential distribution.

2.1. Maximum likelihood estimation

Casella and Berger (1990) discussed the likelihood function is the most frequently used method of parameter estimation. By using (1), the likelihood function of the an extension of the exponential distribution is

$$L_{ML} = (\lambda\alpha)^n \, e^{\sum_{i=1}^{n}(1-(1+\lambda x_i)^{\alpha})} \prod_{i=1}^{n}(1+\lambda x_i)^{\alpha-1}, \tag{5}$$

and by using (5), the log likelihood function is given as

$$\ln L_{ML} = n(\ln\alpha + \ln\lambda) + \sum_{i=1}^{n}(1-(1+\lambda x_i)^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\ln(1+\lambda x_i), \tag{6}$$

respectively. To obtain the normal equations for the unknown parameters, we differentiate Equation (6) partially with respect to the parameters λ and α, and equate them to zero. The estimators $\hat{\lambda}_{ML}$ and $\hat{\alpha}_{ML}$ can be obtained as the solution of the following equations:

$$\frac{\partial \ln L_{ML}}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}(1+\lambda x_i)^{\alpha}\ln(1+\lambda x_i) + \sum_{i=1}^{n}\ln(1+\lambda x_i)$$

$$\frac{\partial \ln L_{ML}}{\partial\lambda} = \frac{n}{\lambda} - \alpha\sum_{i=1}^{n}x_i(1+\lambda x_i)^{\alpha-1} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i}{(1+\lambda x_i)}.$$

The partial derivatives of MLE with respect to the unknown parameters cannot be solved directly, so we utilize numerical methods like the Newton-Raphson algorithms to calculate the MLE estimators.

2.2. Maximum product spacing

According to Cheng and Amin (1983) introduced maximum product spacing as following

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}, \tag{7}$$

where G is defined as the geometric mean of the product spacing function and where

$$D_i = \begin{cases} D_1 = F(x_1), \\ D_i = F(x_i) - F(x_{i-1});\, i = 2,\dots,n, \\ D_{n+1} = 1 - F(x_n), \end{cases}$$

such that $\sum_{i=1}^{n+1} D_i = 1$ by using (2) and (7), the product spacing function of ExEx distribution is

$$G = \left((1-e^{(1-(1+\lambda x_1)^{\alpha})})(e^{(1-(1+\lambda x_n)^{\alpha})}) \prod_{i=2}^{n}\left[e^{(1-(1+\lambda x_{i-1})^{\alpha})} - e^{(1-(1+\lambda x_i)^{\alpha})} \right] \right)^{\frac{1}{n+1}}. \tag{8}$$

By using (8), the natural logarithm of the product spacing function is

$$\ln G = \frac{1}{n+1}\left[\ln(1-e^{(1-(1+\lambda x_1)^{\alpha})}) + (1-(1+\lambda x_n)^{\alpha}) + \sum_{i=2}^{n}\ln\left(e^{(1-(1+\lambda x_{i-1})^{\alpha})} - e^{(1-(1+\lambda x_i)^{\alpha})} \right) \right]. \tag{9}$$

To obtain the normal equations for the unknown parameters, we differentiate Equation (9) partially with respect to the parameters λ and α and equate them to zero. The estimators $\hat{\lambda}_{MPS}$ and $\hat{\alpha}_{MPS}$ of λ and α can be obtained as the solution of the following equations:

$$\frac{\partial \ln G}{\partial\lambda} = \frac{1}{n+1}\left[\begin{gathered} \frac{(e^{(1-(1+\lambda x_1)^{\alpha})})\alpha x_1\left(1+\lambda x_1\right)^{\alpha-1}}{\left(1-(e^{(1-(1+\lambda x_1)^{\alpha})})\right)} - \alpha x_n\left(1+\lambda x_n\right)^{\alpha-1} \\ + \sum_{i=2}^{n}\frac{\left(e^{(1-(1+\lambda x_i)^{\alpha})}\right)\alpha x_i\left(1+\lambda x_i\right)^{\alpha-1}\Big) - (e^{(1-(1+\lambda x_{i-1})^{\alpha})})\alpha x_{i-1}\left(1+\lambda x_{i-1}\right)^{\alpha-1}}{(e^{(1-(1+\lambda x_{i-1})^{\alpha})}) - (e^{(1-(1+\lambda x_i)^{\alpha})})} \end{gathered} \right],$$

$$\frac{\partial \ln G}{\partial \alpha}=\frac{1}{n+1}\left[\begin{array}{l}\dfrac{(e^{(1-(1+\lambda x_1)^{\alpha})})\ln\left(1+\lambda x_1\right)}{\left(1-e^{(1-(1+\lambda x_1)^{\alpha})}\right)}-\left(1+\lambda x_n\right)^{\alpha}\ln\left(1+\lambda x_n\right)\\ +\displaystyle\sum_{i=2}^{n}\frac{\left(e^{(1-(1+\lambda x_i)^{\alpha})}\ln\left(1+\lambda x_i\right)\right)-\left(e^{(1-(1+\lambda x_{i-1})^{\alpha})}\ln\left(1+\lambda x_{i-1}\right)\right)}{(e^{(1-(1+\lambda x_{i-1})^{\alpha})})-(e^{(1-(1+\lambda x_i)^{\alpha})})}\end{array}\right].$$

The partial derivatives of MPS with respect to the unknown parameters cannot be solved directly, so we utilize numerical methods like the Newton-Raphson algorithms to calculate the MPS estimators.

2.3. Bayesian estimation

In this section we consider the Bayesian estimation of the unknown parameters λ and α. The Bayes estimates is considered under the assumption that the random variables λ and α have an independent gamma distribution is a conjugate prior to the extension of the exponential distribution distributions. Assumed that $\lambda \sim \text{Gamma}(a,b)$ and $\alpha \sim \text{Gamma}(c,d)$, then, the joint prior density of λ and α can be written as

$$g(\lambda,\,\alpha)\propto \lambda^{\alpha-1}\,e^{-\frac{\lambda}{b}}\,\alpha^{c-1}\,e^{-\frac{\alpha}{d}};\quad a,b,c,\text{ and } d\succ 0, \tag{10}$$

here all the hyper parameters a,b,c and d are known and non-negative. For the choice of hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. Such that $E(\alpha)=ba;\ \text{var}\left(\alpha\right)=ab^2.$

Based on the likelihood function (5) and the joint prior density (10), the joint posterior of α and λ is

$$g(\alpha,\lambda\,|\,x)=K\,\lambda^{n+a-1}\,\alpha^{n+c-1}e^{-\frac{\lambda}{b}-\frac{\alpha}{d}}\,e^{\sum_{i=1}^{n}\left(1-(1+\lambda x_i)^{\alpha}\right)}\prod_{i=1}^{n}\left(1+\lambda x_i\right)^{\alpha-1}, \tag{11}$$

here the normalizing constant K is

$$K_{ML}=\left[\iint_0 \lambda^{n+\alpha-1}\,\alpha^{n+c-1}\,e^{-\frac{\lambda}{b}-\frac{\alpha}{d}}\,e^{\sum_{i=1}^{n}\left(1-(1+\lambda x_i)^{\alpha}\right)}\prod_{i=1}^{n}\left(1+\lambda x_i\right)^{\alpha-1}d\theta\,d\alpha\right]^{-1}.$$

