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Cumulative Sum Control Chart Applied to Monitor Shifts in the Mean of a Long-memory ARFIMAX(p,d*,q,r) Process with Exponential White Noise

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Abstract

The aim of this study is to derive the average run length (ARL) for detecting a changes in the process mean of a long-memory autoregressive fractionally integrated moving-average model with exogenous variables (ARFIMAX(p, d*, q, r)) process with exponential white noise on a cumulative sum (CUSUM) control chart. ARLs are derived using explicit formulas and the numerical integral equation (NIE) method, which is the solution for the integral equation. Moreover, proof of the existence and uniqueness of the proposed ARL based on Banach's fixed-point theorem are presented. The performances of the two ARLs were evaluated in terms of accuracy and computational time for monitoring shifts in the process mean for an $\text{ARFIMAX}(p,d*,q,r)$ process on a CUSUM control chart. The results reveal that although their accuracies were similar, the explicit formula method consumed less computational time than the NIE method and so is recommended as a good alternative for this scenario.

Keywords: ARFIMA with exogenous variables, numerical integral equation (NIE) method, exponential white noise.

1. Introduction

Statistical process control is a popular approach for monitoring processes, with the key tool being the control chart. The Shewhart control chart (Shewhart 1931) was the first, followed by the Cumulative Sum (CUSUM) control chart initially suggested by Page (1954) and the exponentially weighted moving-average (EWMA) control chart first reported by Roberts (1959). The Shewhart control chart is appropriate for when there is a large shift size in the statistic parameter on interest (the mean or variance) of a process when the observations follow a normal distribution. Concurrently, small shift sizes in a statistic parameter can be detected by using the CUSUM and EWMA control charts; these are also more applicable for when the observations follow complex patterns such as autocorrelation or changing point (Yashchin 1993, Wardell et al. 1994).

The prominent feature of the CUSUM control chart is that it can detect unstable processes appropriately for subgroups of data and single observations. The use of the CUSUM control chart for real observations was reported by Sheng-Shu and Fong-Jung (2013) who suggested that for monitoring a quality control process for wafer production, the CUSUM control chart is better than the EWMA control chart. In a study by Benoit and Pierre (2009), the CUSUM control chart was found to be suitable for small shifts and persistent changes in the mean and variance of North Sea cod trawl numbers for an international bottom trawl survey. In the present study, the effectiveness of the CUSUM control chart and its application to real-world situations is of interest.

Observations collected from real stochastic processes often have a time-series component. In particular, econometric observations in a time-series model can have autoregressive (AR) and moving average (MA) components. When establishing a model, the error (the difference between the actual and approximated value) should be as small as possible for maximum accuracy. The error of a time-series model is called white noise that is normally distributed arising from autocorrelated observations. Sometimes, the white noise in autocorrelated data is not normally distributed, thus the form of the time-series model where the white noise is exponentially distributed, such as wind speed, the amount of dissolved oxygen in a river, and daily flow rates of a river, is especially interesting. For example, Jacob and Lewis (1977) considered an ARMA(1,1) process with exponential white noise. At a later time, Mohamed and Hocine (2003) conducted a Bayesian analysis of AR(1) with exponential white noise, while Pereira and Turkrman (2004) used exponential white noise to develop a Bayesian analysis of threshold AR models. Recently, Suparman (2018) proposed parameter estimations for an AR model with exponential white noise when the order is unknown.

Conventional models used for short-memory processes $\mathrm{AR}(p)$, $\mathrm{MA}(q)$, $\mathrm{ARMA}(p,q)$, and AR integrated MA $(\mathrm{ARIMA}(p,d,q))$ cannot be applied for long-memory ones. This problem has been solved by the establishment of several models, the best-known being the AR fractionally integrated MA $(\mathrm{ARFIMA}(p,d,q))$ model. In addition, the ARFIMAX model is an extension of ARFIMA through the inclusion of exogenous variable, i.e. $\mathrm{ARFIMAX}(p,d^*,q,r)$. Granger and Joyeux (1980) and Hosking (1981) created ARFIMA models for a long-memory process for realized volatilities. Moreover, Ebens (1999) used ARFIMAX models for estimating the realized volatilities in a Dow Jones Industrial Average portfolio. There is a relationship between econometric models and economic indicators (variables affecting economic forecasting). However, an exogenous variable is not affected by other variables in the system, only by external influences (for instance, the investment policies of the government) and include exchange, interest, and inflation rates, etc. Exogenous variables affect an econometric model when forecasting economic situations. For economic forecasting and other fields, if the forecasting model includes an exogenous variable, the model is usually more accurate than one without it. This type of time-series model is an interesting concept. Many control charts have been used for time-series modeling with ARFIMA. Ramjee (2000) analyzed the performance of Shewhart and EWMA control charts with correlated observations in an ARFIMA model; the results indicate that the control charts did not perform well for detecting process shifts, and so proposed the hyperbolic weighted MA (HWMA) control chart. Later, Ramjee et al. (2002) presented an HWMA forecast-based control chart especially designed for a non-stationary ARFIMA model with autocorrelated data. ARFIMA and ARIMA models have been applied to applications for monitoring the air quality data in Taiwan (Pan and Chen 2008); the authors concluded that residual control charts using ARFIMA models were more suitable than an ARIMA model. The EWMA control chart to detect a change in the mean of a long-memory process was recently introduced by Rabyk and Schmid (2016), with the control chart's design based on an $\mathrm{ARFIMA}\left(p,d,q\right)$ process.