In the method of the Markov chain Monte Carlo (MCMC) can be used to generate samples from the posterior density function (11) and in turn to compute the Bayes estimates of the unknown parameters. To generate samples from (11), we can rewrite the posterior density (11) as $g(\alpha,\,\lambda\,|\,x)\propto g_1(\alpha\,|\,\lambda,\,x)\,g_2(\lambda\,|x).$

For the extension of the exponential distribution distributions, the full conditional posterior distributions of the parameters are given by

$$g_1(\alpha;\,\lambda,\,x)\propto \alpha^{n+c-1}\,e^{-\frac{\alpha}{d}}\,e^{\sum_{i=1}^{n}\left(1-(1+\lambda x_i)^{\alpha}\right)}\prod_{i=1}^{n}\left(1+\lambda x_i\right)^{\alpha-1},$$

$$g_2(\lambda;\,\alpha,\,x)\propto \lambda^{n+a-1}\,e^{-\frac{\lambda}{b}}\,e^{\sum_{i=1}^{n}\left(1-(1+\lambda x_i)^{\alpha}\right)}\prod_{i=1}^{n}\left(1+\lambda x_i\right)^{\alpha-1}.$$

Since the full conditional posterior distributions do not have simple forms in perspective of sampling, we use the Metropolis-Hastings algorithm within each Gibbs chain. To more example see (Almetwally et al. 2018, Almetwally et al. 2019).

In our simulation study presented in the next section, MCMC procedure is used to generate the full conditional posterior distributions of λ and α under squared error loss function, obtain the

Bayes estimates of λ and α as $\tilde{\lambda}=\sum_{i=1}^{N}\frac{\lambda^{\langle i\rangle}}{N}$ and $\tilde{\alpha}=\sum_{i=1}^{N}\frac{\alpha^{\langle i\rangle}}{N}$, respectively. We set the number of periods in the MCMC process to be $N=10,000$.

3. Stress-Strength Model

Let X and Y are the independent strength and stress random variables observed from ExEx distribution. Then, the stress-strength reliability R is calculated as

$$R=P(Y\prec X)=\int_{X=0}^{\infty}\left(\int_{y=0}^{\infty}f(y;\,\alpha_2,\lambda_2)dy\right)f(x;\alpha_1,\lambda_1)\,dx.$$

Then, we have

$$\hat{R}=1-\hat{\lambda}_1\hat{\alpha}_1\int_0^{\infty}(1+\hat{\lambda}_1x)^{\hat{\alpha}_1-1}e^{\left(2-(1+\hat{\lambda}_1x)^{\hat{\alpha}_1}-(1+\hat{\lambda}_2x)^{\hat{\alpha}_2}\right)}dx. \tag{12}$$

If $X\sim\mathrm{ExEx}(\alpha_1,\lambda_1)$ and $Y\sim\mathrm{ExEx}(\alpha_2,\lambda_2)$,

$$\hat{R}=1-\alpha_1e^2\sum_{p=0}^{\infty}\binom{\alpha_1-1}{p}(\lambda_1)^{1+p}\int_0^{\infty}x^pe^{-\sum_{K-0}^{\infty}\binom{\alpha_1}{K}(\lambda_1\,x)^K-\sum_{L-0}^{\infty}\binom{\alpha_2}{L}(\lambda_2\,x)^L}dx.$$

Figure 3 displays plots of the reliability functions of stress-strength for the ExEx distribution by (12) for different values of parameters as follows

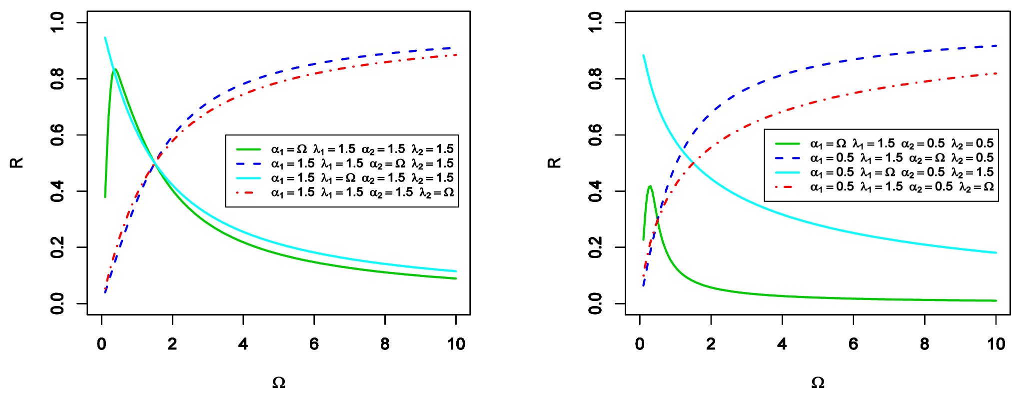

Figure 3 Plots of the ExEx reliability functions of stress-strength by some values of parameters

The MLE, MPS and Bayesian estimation will be obtained to get the estimated model parameters as well as the stress-strength system reliability in case of complete sample of observations.

3.1. Maximum likelihood estimation

The likelihood function of an extension of the exponential distribution for stress-strength model is

$$L(\Omega)=\prod_{i=1}^{n}f(x_i;\,\alpha_1,\lambda_1)\prod_{j=1}^{m}f(y_j;\alpha_2,\lambda_2),$$

and the log-likelihood function of ExEx distribution under stress-strength model is given as

$$l(\Omega)=\ln L(\Omega)=n(\ln\alpha_1+\ln\lambda_1)+\sum_{i=1}^{n}(1-(1+\lambda_1 x_i)^{\alpha_1})+(\alpha_1-1)\sum_{i=1}^{n}\ln(1+\lambda_1 x_i)$$

$$+m(\ln\alpha_2+\ln\lambda_2)+\sum_{j=1}^{m}(1-(1+\lambda_2 y_j)^{\alpha_2})+(\alpha_2-1)\sum_{j=1}^{m}\ln(1+\lambda_2 y_j), \tag{13}$$

respectively. To obtain the normal equations for the unknown parameters, we differentiate Equation (13) partially with respect to the parameters Ω where $\Omega=\left(\alpha_1,\lambda_1,\alpha_2,\lambda_2\right)$ and equate them to zero. The estimators $\hat{\Omega}$ can be obtained as the solution of the following equations.

$$\frac{\partial l(\Omega)}{\partial\alpha_j}=\frac{S_j}{\alpha_j}-\sum_{i=1}^{S_j}(1+\lambda_j z_{ij})^{\alpha_j}\ln(1+\lambda_j z_{ij})\sum_{i=1}^{S_j}\ln(1+\lambda_j z_{ij});\quad j=1,2,$$

$$\frac{\partial l(\Omega)}{\partial\lambda_j}=\frac{S_j}{\lambda_j}-\alpha_j\sum_{i=1}^{S_j}z_{ij}(1+\lambda_j z_{ij})^{\alpha_j-1}+(\alpha_j-1)\sum_{i=1}^{S_j}\frac{z_{ij}}{(1+\lambda_j z_{ij})},$$

where $z_{ij}=\left(x_i,y_j\right),\ s_j=\left(n,m\right)$.

3.2. Maximum product spacing

The Maximum product spacing for stress-strength model is denoted as following

$$GS(\Omega)=\left(\prod_{i=1}^{n+1}D_i(x_i;\ \alpha_1,\ \lambda_1)\right)^{\frac{1}{n+1}}\left(\prod_{j=1}^{m+1}D_i(y_j;\ \alpha_2,\ \lambda_2)\right)^{\frac{1}{m+1}}, \tag{14}$$

such that $\sum_i D_i\left(z_{ij},\alpha_j,\lambda_j\right)=1,$ where

$$D_i(z_{ji};\alpha_j,\lambda_j)=\begin{cases} D_1=F(z_{j1};\alpha_j,\lambda_j),\\ D_i=F(z_{ji};\alpha_j,\lambda_j)-F(z_{j(i-1)};\alpha_j,\lambda_j); i=2,\ldots,s_j, j=1,2,\\ D_{n+1}=1-F(z_{jn};\alpha_j,\lambda_j).\end{cases}$$

By using (14), the natural logarithm of the product spacing function of the ExEx distribution for stress-strength model is denoted as following

$$gs(\Omega)=\ln GS(\Omega)=\frac{1}{n+1}\left[\ln\left(1-e^{\left(1-(1+\lambda_1 x_1)^{\alpha_1}\right)}\right)+\left(1-(1+\lambda_1 x_n)^{\alpha_1}\right)+\sum_{i=2}^{n}\ln\left(e^{\left(1-(1+\lambda_1 x_{i-1})^{\alpha_1}\right)}-e^{\left(1-(1+\lambda_1 x_i)^{\alpha_1}\right)}\right)\right]$$

$$+\frac{1}{m+1}\left[\ln\left(1-e^{\left(1-(1+\lambda_2 y_1)^{\alpha_2}\right)}\right)+\left(1-(1+\lambda_2 y_n)^{\alpha_2}\right)+\sum_{j=2}^{m}\ln\left(e^{\left(1-(1+\lambda_2 y_{j-1})^{\alpha_2}\right)}-e^{\left(1-(1+\lambda_2 y_j)^{\alpha_2}\right)}\right)\right]. \tag{15}$$

To obtain the normal equations for the unknown parameters, we differentiate Equation (15) partially with respect to the parameters Ω and equate them to zero. The estimators $\hat{\Omega}$ can be obtained as the solution of the following equations.

$$\frac{\partial gs(\Omega)}{\partial\lambda_j}=\frac{1}{s_j+1}\left[\frac{e^{\left(1-(1+\lambda_j z_{j1})^{\alpha_j}\right)}\alpha_j z_{j1}\left(1+\lambda_j z_{j1}\right)^{\alpha_j-1}}{\left(1-e^{\left(1-(1+\lambda_j z_{j1})^{\alpha_j}\right)}\right)}-\alpha_j z_{jn}\left(1+\lambda_j z_{jn}\right)^{\alpha_j-1}\right.$$

$$\left.+\sum_{i=2}^{s_j}\frac{\left(e^{\left(1-(1+\lambda_j z_{ji})^{\alpha_j}\right)}\alpha_j z_{ji}\left(1+\lambda_j z_{ji}\right)^{\alpha_j-1}\right)-\left(e^{\left(1-(1+\lambda_j z_{ji-1})^{\alpha_j}\right)}\alpha_j z_{ji-1}\left(1+\lambda_j z_{ji-1}\right)^{\alpha_j-1}\right)}{\left(e^{\left(1-(1+\lambda_j z_{ji-1})^{\alpha_j}\right)}-e^{\left(1-(1+\lambda_j z_{ji})^{\alpha_j}\right)}\right)}\right]$$

$$\frac{\partial gs(\Omega)}{\partial \alpha_j} = \frac{1}{s_j+1}\left[\frac{e^{\left(1-(1+\lambda_j z_{j1})^{\alpha_j}\right)}\ln\left(1+\lambda_j z_{j1}\right)}{\left(1-e^{\left(1-(1+\lambda_j z_{j1})^{\alpha_j}\right)}\right)} - \left(1+\lambda_j z_{jn}\right)^{\alpha_j}\ln\left(1+\lambda_j z_{jn}\right)\right.$$
$$\left.+\sum_{i=2}^{s_j}\frac{\left(e^{\left(1-(1+\lambda_j z_{ji})^{\alpha_j}\right)}\ln\left(1+\lambda_j z_{jn}\right)\right)-\left(e^{\left(1-(1+\lambda_j z_{ji-1})^{\alpha_j}\right)}\ln\left(1+\lambda_j z_{ji-1}\right)\right)}{\left(e^{\left(1-(1+\lambda_j z_{ji-1})^{\alpha_j}\right)}-e^{\left(1-(1+\lambda_j z_{ji})^{\alpha_j}\right)}\right)}\right].$$

The above nonlinear equations cannot be solved analytically so, $\hat{\Omega}$ of Ω.

3.3. Bayesian estimation

In this subsection, we discuss the Bayesian inference of the unknown parameters of an ExEx distribution under stress-strength model. For Bayesian parameter estimation we will consider squared error loss function. When the parameters of the model are unknown, a joint conjugate prior for the parameters does not exist. We suggest using independent gamma priors for Ω having pdfs. The joint prior density of λ and α can be written as

$$g(\Omega) \propto \prod_{j=1}^{2}\lambda_j^{\alpha_j-1}e^{-\frac{\lambda_j}{b_j}}\alpha_j^{c_{j}-1}e^{-\frac{\alpha_j}{d_j}};\quad a,b,c \text{ and } d \succ 0, j=1,2, \tag{16}$$

where the hyper-parameters $a_1,b_1,a_2,b_2,c_1,d_1,c_2$ and d_2 are selected to reflect the prior knowledge about the unknown parameters and

$$\alpha_j = \frac{\left(\frac{1}{K}\sum_{i=1}^{K}\frown{n}_1^i\right)^2}{\frac{1}{K-1}\sum_{i=1}^{K}\frown{n}_1^i\left(\frown{n}_1^i-\frac{1}{K}\sum_{i=1}^{K}\frown{n}_1^i\right)^2} \text{ and } \lambda_j = \frac{\frac{1}{K}\sum_{i=1}^{K}\frown{n}_1^i}{\frac{1}{K-1}\sum_{i=1}^{K}\frown{n}_1^i\left(\frown{n}_1^i-\frac{1}{K}\sum_{i=1}^{K}\frown{n}_1^i\right)^2}.$$

4. Simulation Study

In this section, we provide a complete algorithm of Monte Carlo simulation (MCS) study. We explain our algorithm through an application in estimation methods framework, especially; we will use Monte Carlo technique to compare between MLE, MPS and Bayesian estimation methods based on complete data for estimating ExEx distribution in life time by R language.

Simulation Algorithm: We can build our model by generate all simulation controls. In this stage, we must follow the following steps by order:

Step 1. Suppose different values of the parameters vector of ExEx distribution.

Step 2. Choose the sample size $n = 30, 50$ and 100.

Step 3. Generate the sample random values of ExEx distribution by using quantile function in equation $x_i = \frac{1}{\lambda}\left\{\left[1-\ln\left(1-u_i\right)\right]^{1/\alpha}-1\right\}; 0 < u < 1.$

Step 4. Solve differential equations for each estimation methods, to obtain the estimators of the parameters for ExEx distribution, we calculate $\hat{\alpha}$ and $\hat{\lambda}$.

Step 5. Repeat this experiment $L-1$ times. In each experiment, the same values of the parameters.