The statistical performances of control charts are generally evaluated in terms of the average run length (ARL) based on the expectation of the run length distribution. In other words, ARL represents the average number of observations plotted on a control chart until it provides an out-of-control signal or a false alarm. The ARLs when the process is in- or out-of-control are denoted by ARL_0 and ARL_1, respectively. ARL_0 should be large when the process is operating on target (the mean is at the desired level) but on the other hand, ARL_1 should be small to bestow the capability of rapidly detecting a change in the process mean. This performance measure can be computed by using the integral equation (IE) approach originally introduced by Page (1954), the Markov chain approach (MCA) by Brook and Evans (1972), Monte Carlo (MC) simulation by Hawkins (1981), or the numerical IE (NIE) method by Peerajit et al. (2018), among others. The IE method is the most advanced but requires a great deal of programming and computation. MCA requires discretization of the process continuity into many steps and calculations of a matrix inverse. MC is simple to program and good for checking accuracy but requires a large number of sample trajectories. NIE is good for checking the accuracy of explicit formulas, so it is usually very time consuming and it is also difficult and laborious to find the optimal design. In this paper, we derive explicit formulas and the NIE method through an algorithm developed in the Mathematica program.

The rest of this paper is organized as follows. First, characteristics of the generalized ARFIMAX process and the CUSUM control charts are reported in Section 2. Derivations of the ARLs using explicit formulas and the NIE method for a long-memory process with exponential white noise on a CUSUM control chart are presented in Section 3. The solution of the IE to derive an exact expression for the ARL for monitoring shifts in the process mean on CUSUM control charts and the existence and uniqueness of the ARL via Banach's fixed-point theorem is confirmed and discussed in Section 4. The ARL results to compare the performances of the ARLs based on explicit formulas and the NIE method for monitoring changes in the process mean are reported in Section 5. In section 6, the application of the ARL derived from explicit formulas and the NIE method on a CUSUM control chart is illustrated using the US dollar (USD) exchange rate data with an exogenous variable. ARLs using the two methods are compared in terms of their out-of-control performance. Finally, a discussion and conclusions of the study are provided in Section 7.

2. Analysis and Characteristics of the Generalized ARFIMAX Process and the CUSUM Control Chart

In this section, we describe the relevant basics of a long-memory ARFIMAX model with exponential white noise. We also define the generalized ARFIMAX process used on a CUSUM control chart appropriate for monitoring shifts in the process mean. In the last subsection, we explore the characteristics of the ARL related to the evaluation of control chart performance.

2.1. The generalized ARFIMAX process

The ARFIMAX model is an extension of the AR fractionally integrated MA model suggested by Granger and Joyeux (1980) and Hosking (1981). The $\text{ARFIMAX}(p,d^*,q,r)$ process, where p is the AR order, q is the MA order, d^* is the fractional order of integration, and r means there are exogenous variable order in the model. Ebens (1999) was applied in the research. The $\text{ARFIMAX}(p,d^*,q,r)$ process can be generalized as

$$\phi_p(B)(1-B)^{d^*}Y_t = \mu + \sum_{k=1}^{r}\beta_k X_k + \theta_q(B)\varepsilon_t, \tag{1}$$

where $\{Y_t, t=1,2,...\}$ is a sequence of ARFIMAX process, μ is the constant process mean of $\{Y_t\}$, X_k is an exogenous variable, β_k is an unknown parameters., ε_t is a white noise process assumed to be exponentially distributed, $\phi_p(B)$ and $\theta_q(B)$ are AR and MA polynomials in B, respectively, when B is a backward-shift operator defined by $BY_t = Y_{t-1}$. The differencing operator $(1-B)$ is raised to a fraction power, $d*$, denoted by the fractional order of integration. The fractional differencing operator $(1-B)^{d^*}$ can be defined according to the binomial expansion as described by Granger and Joyeux (1980), and Hosking (1981) as follows

$$(1-B)^{d^*} = 1 - d^*B + \frac{d^*(d^*-1)}{2!}B^2 - \frac{d^*(d^*-1)(d^*-2)}{3!}B^3 - ..., \tag{2}$$

where the parameter $d*$ is not an integer and represents the fractional order of integration. When the autocorrelations are all positive, process Y_t is intermediate-memory for $-0.5 < d* < 0$, short-memory (or short-range dependence) corresponding to a standard ARMA process for $d* = 0$, and long-memory (or long-range dependence) for $d* < 0.5$.

Both Equations (1) and (2) can be respectively rearranged for the generalized ARFIMAX$(p, d*, q, r)$ process with exponential white noise on a CUSUM control chart as follows

$$\begin{aligned} Y_t = \mu &+ \left(\phi_1 Y_{t-1} - d^*\phi_1 Y_{t-2} + \frac{d^*(d^*-1)}{2!}\phi_1 Y_{t-3} - \frac{d^*(d^*-1)(d^*-2)}{3!}\phi_1 Y_{t-3} + ... \right) \\ &+ \left(\phi_2 Y_{t-2} - d^*\phi_2 Y_{t-3} + \frac{d^*(d^*-1)}{2!}\phi_2 Y_{t-4} - \frac{d^*(d^*-1)(d^*-2)}{3!}\phi_2 Y_{t-5} + ... \right) \\ &\vdots \\ &+ \left(\phi_p Y_{t-p} - d^*\phi_p Y_{t-p-1} + \frac{d^*(d^*-1)}{2!}\phi_p Y_{t-p-2} - \frac{d^*(d^*-1)(d^*-2)}{3!}\phi_p Y_{t-p-3} + ... \right) \\ &+ \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - ... - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_i \\ &+ \left(d^*Y_{t-1} - \frac{d^*(d^*-1)}{2!}Y_{t-2} + \frac{d^*(d^*-1)(d^*-2)}{3!}Y_{t-3} - ... \right) \end{aligned}$$