It is certain that, the values of generating random are varying from experiment to experiment even though n are not changed. In the end, we have L-values of mean, MSE survival by (3, 12) and hazard by (4), we restricted the number of repeat this experiment to 10,000. Take the averages of these values and call them Monte Carlo estimates, where $\hat{\delta}$ is the estimated value of $\hat{\delta}=(\alpha,\lambda)$ and the mean squared error (MSE) of the estimator, $\text{MSE}=\text{mean}(\hat{\delta}-\delta)^2$. After ending the treatment stage, we must check and evaluate the simulation result before put or discuss (display) it in our paper (research).

All the estimates reveal the property of consistency, i.e., the MSE and reliability measure decrease when n increases. Keeping $\alpha<1$ and fixed, the MSE of $\hat{\lambda}$ decreases, the MSE of $\hat{\alpha}$ increases and the MSE of reliability measure decreases when λ increases, as shown in Table 1. In most cases, the MPS method is superior to the MLE method with an increase hazard. While the Bayesian method remains in the first place in estimation methods.

All the estimates reveal the property of consistency, i.e., the MSE and reliability measure decrease when n increases. Keeping λ fixed, the MSE of $\hat{\alpha}$ decreases, the MSE of $\hat{\alpha}$ decreases and the MSE of reliability measure decreases when λ increases, as shown in Tables 2-4. In most cases, the MPS method is superior to the MLE method with an increase hazard, except in small samples and this is so frequent when $\alpha\le 1.5$. While the Bayesian method remains in the first place in estimation methods, as shown in Figures 4 and 5.

From the Tables 5 and 6, we observe the following: In general, for increasing the sample size, the MSE of the considered parameters and reliability measure of stress-strength decreases. The method of Bayesian shows its absolute efficiency in estimating the parameters and reliability measure of stress-strength model, where MSE values are lower than other methods, as shown in Figure 6.

Table 1 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters when $\alpha=0.5$ and with different values of λ

$\alpha=0.5,\lambda=0.5$							
n		MLE		MPS		MCMC	
		Mean	MSE	Mean	MSE	Mean	MSE
30	$\hat{\alpha}$	0.59488	0.05322	0.48937	0.02388	0.57361	0.02794
	$\hat{\lambda}$	0.51300	0.11205	0.68778	0.19861	0.52821	0.06184
	survival	0.30399	0.00169	0.31374	0.00166	0.26807	0.00135
	hazard	8.96156	8.47578	10.16688	13.03217	8.23178	8.64200
50	$\hat{\alpha}$	0.56512	0.03943	0.49266	0.01601	0.53139	0.00742
	$\hat{\lambda}$	0.50010	0.06144	0.62249	0.10555	0.48734	0.01636
	survival	0.30040	0.00112	0.30682	0.00109	0.29246	0.00057
	hazard	8.90739	5.56718	9.73008	7.56715	8.80939	5.35966
100	$\hat{\alpha}$	0.52436	0.00807	0.48658	0.00596	0.51573	0.00348
	$\hat{\lambda}$	0.49998	0.02901	0.56934	0.04172	0.49516	0.01037
	survival	0.29487	0.00050	0.29890	0.00049	0.29002	0.00028
	hazard	8.98618	2.56423	9.47379	3.22060	8.89299	2.37430

Table 1 (Continued)

n		MLE		MPS		MCMC	
		Mean	MSE	Mean	Mean	MSE	Mean
$\alpha = 0.5, \lambda = 1.5$							
30	$\hat{\alpha}$	0.63314	0.11880	0.51051	0.02151	0.57599	0.02172
	$\hat{\lambda}$	1.37657	0.52672	1.66502	0.43466	1.35241	0.19312
	survival	0.30875	0.00179	0.32058	0.00167	0.29147	0.00068
	hazard	2.89543	0.93810	3.24347	1.27990	2.79876	0.93917
50	$\hat{\alpha}$	0.56224	0.03184	0.49628	0.00862	0.51572	0.00311
	$\hat{\lambda}$	1.39775	0.30360	1.61921	0.24352	1.46457	0.03334
	survival	0.30169	0.00089	0.31011	0.00087	0.29608	0.00054
	hazard	2.94649	0.60509	3.19485	0.78147	2.98323	0.58323
100	$\hat{\alpha}$	0.52162	0.00509	0.49063	0.00363	0.50593	0.00134
	$\hat{\lambda}$	1.44668	0.12824	1.58901	0.14198	1.48368	0.01973
	survival	0.29478	0.00035	0.30057	0.00038	0.29172	0.00020
	hazard	2.99828	0.26497	3.14837	0.32793	3.01095	0.26210
$\alpha = 0.5, \lambda = 3.0$							
30	$\hat{\alpha}$	0.55929	0.05379	0.48941	0.00931	0.53064	0.00707
	$\hat{\lambda}$	2.81155	0.60979	3.11389	0.34674	2.85008	0.27680
	survival	0.29914	0.00071	0.31868	0.00113	0.29594	0.00045
	hazard	1.48963	0.21478	1.65395	0.29913	1.48997	0.21031
50	$\hat{\alpha}$	0.54103	0.03020	0.48654	0.00301	0.50762	0.00212
	$\hat{\lambda}$	2.84105	0.62670	3.09685	0.18276	2.97411	0.03174
	survival	0.29729	0.00052	0.30933	0.00056	0.29410	0.00043
	hazard	1.49297	0.13340	1.60999	0.17023	1.50995	0.12889
100	$\hat{\alpha}$	0.51128	0.00257	0.49054	0.00148	0.50344	0.00103
	$\hat{\lambda}$	2.94140	0.12712	3.07084	0.10072	2.99475	0.01838
	survival	0.29263	0.00018	0.30126	0.00024	0.29229	0.00019
	hazard	1.49281	0.06736	1.55879	0.07885	1.50084	0.06573

Table 2 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters when $\alpha = 1.5$ and with different values of λ

n		MLE		MPS		MCMC	
		Mean	MSE	Mean	MSE	Mean	MSE
$\alpha = 1.5, \lambda = 0.5$							
30	$\hat{\alpha}$	1.66809	0.35623	1.30389	0.29091	1.47989	0.11180
	$\hat{\lambda}$	0.59847	0.18783	0.81946	0.35808	0.60276	0.05966
	survival	0.38658	0.00036	0.39288	0.00032	0.36998	0.00140
	hazard	1.05901	0.02960	1.14573	0.04058	1.01657	0.03136
50	$\hat{\alpha}$	1.62096	0.22994	1.34259	0.20657	1.49200	0.02307
	$\hat{\lambda}$	0.55412	0.09087	0.71726	0.19092	0.52004	0.00835
	survival	0.38945	0.00023	0.39174	0.00025	0.39215	0.00048
	hazard	1.06178	0.01602	1.11830	0.02003	1.06944	0.01560
100	$\hat{\alpha}$	1.58618	0.16217	1.39871	0.12642	1.50184	0.01477
	$\hat{\lambda}$	0.52771	0.04324	0.62067	0.07552	0.51056	0.00477
	survival	0.39102	0.00014	0.39179	0.00016	0.39152	0.00021
	hazard	1.06475	0.00883	1.09711	0.01020	1.06562	0.00867
$\alpha = 1.5, \lambda = 1.5$							
30	$\hat{\alpha}$	1.94743	0.87027	1.46045	0.36492	1.59284	0.10808
	$\hat{\lambda}$	1.51318	0.96218	1.97679	1.18410	1.49941	0.14194
	survival	0.39353	0.00028	0.40161	0.00020	0.39253	0.00081
	hazard	0.35180	0.00333	0.37958	0.00438	0.35430	0.00321
50	$\hat{\alpha}$	1.80966	0.57281	1.45225	0.27681	1.50631	0.01714
	$\hat{\lambda}$	1.52989	0.64473	1.89544	0.84467	1.49713	0.02394
	survival	0.39359	0.00021	0.39753	0.00015	0.40077	0.00114
	hazard	0.35304	0.00215	0.37169	0.00262	0.36355	0.00205
100	$\hat{\alpha}$	1.70656	0.33274	1.46307	0.16467	1.50406	0.01036
	$\hat{\lambda}$	1.48624	0.36150	1.73747	0.42839	1.50190	0.01612
	survival	0.39468	0.00014	0.39531	0.00010	0.39571	0.00044
	hazard	0.35562	0.00105	0.36617	0.00120	0.35948	0.00098
$\alpha = 1.5, \lambda = 3.0$							
30	$\hat{\alpha}$	2.00318	1.11189	1.46458	0.17177	1.57516	0.06323
	$\hat{\lambda}$	2.74916	1.92576	3.37882	1.20923	2.94043	0.23351
	survival	0.39709	0.00011	0.40946	0.00031	0.39980	0.00107
	hazard	0.17367	0.00081	0.18797	0.00103	0.17846	0.00079
50	$\hat{\alpha}$	1.88991	0.82655	1.46449	0.11353	1.51080	0.01408
	$\hat{\lambda}$	2.84384	1.81114	3.28059	0.81588	2.98196	0.03063
	survival	0.39657	0.00015	0.40400	0.00013	0.40000	0.00134
	hazard	0.17583	0.00048	0.18545	0.00057	0.18053	0.00042
100	$\hat{\alpha}$	1.80891	0.50709	1.47446	0.06027	1.50664	0.00802
	$\hat{\lambda}$	2.72458	1.09764	3.15668	0.43294	2.99301	0.01831
	survival	0.39840	0.00014	0.39921	0.00004	0.39491	0.00052
	hazard	0.17775	0.00025	0.18309	0.00029	0.17919	0.00024

Table 3 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters when $\alpha = 3.0$ and with different values of λ

n		MLE		MPS		MCMC	
		Mean	MSE	Mean	MSE	Mean	MSE
$\alpha=3.0, \lambda=0.5$							
30	$\hat{\alpha}$	3.03467	0.31470	2.74217	0.38518	2.87975	0.22365
	$\hat{\lambda}$	0.56012	0.11058	0.62347	0.17813	0.56502	0.02497
	survival	0.41074	0.00028	0.43008	0.00036	0.40490	0.00092
	hazard	0.43445	0.00349	0.46753	0.00489	0.42813	0.00362
50	$\hat{\alpha}$	3.07436	0.33960	2.81452	0.21761	2.97892	0.02981
	$\hat{\lambda}$	0.52749	0.05101	0.56431	0.06492	0.50991	0.00373
	survival	0.41407	0.00014	0.42749	0.00023	0.41706	0.00052
	hazard	0.43610	0.00206	0.45880	0.00273	0.44003	0.00209
100	$\hat{\alpha}$	2.94291	0.25166	2.85951	0.13069	2.99047	0.01867
	$\hat{\lambda}$	0.55194	0.04894	0.54353	0.02662	0.50835	0.00218
	survival	0.41479	0.00010	0.42437	0.00012	0.41795	0.00023
	hazard	0.43486	0.00105	0.44851	0.00120	0.43685	0.00105
$\alpha=3.0, \lambda=1.5$							
30	$\hat{\alpha}$	3.34024	1.23929	2.72874	0.56213	2.99092	0.18260
	$\hat{\lambda}$	1.66399	0.94957	1.82715	0.59593	1.55702	0.08917
	survival	0.41057	0.00017	0.42993	0.00026	0.42093	0.00105
	hazard	0.14280	0.00036	0.15403	0.00046	0.14756	0.00037
50	$\hat{\alpha}$	3.28550	1.15310	2.75542	0.41808	2.99058	0.02650
	$\hat{\lambda}$	1.63631	0.69903	1.75207	0.35555	1.49699	0.01872
	survival	0.41284	0.00009	0.42594	0.00013	0.42476	0.00140
	hazard	0.14441	0.00022	0.15213	0.00028	0.14965	0.00027
100	$\hat{\alpha}$	3.04631	0.74717	2.80515	0.23996	2.99570	0.01723
	$\hat{\lambda}$	1.72029	0.63023	1.67284	0.17363	1.50368	0.01100
	survival	0.41317	0.00008	0.42311	0.00005	0.42167	0.00061
	hazard	0.14486	0.00012	0.14960	0.00014	0.14754	0.00012
$\alpha=3.0, \lambda=3.0$							
30	$\hat{\alpha}$	3.77067	2.78647	2.85302	0.25646	3.02692	0.15340
	$\hat{\lambda}$	2.91819	2.67049	3.18848	0.45241	2.96689	0.16916
	survival	0.41378	0.00017	0.43404	0.00060	0.42841	0.00163
	hazard	0.07194	0.00010	0.07787	0.00013	0.07604	0.00011
50	$\hat{\alpha}$	3.50924	1.86948	2.86380	0.16909	2.99510	0.02509
	$\hat{\lambda}$	3.00010	1.91814	3.18641	0.30135	2.98844	0.02913
	survival	0.41610	0.00007	0.43019	0.00035	0.42680	0.00202
	hazard	0.07186	0.00006	0.07592	0.00007	0.07462	0.00005
100	$\hat{\alpha}$	3.33708	1.09460	2.89536	0.10194	3.00130	0.01522
	$\hat{\lambda}$	3.01925	1.34534	3.11719	0.16516	2.98729	0.01779
	survival	0.41646	0.00003	0.42498	0.00015	0.42054	0.00085
	hazard	0.07281	0.00003	0.07526	0.00004	0.07386	0.00002

Table 4 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters when $\alpha = 5.0$ and with different values of λ