or

$$\begin{aligned} Y_t &= \mu + \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - ... \right) \\ &+ \varepsilon_t - \sum_{j=1}^{q}\theta_j\varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k + \left(d^*Y_{t-1} - \frac{d^*(d^*-1)}{2!}Y_{t-2} + ... \right), \end{aligned} \tag{3}$$

where ε_t is a white noise process assumed to be an independently identically distributed (i.i.d.) observed sequence in exponential distribution ($\varepsilon_t \sim \text{Exp}(\lambda)$). Furthermore, $-1 \le \phi_i \le 1$ and $-1 \le \theta_j \le 1$ are the constraints of the AR and MA coefficients, respectively. In this research, the initial value of the ARFIMAX$(p, d*, q, r)$ process $Y_{t-1}, Y_{t-2}, ..., Y_{t-p}, Y_{t-(p+1)}, ...$ and $\varepsilon_{t-1}, \varepsilon_{t-2}, ..., \varepsilon_{t-q}$ is assumed to be 1.

2.2. The CUSUM control chart for monitoring shifts in the process mean

The concept of the CUSUM control chart was first proposed by Page (1954). It is well known that the CUSUM control chart performs better than the Shewhart control chart for small-to-medium-sized changes in the process mean. The commonly used form of the upper-sided CUSUM control chart is based on the sequence

$$S_t = \max\{S_{t-1} + Y_t - a,\ 0\},\ \text{for}\ \ t = 1,2,\ldots, \tag{4}$$

where S_t is the CUSUM control chart statistic used for detecting upward shifts, quantity S_{t-1} denotes the previous value of the statistic (its initial value S_0 is set to u and the parameter of process Y_t is the sequence of the generalized $\text{ARFIMAX}(p,d^*,q,r)$ process with exponential white noise), and a is a suitably chosen positive constant.

The corresponding stopping time (τ_h) for a CUSUM control chart with predetermined threshold h can be written as

$$\tau_h = \inf\{t > 0;\ S_t > h\},\ \text{for}\ u \le h, \tag{5}$$

where h is a constant parameter-the upper control limit (UCL) of CUSUM chart.

2.3. Characteristics of the average run length (ARL)

Let ε_t, $t = 1,2,\ldots,$ be a sequence of continuous i.i.d. random variables taken from an exponential distribution. This sequence has a distribution function $(F(x,\lambda))$ by following the change point model

$$\varepsilon_t = \begin{cases} \text{Exp}(\lambda_0), & t = 1,\ 2,\ \ldots,\ m-1 \\ \text{Exp}(\lambda_1 > \lambda_0), & t = m,\ m+1,\ \ldots\ , \end{cases} \tag{6}$$

where λ_0 and λ_1 are known parameters. By considering the change point in Equation (6), the ARL is defined more rigorous with $E_m(.)$ as the expectation for a fixed change point m. Thus,

$$\text{ARL} = \begin{cases} E_\infty(\tau_h), & \lambda = \lambda_0 \\ E_1(\tau_h), & \lambda \ne \lambda_0. \end{cases} \tag{7}$$

As previously mentioned, the ARL denotes the average number of observations until the signal for a sequence with a constant expectation indicates the out-of-control state. Meanwhile, $m = \infty$ indicates no change in the statistical process: the so-called in-control ARL (ARL_0). On the other hand, $m = 1$ marks the first time point that a change takes place from λ_0 to λ in the statistical process: the so-called out-of-control ARL (ARL_1).

3. Derivation of the Explicit and Approximated ARLs for a Long-memory ARFIMAX Process with Exponential White Noise on a CUSUM Control Chart

In this section, the explicit and approximated ARLs are derived as the solution of the IE for a long-memory ARFIMAX process with exponential white noise on a CUSUM control chart to monitor changes in the process mean.

Let $\mathbb{P}_S(\mathbb{E}_S)$ be the probability measure (expectation) corresponding to the initial value ψ. Formally, let $L(\psi)$ be the ARL for the upper-sided CUSUM control chart. The initial value of the in monitoring statistic $S_0 = \psi$ for a long-memory ARFIMAX model process after it has been determined at $\psi \in [0,h]$.

The ARL is defined as a function of $L(\psi) = \mathrm{E}_m(\tau_h) < \infty.$ Hence, it can be shown that the IE is in the form

$$L(\psi) = 1 + \mathbb{P}_s\{S_1 = 0\}L(0) + \mathbb{E}_s[I\{0 < S_1 < h\}L(S_1)], \tag{8}$$

where $I\{0 < S_1 < h\}$ is the indicator function.

The IE derived from the Fredholm integral equation of the second kind can be written in the following form

$$L(\psi) = 1 + F\left(a - \psi - Y_t\right)L(0) + \int_0^h L(z) f(z + a - \psi - Y_t)dz, \tag{9}$$

for $\psi \in [0, h]$ is UCL of CUSUM chart.

Let $\varepsilon_t, t = 1, 2, ...,$ be a sequence of continuous i.i.d random variables with an exponential distribution. Therefore, in (9) becomes

$$L(\psi) = 1 + \left(1 - \exp\left\{\begin{matrix} -\lambda\left(a - \psi - \mu - \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \dfrac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - ...\right)\right. \\ \left. -\varepsilon_t + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j} - \sum_{k=1}^{r}\beta_k X_k + \left(d^* Y_{t-1} - \dfrac{d^*(d^*-1)}{2!}Y_{t-2} + ...\right)\right) \end{matrix}\right\}\right)L(0)$$

$$+\lambda\left(\exp\left\{\begin{matrix} \lambda\left(\psi - a + \mu + \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \dfrac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - ...\right)\right. \\ \left. +\varepsilon_t - \sum_{j=1}^{q}\theta_j\varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k - \left(d^* Y_{t-1} - \dfrac{d^*(d^*-1)}{2!}Y_{t-2} + ...\right)\right) \end{matrix}\right\}\right)\int_0^h L(z)\exp\{-\lambda z\}dz. \tag{10}$$