n		MLE		MPS		MCMC	
		Mean	MSE	Mean	MSE	Mean	MSE
$\alpha = 5.0, \lambda = 1.5$							
30	$\hat{\alpha}$	4.96935	1.51168	4.80995	0.24573	4.93800	0.26088
	$\hat{\lambda}$	1.81123	1.19999	1.54742	0.06261	1.55281	0.06277
	survival	0.41935	0.00037	0.44636	0.00091	0.43237	0.00116
	hazard	0.07942	0.00011	0.08629	0.00015	0.08240	0.00011
50	$\hat{\alpha}$	5.05866	0.78220	4.83439	0.15603	4.98159	0.03082
	$\hat{\lambda}$	1.59157	0.26579	1.53596	0.03578	1.50504	0.01646
	survival	0.42430	0.00022	0.44159	0.00052	0.43321	0.00150
	hazard	0.08045	0.00006	0.08512	0.00008	0.08277	0.00007
100	$\hat{\alpha}$	5.00921	0.98968	4.89491	0.07757	5.00092	0.01918
	$\hat{\lambda}$	1.60894	0.29749	1.51415	0.01512	1.49803	0.00885
	survival	0.42435	0.00009	0.43573	0.00022	0.42901	0.00058
	hazard	0.08110	0.00003	0.08399	0.00004	0.08212	0.00004
$\alpha = 5.0, \lambda = 3.0$							
30	$\hat{\alpha}$	5.49853	2.95436	4.83599	0.17311	4.95377	0.22890
	$\hat{\lambda}$	3.23043	2.61829	3.02878	0.12096	2.99933	0.14102
	survival	0.42116	0.00032	0.44689	0.00102	0.44064	0.00212
	hazard	0.03998	0.00003	0.04349	0.00004	0.04264	0.00004
50	$\hat{\alpha}$	5.30403	2.56847	4.86101	0.11422	4.97478	0.02804
	$\hat{\lambda}$	3.31764	2.56027	3.02739	0.07729	2.99138	0.02511
	survival	0.42248	0.00016	0.44159	0.00059	0.43320	0.00205
	hazard	0.04026	0.00002	0.04275	0.00002	0.04153	0.00001
100	$\hat{\alpha}$	5.15858	1.02403	4.90820	0.05516	4.99159	0.01852
	$\hat{\lambda}$	3.08877	0.74298	3.01974	0.03589	2.99856	0.01700
	survival	0.42551	0.00009	0.43606	0.00025	0.43009	0.00086
	hazard	0.04050	0.00001	0.04192	0.00001	0.04107	0.00001
$\alpha = 5.0, \lambda = 5.0$							
30	$\hat{\alpha}$	5.680840	2.860923	4.865118	0.178747	4.968343	0.209533
	$\hat{\lambda}$	4.950917	3.120347	5.029074	0.118472	4.946462	0.223036
	survival	0.422584	0.000378	0.446757	0.001058	0.444833	0.002616
	hazard	0.023862	0.000009	0.025931	0.000014	0.025774	0.000014
50	$\hat{\alpha}$	5.532185	2.739056	4.879961	0.107061	4.984977	0.028485
	$\hat{\lambda}$	5.052125	3.602149	5.028304	0.083244	4.984005	0.030100
	survival	0.424315	0.000179	0.441760	0.000634	0.433633	0.002174
	hazard	0.024093	0.000005	0.025566	0.000007	0.024838	0.000003
100	$\hat{\alpha}$	5.519398	2.892982	4.919757	0.053511	4.996870	0.019168
	$\hat{\lambda}$	5.027814	3.143470	5.019490	0.040923	4.991419	0.018846
	survival	0.426187	0.000067	0.436422	0.000291	0.430631	0.001014
	hazard	0.024273	0.000003	0.025138	0.000004	0.024634	0.000002

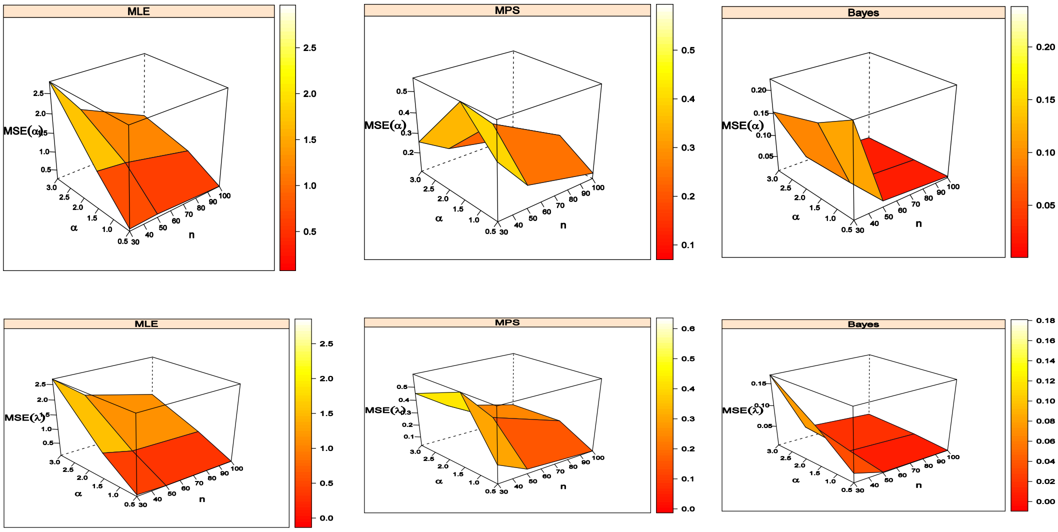

Figure 4 Plots of the MSE of parameters when $\alpha = 3.0$ and with different values of λ

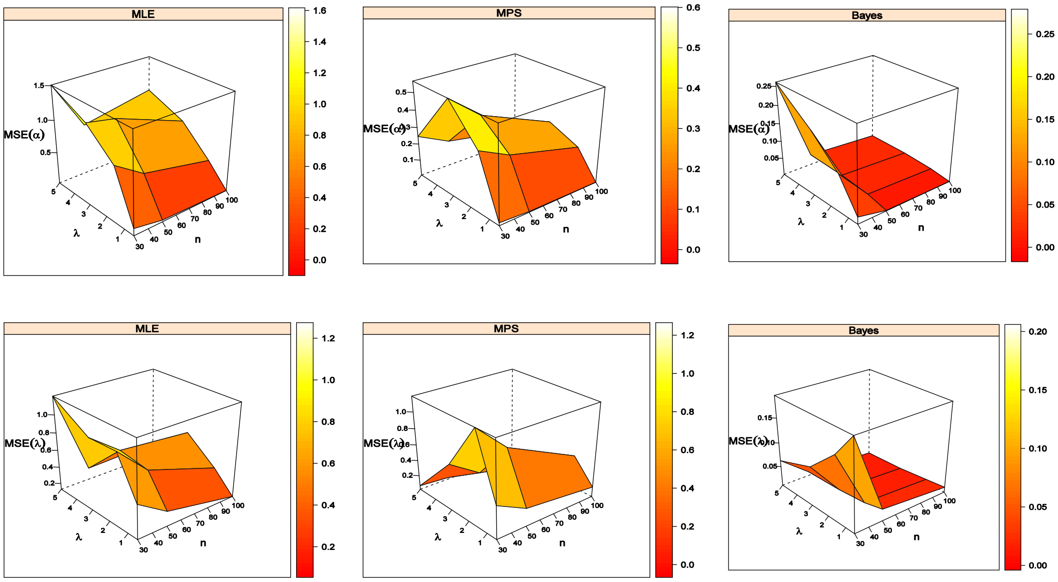

Figure 5 Plots of the MSE of parameters when $\lambda = 1.5$ and with different values of α

Table 5 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters for stress-strength model when $\alpha_1 = 0.5, \lambda_1 = 0.9, \alpha_2 = 0.9, \lambda_2 = 1.5$