3.1. The ARL based on explicit formulas

Theorem 3.1 *Let $L(\psi)$ be the ARL of the IE in (9) corresponding to the long-memory* $\mathrm{ARFIMAX}(p, d^*, q, r)$ *process on a CUSUM control chart, then*

$$L(\psi) = \exp\{\lambda h\}\left(1 + \exp\left\{\begin{matrix} \lambda\left(a - \mu - \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \dfrac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - ...\right)\right. \\ \left. -\varepsilon_t + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j} - \sum_{k=1}^{r}\beta_k X_k + \left(d^* Y_{t-1} - \dfrac{d^*(d^*-1)}{2!}Y_{t-2} + ...\right)\right) \end{matrix}\right\} - \lambda h\right)$$
$$- \exp\{\lambda\psi\}; \;\; \psi \geq 0.$$

Proof: First, by introducing the IE and defining $c = \int_0^h L(z)\exp\{-\lambda z\}dz,$ Equation (10) can be simplified as follows

$$L(\psi)=1+\left(1-\exp\left\{\begin{array}{l}-\lambda\left(a-\psi-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}\right)L(0)$$

$$+c\lambda\left(\exp\left\{\begin{array}{l}\lambda\left(\psi-a+\mu+\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.+\varepsilon_t-\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}+\sum_{k=1}^{r}\beta_k X_k-\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}\right). \quad (11)$$

By replacing $\psi=0$ in (11), we obtain

$$L(0)=1+\left(1-\exp\left\{\begin{array}{l}-\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}\right)L(0)$$

$$+c\lambda\left(\exp\left\{\begin{array}{l}\lambda\left(-a+\mu+\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.+\varepsilon_t-\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}+\sum_{k=1}^{r}\beta_k X_k-\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}\right).$$

$$\therefore L(0)=\exp\left\{\begin{array}{l}\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}+c\lambda. \quad (12)$$

Subsequently, by substituting $L(0)$ from (12) into (11), it follows that

$$L(\psi)=1+c\lambda+\exp\left\{\begin{array}{l}\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}-\exp\{\lambda\psi\}. \quad (13)$$

Constant c can be formed as

$$c=\int_0^h L(z)\exp\{-\lambda z\}dz$$

$$=\int_0^h\left(1+c\lambda+\exp\left\{\begin{array}{l}\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right)\end{array}\right\}-\exp\{\lambda z\}\right)\exp\{-\lambda z\}dz$$

$$c=\frac{\exp\{\lambda h\}}{\lambda}\left(1+\exp\left\{\begin{array}{l}\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-...\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+...\right)\right)\end{array}\right\}\right)\left(1-\exp\{-\lambda h\}\right)$$
$$-h\exp\{\lambda h\}. \tag{14}$$

Finally, substituting constant c from (14) into (13) results in

$$L(\psi)=\exp\{\lambda h\}\left(1+\exp\left\{\begin{array}{l}\lambda\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-...\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+...\right)\right)\end{array}\right\}-\lambda h\right) \tag{15}$$
$$-\exp\{\lambda\psi\};\ \ \psi\geq 0.$$

Therefore, the proof is complete.

As was shown in the preceding equation, a process in the in-control state depends on the exponential parameter $(\lambda=\lambda_0)$. For this reason, ARL$_0$ of the CUSUM control chart is then determined by

$$\mathrm{ARL}_0=\exp\{\lambda_0 h\}\left(1+\exp\left\{\begin{array}{l}\lambda_0\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-...\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+...\right)\right)\end{array}\right\}-\lambda_0 h\right)$$
$$-\exp\{\lambda_0\psi\};\ \ \psi\geq 0. \tag{16}$$

On the other hand, a process in the out-of-control state depends on the exponential parameter $(\lambda=\lambda_1)$, which represents a shift in the process level (δ) by $\lambda_1=\lambda_0+\delta$, where $\lambda_0=1$. Therefore, ARL$_1$ of the CUSUM control chart is then determined by detecting the change in the process mean.

$$\mathrm{ARL}_1=\exp\{\lambda_1 h\}\left(1+\exp\left\{\begin{array}{l}\lambda_1\left(a-\mu-\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-...\right)\right.\\ \left.-\varepsilon_t+\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}-\sum_{k=1}^{r}\beta_k X_k+\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+...\right)\right)\end{array}\right\}-\lambda_1 h\right)$$
$$-\exp\{\lambda_1\psi\};\ \ \psi\geq 0. \tag{17}$$

3.2. The NIE method

The quadrature method is usually based on Equation (9). The integral $\int_0^h f(z)dz$ can be approximated by summing the areas of the rectangles with the integral f value. It is chosen by base h/m with heights at the midpoints of the intervals of length h/m beginning at zero. There are division points $a_1 \le \ldots \le a_m$ within the interval $[0,h]$, and w_j are the weights defined for different quadrature rules for which $h/m \ge 0$ can be written in the following form

$$\int_0^h W(z)f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j),$$

where $W(z)$ is a weight function and a_j is a set of points for $a_j = h(2j-1)/2m,\ j = 1,2,\ldots,m.$

Let $\tilde{L}(\psi)$ be the approximated ARL of the NIE method using the Gauss-Legendre quadrature rule. Consequently, the NIE method for the ARL of the CUSUM control chart in detecting a shift in the process mean for a long-memory ARFIMAX(p,d^*,q,r) process can be written as

$$\begin{aligned}\tilde{L}(\psi) = 1 + \tilde{L}(a_1)F\Bigg(& a-\psi-\mu-\sum_{i=1}^{p}\Big(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - \ldots\Big) - \varepsilon_t \\ & +\sum_{j=1}^{q}\theta_j\varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k + \Big(d^*Y_{t-1} - \frac{d^*(d^*-1)}{2!}Y_{t-2} + \ldots\Big)\Bigg) \\ & +\sum_{j=1}^{m} w_j\tilde{L}(a_j)f\Bigg(a_j + a-\psi-\mu-\sum_{i=1}^{p}\Big(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - \ldots\Big) - \varepsilon_t \\ & +\sum_{j=1}^{q}\theta_j\varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k + \Big(d^*Y_{t-1} - \frac{d^*(d^*-1)}{2!}Y_{t-2} + \ldots\Big)\Bigg).\end{aligned} \quad (18)$$

4. The Existence and Uniqueness of the IE for the ARL Based on Explicit Formulas

The derivation of the ARL through an IE for an upper-sided CUSUM control chart was discussed by Venkateshwara et al. (2001). Based on this, Banach's fixed-point theorem was applied as the source for the existence and uniqueness theorem (Wilasinee et al. 2019). The theoretical proof for its existence and uniqueness ensures that solving the IE results in the ARL based on explicit formulas has the same accuracy as the ARL based on the NIE method.

Definition 4.1 (Metric space) *A metric space is a pair $(\mathcal{K},d)$ where $\mathcal{K}$ is a non-empty set and distance function on P_n (or metric on $\mathcal{K}$). If a function $d:\mathcal{K}\times\mathcal{K}\to\mathbb{R}$ satisfied, for all $P,Q,R\in\mathcal{K}$, the following properties:*

1) $d(P,Q)=0$ implies $P=Q$,

2) $d(P,Q)=d(Q,P)$; *(Symmetry),*

3) $d(P,Q)\le d(P,R)+d(R,Q)$ *(Triangle inequality),*

where d is called distance function, which associates a distance $d(P,Q)=\|P-Q\|$ with every pair of points $P,Q\in\mathbb{R}$, and the pair $(\mathcal{K},d)$ is said to be a metric space.

Note: The definition for a metric is always positive, for all $P, Q \in \boldsymbol{K}$.

$$0 = d(P,P) \le d(P,Q) + d(Q,P) = 2d(P,Q).$$

Now, the author recalls the definition of Cauchy sequence, which is the formalization of the concept of a sequence.

Definition 4.2 *A sequence* $\{P_n\}_{n\ge 0}$ *elements in metric space* $\boldsymbol{K}$ *is a Cauchy sequence if for each* $\varepsilon > 0$, *there exists* N *such that all* $n, m > N$, *then* $d(P_n, P_m) \le \varepsilon$.

Definition 4.3 *The metric space is complete if each Cauchy sequence* $\{P_n\}_{n\ge 0}$ *converges, i.e. there exists* $P \in \boldsymbol{K}$ *so that* $d(P_n, P_m) \to 0$ *as* $n \to \infty$.

Definition 4.4 (Fixed points of mappings) *Let* $T : \boldsymbol{K} \to \boldsymbol{K}$, *then* $P \in \boldsymbol{K}$ *is called a fixed point of* T *if* $T(P) = P$.

Theorem 4.1 (Banach's fixed-point theorem or the contraction theorem) *Let* $\boldsymbol{K} = (\boldsymbol{K}, d)$ *be a complete metric space, then mapping* $T : \boldsymbol{K} \to \boldsymbol{K}$ *is said to be a contraction (or a contraction mapping) on* $\boldsymbol{K}$ *if there exists real number* ρ; $\rho \in [0,1)$ *such that*

$$d(T(P), T(Q)) \le \rho d(P,Q) \text{ for every } P, Q \in \boldsymbol{K},$$

Thus, T *has a precisely unique fixed point (e.g. unique* $P \in \boldsymbol{K}$ *such that* $P = T(P)$*).*

Theorem 4.2 *Suppose that* $L(\psi)$ *in Theorem 4.1, the ARL based on explicit formulas corresponding to the CUSUM control chart for a long-memory ARFIMAX process exists and is unique.*

Proof: To prove the existence of the ARL derived from explicit formulas, let T be a contraction in complete metric space $(\boldsymbol{K}, d)$, $\mathbf{C}([0,h])$ be a set of continuous functions of the ARL on interval $[0,h]$ and ARL_0 be an arbitrary but fixed element in $\boldsymbol{K}$. Define a sequence of iterates $\{\text{ARL}_n\}_{n\ge 0}$ in $\boldsymbol{K}$ by

$$\text{ARL}_{n+1} = T(\text{ARL}_n), \text{ for all } n \ge 1.$$

Since T is a contraction, then

$$d(\text{ARL}_2, \text{ARL}_1) = d(T(\text{ARL}_1), T(\text{ARL}_0)) \le \rho d(\text{ARL}_1, \text{ARL}_0), \text{ for some } \rho \in (0,1).$$

Continuing inductively, we obtain

$$d(\text{ARL}_{n+1}, \text{ARL}_n) \le \rho d(\text{ARL}_n, \text{ARL}_{n-1}) \le \rho^2 d(\text{ARL}_{n-1}, \text{ARL}_{n-2}) \le \ldots \le \rho^n d(\text{ARL}_1, \text{ARL}_0).$$

Repeatedly applying the triangle inequality into this formula when $n < m$ implies that

$$d(\text{ARL}_n, \text{ARL}_m) \le d(\text{ARL}_n, \text{ARL}_{n+1}) + \ldots + d(\text{ARL}_{m-1}, \text{ARL}_m),$$

and from above, it follows that

$$d(\text{ARL}_n, \text{ARL}_m) \le (\rho^n + \rho^{n+1} + \ldots + \rho^{m-1}) d(\text{ARL}_1, \text{ARL}_0).$$