(n,m)		MLE		MPS		Bayes	
		Mean	MSE	Mean	MSE	Mean	MSE
(30,20)	α_1	0.59667	0.06552	0.48769	0.02816	0.52360	0.00592
	λ_1	0.92999	0.33065	1.24318	0.59749	0.86891	0.02810
	α_2	1.30318	0.57414	0.95423	0.24378	0.91719	0.01350
	λ_2	1.35833	0.85819	1.99598	1.69263	1.49766	0.02967
	R	0.77532	0.00341	0.75721	0.00340	0.77329	0.00154
(50,40)	α_1	0.56288	0.02821	0.49270	0.01479	0.51748	0.00334
	λ_1	0.88419	0.17273	1.09654	0.28852	0.87468	0.01856
	α_2	1.11403	0.24489	0.91240	0.14754	0.91013	0.00779
	λ_2	1.41357	0.48277	1.85933	0.90450	1.50156	0.01765
	R	0.77561	0.00218	0.76402	0.00220	0.77360	0.00102
(100,70)	α_1	0.52284	0.00769	0.48530	0.00565	0.50556	0.00127
	λ_1	0.91027	0.09965	1.03348	0.13892	0.89060	0.00681
	α_2	1.02257	0.10911	0.89456	0.07499	0.90377	0.00352
	λ_2	1.44568	0.30869	1.74873	0.51307	1.49707	0.00734
	R	0.77413	0.00110	0.76706	0.00112	0.77359	0.00045
(30,30)	α_1	0.59748	0.067011	0.488782	0.030782	0.523678	0.005434
	λ_1	0.939538	0.353553	1.257958	0.647879	0.861523	0.02734
	α_2	1.236736	0.488852	0.941885	0.199651	0.915826	0.010885
	λ_2	1.414832	0.791999	1.892941	1.141962	1.49347	0.027591
	R	0.77553	0.003243	0.759996	0.003181	0.77435	0.001349
(50,50)	α_1	0.556114	0.02554	0.486826	0.013375	0.513975	0.003436
	λ_1	0.885732	0.166738	1.098986	0.277789	0.871903	0.017735
	α_2	1.104154	0.206331	0.913773	0.105485	0.909875	0.006627
	λ_2	1.384466	0.440184	1.772895	0.725963	1.493925	0.016915
	R	0.7759	0.001769	0.765103	0.001796	0.775157	0.000807
(100,100)	α_1	0.521714	0.007302	0.484404	0.005393	0.505193	0.001346
	λ_1	0.905112	0.088704	1.029167	0.126605	0.892328	0.006477
	α_2	0.992143	0.078847	0.884044	0.045039	0.907428	0.002934
	λ_2	1.477091	0.279797	1.708956	0.36437	1.496547	0.007238
	R	0.775071	0.000928	0.768714	0.000933	0.77452	0.000406
(30,40)	α_1	0.60304	0.06613	0.49260	0.02942	0.52257	0.00553
	λ_1	0.90608	0.31412	1.21602	0.55916	0.86186	0.02659
	α_2	1.15014	0.33520	0.90121	0.12074	0.91323	0.00872
	λ_2	1.45673	0.70000	1.87261	0.96745	1.49814	0.02869
	R	0.77603	0.00292	0.76185	0.00282	0.77506	0.00118
(50,60)	α_1	0.55659	0.02923	0.48735	0.01657	0.51382	0.00316
	λ_1	0.91008	0.19316	1.13160	0.33355	0.87853	0.01732
	α_2	1.03542	0.13775	0.88192	0.07946	0.90600	0.00605
	λ_2	1.47414	0.39998	1.80985	0.62388	1.50327	0.01758
	R	0.77515	0.00189	0.76465	0.00194	0.77387	0.00085
(100,150)	α_1	0.52657	0.00832	0.48847	0.00590	0.50721	0.00131
	λ_1	0.89933	0.09393	1.02243	0.13119	0.88908	0.00729
	α_2	0.97715	0.05776	0.88675	0.03394	0.90363	0.00245
	λ_2	1.47308	0.23966	1.66306	0.28204	1.50096	0.00714
	R	0.77468	0.00091	0.76879	0.00091	0.77389	0.00037

Table 6 Mean and MSE for MLE, MPS and Bayesian of ExEx parameters for stress-strength model when $\alpha_1 = 1.5, \lambda_1 = 2.5, \alpha_2 = 1.5, \lambda_2 = 2.5$

(n,m)		MLE		MPS		Bayes	
		Mean	MSE	Mean	MSE	Mean	MSE
(30,20)	α_1	2.3248	2.1587	1.5459	0.5311	1.4967	0.0189
	λ_1	1.9065	2.2625	2.5406	2.1210	1.9720	0.0300
	α_2	3.8310	3.2068	2.7277	1.0673	3.0056	0.0295
	λ_2	1.8831	2.0880	2.3068	1.8002	1.7123	0.0258
	R	0.6820	0.0064	0.6702	0.0031	0.6822	0.0014
(50,40)	α_1	2.0187	1.2575	1.4991	0.3654	1.5036	0.0114
	λ_1	1.9805	1.4401	2.4756	1.4509	1.9888	0.0182
	α_2	3.5306	2.1880	2.7547	0.7631	3.0005	0.0190
	λ_2	1.8893	1.4777	2.1375	1.0197	1.7003	0.0157
	R	0.6785	0.0031	0.6689	0.0020	0.6769	0.0009
(100,70)	α_1	1.7974	0.5949	1.4697	0.1815	1.5001	0.0053
	λ_1	1.9967	0.9343	2.3235	0.8052	1.9946	0.0075
	α_2	3.4389	1.5293	2.7913	0.4915	3.0028	0.0080
	λ_2	1.7908	0.9414	2.0060	0.5804	1.7024	0.0065
	R	0.6788	0.0014	0.6711	0.0012	0.6777	0.0004
(30,30)	α_1	2.2217	2.0379	1.5079	0.5426	1.5002	0.0193
	λ_1	2.0500	2.2757	2.6797	2.3374	1.9777	0.0277
	α_2	3.7129	3.1513	2.7164	0.9237	3.0125	0.0277
	λ_2	1.9848	2.3249	2.2521	1.3817	1.7145	0.0241
	R	0.6737	0.0078	0.6687	0.0028	0.6818	0.0014
(50,50)	α_1	1.9365	1.0509	1.4568	0.3303	1.5007	0.0103
	λ_1	2.0382	1.4424	2.5371	1.5181	1.9888	0.0189
	α_2	3.5419	2.1468	2.7633	0.6286	2.9994	0.0176
	λ_2	1.8218	1.1965	2.0650	0.7293	1.7048	0.0144
	R	0.6774	0.0025	0.6690	0.0017	0.6781	0.0008
(100,100)	α_1	1.8019	0.5648	1.4724	0.1727	1.5015	0.0055
	λ_1	1.9541	0.8291	2.2945	0.7305	1.9935	0.0067
	α_2	3.2992	1.4584	2.7667	0.4190	2.9986	0.0074
	λ_2	1.8835	0.9949	1.9896	0.4467	1.6991	0.0066
	R	0.6796	0.0012	0.6726	0.0010	0.6767	0.0004
(30,40)	α_1	2.2321	2.0014	1.5123	0.5247	1.5006	0.0196
	λ_1	2.0135	2.2107	2.6468	2.3022	1.9673	0.0281
	α_2	3.6859	2.7540	2.7507	0.7039	3.0066	0.0285
	λ_2	1.9095	1.9494	2.1273	1.0048	1.6989	0.0213
	R	0.6755	0.0053	0.6693	0.0026	0.6804	0.0013
(50,60)	α_1	2.0102	1.1889	1.4959	0.3511	1.4985	0.0122
	λ_1	1.9463	1.2733	2.4612	1.3705	1.9886	0.0171
	α_2	3.4144	1.9632	2.7138	0.6075	3.0014	0.0173
	λ_2	1.9140	1.2435	2.1110	0.7392	1.7069	0.0141
	R	0.6800	0.0019	0.6713	0.0017	0.6791	0.0008
(100,150)	α_1	1.8126	0.6223	1.4806	0.1889	1.5058	0.0054
	λ_1	1.9711	0.9007	2.2999	0.7584	1.9947	0.0075
	α_2	3.3383	1.3079	2.8225	0.2886	2.9970	0.0069
	λ_2	1.7845	0.6902	1.8996	0.2535	1.7007	0.0056
	R	0.6775	0.0010	0.6722	0.0008	0.6759	0.0004