Using the property of sum is a geometric series in ρ, we obtain

$$d\left(\text{ARL}_n, \text{ARL}_m\right) \le \frac{\rho^n}{1-\rho} d(\text{ARL}_1, \text{ARL}_0).$$

From above, $|\rho| < 1$ implies that $\rho^n/(1-\rho) \to 0$ as $n \to \infty$. Hence, $\{\text{ARL}_n\}_{n\ge 0}$ is a Cauchy sequence. There is a limit point of ARL in $\boldsymbol{K}$ because $(\boldsymbol{K}, d)$ is complete metric space. Hence, there exists a unique point $\text{ARL} \in \boldsymbol{K}$ such that

$$T(\text{ARL}) = \lim_{n\to\infty} T\left(\text{ARL}_n\right) = \lim_{n\to\infty} \text{ARL}_{n+1} = \text{ARL}.$$

Therefore, the ARL has a fixed point.

Proof: To prove the uniqueness of the ARL derived from explicit formulas, it must be shown that the operator T is a contraction mapping. Let ARL_1 and ARL_2 be two arbitrary functions in $\mathbf{C}([0,h])$. The common term for complete metric space is $\left((\mathbf{C}[0,h]), \|.\|_\infty\right)$. That is to say, a set of continuous functions of the ARL defined on $[0,h]$, and $\mathbf{C}([0,h])$ becomes norm space if we define

$$\| \text{ARL} \|_\infty = \sup_{\psi\in[0,h]} \left| \int_0^h k(\psi, z) dz \right|,$$

for all functions $k(\psi, z) \in \mathbf{C}([0,h])$, where $k(\psi, z)$ is a kernel function of the IE for the ARL based on explicit formulas obtained by using Theorem 4.1:

$$\| T(\text{ARL}_1) - T(\text{ARL}_2) \|_\infty$$

$$= \sup_{\psi\in[0,h]} \left| \int_0^h \lambda \left(\exp\left\{ \begin{array}{l} \lambda\left(\psi - a + \mu + \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - \ldots\right)\right. \\ \left. + \varepsilon_t - \sum_{j=1}^{q}\theta_j \varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k - \left(d^* Y_{t-1} - \frac{d^*(d^*-1)}{2!} Y_{t-2} + \ldots\right)\right) \end{array} \right\} \right) \left| \text{ARL}_1(z) - \text{ARL}_2(z) \right| dz \right|$$

$$\le \sup_{\psi\in[0,h]} \int_0^h \left| \lambda \left(\exp\left\{ \begin{array}{l} \lambda\left(\psi - a + \mu + \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - \ldots\right)\right. \\ \left. + \varepsilon_t - \sum_{j=1}^{q}\theta_j \varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k - \left(d^* Y_{t-1} - \frac{d^*(d^*-1)}{2!} Y_{t-2} + \ldots\right)\right) \end{array} \right\} \right) \right| \left| \text{ARL}_1(z) - \text{ARL}_2(z) \right| dz$$

$$\le \sup_{\psi\in[0,h]} \int_0^h \left| \lambda \left(\exp\left\{ \begin{array}{l} \lambda\left(\psi - a + \mu + \sum_{i=1}^{p}\left(\phi_i Y_{t-i} - d^*\phi_i Y_{t-(i+1)} + \frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)} - \ldots\right)\right. \\ \left. + \varepsilon_t - \sum_{j=1}^{q}\theta_j \varepsilon_{t-j} + \sum_{k=1}^{r}\beta_k X_k - \left(d^* Y_{t-1} - \frac{d^*(d^*-1)}{2!} Y_{t-2} + \ldots\right)\right) \end{array} \right\} \right) \right| dy \left\| \text{ARL}_1(z) - \text{ARL}_2(z) \right\|_\infty .$$

Hence, we have

$$\left\| T(\text{ARL}_1) - T(\text{ARL}_2) \right\|_\infty = \rho \left\| \text{ARL}_1(z) - \text{ARL}_2(z) \right\|_\infty,$$

where $0 < \rho = \sup_{\psi\in[0,h]} \int_0^h \left| k(\psi, z) \right| dz < 1$ is a positive constant,

Thus,

$$k(\psi,z)=\lambda\exp\left\{\begin{array}{l}\lambda\left(\psi-a+\mu+\sum_{i=1}^{p}\left(\phi_i Y_{t-i}-d^*\phi_i Y_{t-(i+1)}+\frac{d^*(d^*-1)}{2!}\phi_i Y_{t-(i+2)}-\ldots\right)\right.\\ \left.+\varepsilon_t-\sum_{j=1}^{q}\theta_j\varepsilon_{t-j}+\sum_{k=1}^{r}\beta_k X_k-\left(d^*Y_{t-1}-\frac{d^*(d^*-1)}{2!}Y_{t-2}+\ldots\right)\right.\end{array}\right\}.$$

The triangular inequality is used for the supremum norm as follows:

$$\left|\mathrm{ARL}_1(0)-\mathrm{ARL}_2(0)\right|\le\sup_{s\in[0,h]}\left|\mathrm{ARL}_1(\psi)-\mathrm{ARL}_2(\psi)\right|=\left\|\mathrm{ARL}_1-\mathrm{ARL}_2\right\|_\infty.$$

That is to say, $T:\mathbf{C}([0,h])\to\mathbf{C}([0,h])$ is a contraction mapping in complete metric space $\left((\mathbf{C}[0,h]),\|\cdot\|_\infty\right)$. Hence, by Theorem 4.2, the uniqueness of the ARL based on explicit formulas such that $T(\mathrm{ARL})=\mathrm{ARL}$ is confirmed. This completes the proof.

Therefore, the ARL based on explicit formulas for the CUSUM control chart for a long-memory $\mathrm{ARFIMAX}(p,d^*,q,r)$ process exists and is unique.

5. Numerical Results

The performances of the ARLs from the derived explicit formulas and the NIE method were compared for detecting changes in the process mean on a CUSUM control chart for a long-memory $\mathrm{ARFIMAX}(p,d^*,q,r)$ process. Two in-control ARL values, ARL_0 = 370 and ARL_0 = 500, were considered. The number of division points m = 800 nods was used for the NIE method. The out-of-control process is referred to as ARL_1. For the in-control state, exponential parameter $\lambda_0=1$, while for the out-of-control state, λ_1 = 1.01, 1.02, 1.03, 1.05, 1.10, 1.20, and 1.40. The non-stationary ARFIMAX(1,0.25,1,1), ARFIMAX(1,0.35,1,1), ARFIMAX(2,0.25,1,1), and ARFIMAX(2,0.35,1,1) models were applied to attain a comprehensive view of the long-memory process.

The other model parameters were set as follows:

(i) Autoregressive (AR) coefficient: $\phi_1=\pm\,0.01$ and $\phi_2=0.02$.

(ii) Moving-average (MA) coefficient: $\theta_1=0.01$.

(iii) Fraction order: $d^*=0.25,\ 0.35$.

(iv) Exogenous coefficient: $\beta_1=0.5$.

Definition 5.1 *The percentage error (PE) used to compare the performance of the explicit formulas and NIE ARLs across a range of changes in the process mean from 1.01 to 1.40 is defined as*

$$\text{Percentage error (PE)}=\left|\frac{L(\psi)-\tilde{L}(\psi)}{L(\psi)}\right|\times 100\%, \tag{19}$$

where $L(\psi)$ is ARL derived from the explicit formulas, and $\tilde{L}(\psi)$ is approximated ARL of NIE method.

The following observations are evident in the numerical results reported in Tables 1 and 2:

1. In the out-of-control cases when $\lambda>1$, the ARL_1 results tended to decrease rapidly as the λ_1 level increased.

2. There are two ARL_0 values and the ARL_1 values are affected by the UCL because the out-of-control signals are obtained from the computation in Equation (16). Small ARL_1 values are obtained with small h values for different shifts in the process mean.

3. For all processes, ARL_1 for the explicit formulas and the NIE method were similar. In determining the computed performances for ARL_1 of the explicit formulas and the NIE method in terms of PE were less than 0.25%.

4. For all processes, the computational time of the ARL with explicit formulas was substantially less than that of the NIE method (less than 1 second compared to between 1.23 to 2 hours, respectively).

Figures 1 and 2 show bar graphs of different values of λ_1 (shifts in the process mean) versus their respective ARL_1 value using the explicit formulas. ARL_1 values were derived for each mean shift scenario in different long-memory processes. These graphs exhibit various characteristics of the long-memory ARFIMAX(p,d^*,q,r) processes on a CUSUM chart with coefficients $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\theta_1 = 0.1$, and $\beta_1 = 0.5$ in Figure 1 and $\phi_1 = -0.1$, $\phi_2 = 0.2$, $\theta_1 = 0.1$, and $\beta_1 = 0.5$ in Figure 2. A downward trend can be observed as the shift in the process mean increases, i.e. ARL_1 is reduced by an increase in λ_1. In addition, the ARL of ARFIMAX(1,0.25,1,1) was slightly lower than that of ARFIMAX(1,0.15,1,1), while the process that provided the lowest ARL is ARFIMAX(2,0.25,1,1) for both short-(ARL_0 = 370) and long-term (ARL_0 = 500) detection.

It is evident from the results that the ARL using explicit formulas is a good alternative to the NIE method for evaluating shifts in the process mean on a CUSUM control chart for a long-memory ARFIMAX(p,d^*,q,r) process with exponential white noise because of the substantially reduced computational time along with similarly low PE values (less than 0.25%).

6. Practical Applications

The explicit formulas and the NIE method were applied to evaluate their ARLs using real data to illustrate their practical application on a CUSUM control chart. The real data are the stock prices for Airports of Thailand Public Company Limited (AOT: Bangkok) with the exogenous variable (X) as the exchange rate of Thai baht (THB) per USD (unit: the THB rate). The observations were collected daily (5 days per week) from 8 January 2020 to 4 June 2020 and consist of 101 observations (source: https://th.investing.com).

The dataset was diagnosed and fitted to a long-memory ARFIMAX(1,0.499999,1,1) process with coefficients $\phi_1 = 0.857998$, $\theta_1 = -0.658997$, and $\beta_1 = -7.048698$, and significantly distributed exponential white noise. Using the Kolmogorov-Smirnov test, we determined that the white noise followed an exponential distribution with mean λ_0 = 1.3919. Hence, we treated 1.3919 as the exponential parameter for the in-control state. For the CUSUM chart parameters, a = 1.5 and h (the CUSUM control limit) was selected to give the desired in-control ARL_0 = 370 and 500, for which h = 1.304021 and 1.724015, respectively, as calculated using Equation (16). The ARLs on the CUSUM control chart were derived using the two methods, the results of which are summarized in Table 3; it can be seen that the results are obviously in agreement with those in Tables 1 and 2. The numerical results obtained from the explicit formulas and the NIE method were similar for all cases when detecting small-to-moderate-sized changes in the process mean. However, the computational time of the ARL with explicit formulas was substantially less than that of the NIE method (less than 1 second compared to between 1.3 to 1.4 hours, respectively). To sum up, the explicit formula approach is a good alternative for practical applications in detecting changes process mean on a CUSUM control chart.