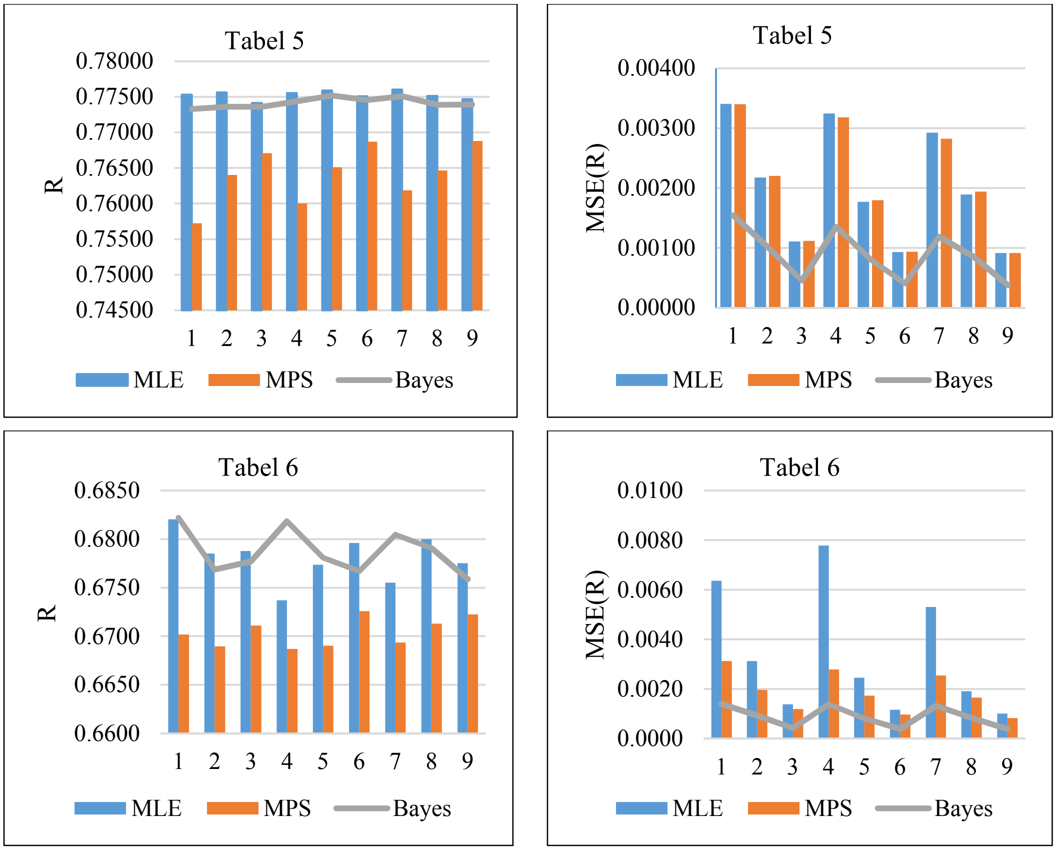

Figure 6 Plots of the reliability functions of stress-strength and MSE

5. Application

The numerical results of the parameter estimation and reliability estimation of ExEx distribution of real data have been presented. And also, we present the numerical results of the stress-strength model of ExEx distribution.

Firstly: We discuss an application of ExEx distribution using real data set to illustrate any estimation methods of ExEx distribution provides significant improvements over. The data represents the remission times (in months) of a random sample from bladder cancer patients reported in Lee and Wang (2003).

The data are as follows: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 7 Estimate, standard error, Kolmogorov-Smirnov test and reliability measure for ExEx distribution

	MLE		MPS		Bayes	
	Estimate	Standard error	Estimate	Standard error	Estimate	Standard error
$\hat{\alpha}$	0.9341502	0.158806	0.84832	0.137941	0.9348	0.00437
$\hat{\lambda}$	0.1163695	0.033877	0.13305	0.038098	0.1123	0.00254
D	0.082065		0.086685		0.07577	
p-value	0.378900		0.313700		0.480100	
survival	0.3626037		0.3644294		0.375822	
hazard	0.1034702		0.09963253		0.0998613	

The results of goodness of fit tests by using Kolmogorov-Smirnov test, based on the p-values for all methods, it is clear that the ExEx distribution fit the data of a bladder cancer study, as seen in Table 7. Depending on stander error, the comparison between the three methods show that the Bayes method estimation is the beast, next MPS then MLE method, because it has the least stander error. Also, the reliability estimation confirms this conclusion as the Bayes has the largest survival which means the probability of remission times in months from bladder cancer.

Secondly: We discuss a stress-strength reliability of ExEx distribution using real data set to illustrate any estimation methods of ExEx based on stress-strength reliability model provides significant improvements over. The real data sets of the waiting times before service of the customers of two banks, A and B, respectively have been used. These data sets have been discussed by Singh et al. (2014a) for estimating the stress-strength reliability in case of the generalized Lindley distribution.

The data of Bank A are as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

The data of Bank B are as follows: 0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0.

Table 8 Estimate, Stander Error, Kolmogorov-Smirnov test and reliability measure for ExEx distribution of Bank A data

	MLE		Bayes	
	Estimate	Standard error	Estimate	Standard error
$\hat{\alpha}$	3.32613	1.78020	3.32445	0.00412
$\hat{\lambda}$	0.02123	0.01356	0.01951	0.00227
D	0.10759		0.09838	
p-value	0.19730		0.28780	
survival	0.4132846		0.450830	
hazard	0.1099570		0.097718	

Table 9 Estimate, Standard Error, Kolmogorov-Smirnov test and reliability measure for ExEx distribution of Bank B data

	MLE		Bayes	
	Estimate	Standard error	Estimate	Standard error
$\hat{\alpha}$	2.06586	1.07900	2.06221	0.00531
$\hat{\lambda}$	0.05754	0.03946	0.05531	0.00227
D	0.08547		0.07789	
p-value	0.77310		0.85980	
survival	0.404120		0.421480	
hazard	0.165820		0.157319	

In this data, we cannot use MPS because there are equal observation in the data, so the spacing will be zero hence the product will also be zero see Equations (3.4) and (3.5) and to compute the MPS estimator, we take the log of product spacing, as we know the log(0) equal-Inf. Despite the effectiveness of the MPS method in the estimate, but this problem hinders their use in the estimation process.

Table 10 Estimate and reliability of stress-strength model for ExEx distribution of Bank data

	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$	$\hat{R}$
MLE	3.328227	0.021210	2.112340	0.055860	0.632156
Bayes	2.184240	0.043160	2.095250	0.076005	0.646155

6. Conclusions

In this paper, parameters estimation for the ExEx distribution are discussed based on the MPS, MLE and the Bayesian methods. In this model, the estimators based on MPS method behave quite better than the estimators based on the MLE but the Bayesian estimation is the best one, where the MSE is less than from the other methods. In case of reliability stress-strength model estimation, we note that when parameters value of stress is small, the estimated model efficiency increases. By checking the previous results, we note that MPS is better than MLE. We can conclude that the MPS method is a good alternative method to the usual MLE method in many situations. We hope that the finding in this paper will be useful for researchers and statistician.

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