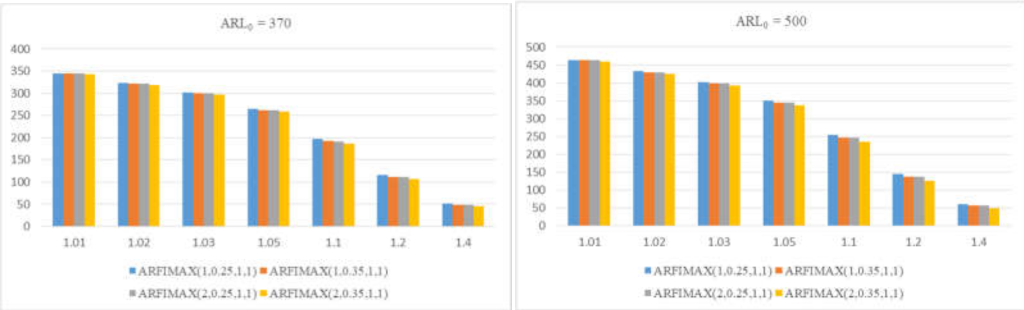

Figure 1 ARL results of the explicit formulas in the out-of-control case on the CUSUM control chart for a long-memory ARFIMAX(p, d^*, q, r) process with $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\theta_1 = 0.1$ and $\beta_1 = 0.5$

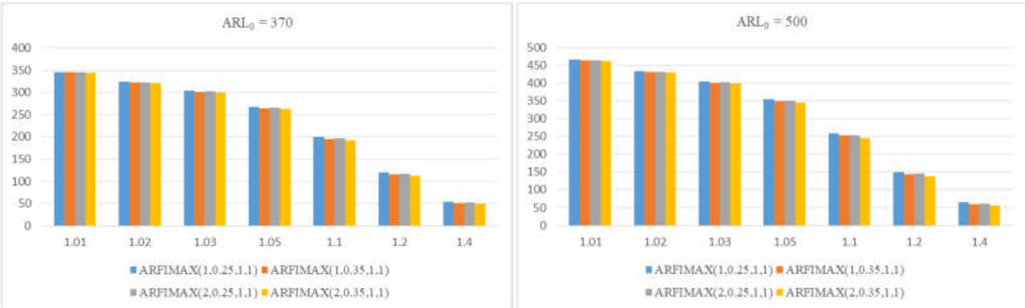

Figure 2 ARL results of the explicit formulas in the out-of-control case on the CUSUM control chart for a long-memory ARFIMAX(p,d^*,q,r) process with $\phi_1 = -0.1$, $\phi_2 = 0.2$, $\theta_1 = 0.1$ and $\beta_1 = 0.5$

7. Discussion and Conclusions

ARLs based on explicit formulas and the NIE method to detect shifts in the process mean of a long-memory ARFIMAX process with exponential white noise on a CUSUM control chart were derived and evaluated. Furthermore, the existence and uniqueness of the ARL based on explicit formulas were proved. In a numerical study, the in-control ARL was established with different parameter settings and levels of process mean shift with which a comparison of the ARL with explicit formulas and the NIE method in monitoring the mean shifts was demonstrated. In conclusion, the PEs of the ARLs based on explicit formulas and the NIE method were similar, but the former consumed much less computational time than the latter. Herein, the ARL is derived using explicit formulas for a long-memory ARFIMAX process on a CUSUM control chart. The focus of the study was on the exogenous variable in an ARFIMAX process, which affects econometric models when forecasting since a forecasting model including an exogenous variable is usually more accurate than one without it. In future studies, it would be interesting to determine more than one criterion can be used for measuring control chart performance. This approach could be extended to other performance measures such as the average extra quadratic loss (AEQL), the average ratio of ARL (ARARL), and the probability of a false alarm (PFA) (see Abujiya et al. 2015, Abujiya et al. 2016).

Table 1 ARL results for the explicit formulas and the NIE method in the out-of-control case on the CUSUM control chart for a long-memory ARFIMAX(p, d^*, q, r) process with $a = 3$ and $ARL_0 = 370$

Models	h	Parameters	ARL	λ_1						
				1.01	1.02	1.03	1.05	1.10	1.20	1.40
ARFIMAX (1, 0.25, 1, 1)	4.262875	$\phi_1 = 0.1$,	Explicit	**345.3398**	**322.7642**	**302.0659**	**265.5874**	**196.6006**	**116.4961**	**51.9757**
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	344.5824	322.0689	301.4267	265.0449	196.2324	116.3118	51.9162
			(Hrs.)	(1.24)	(1.24)	(1.23)	(1.23)	(1.23)	(1.23)	(1.23)
			PE(%)	0.22	0.22	0.21	0.20	0.19	0.16	0.11
	4.039879	$\phi_1 = -0.1$,	Explicit	345.9915	323.9673	303.734	267.9719	199.9146	119.9389	54.2965
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	345.2445	323.2789	303.0989	267.4290	199.5397	119.7452	54.2305
			(Hrs.)	(1.23)	(1.23)	(1.23)	(1.24)	(1.23)	(1.23)	(1.23)
			PE(%)	0.22	0.21	0.21	0.20	0.19	0.16	0.12
ARFIMAX (1, 0.35, 1, 1)	4.51753	$\phi_1 = 0.1$,	Explicit	**344.4419**	**321.1091**	**299.7754**	**262.3266**	**192.1142**	**111.9209**	**48.9882**
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	343.6858	320.4190	299.1447	261.7973	191.7647	111.7548	48.9395
			(Hrs.)	(1.24)	(1.23)	(1.23)	(1.24)	(1.24)	(1.24)	(1.24)
			PE(%)	0.22	0.21	0.21	0.20	0.18	0.15	0.10
	4.304278	$\phi_1 = -0.1$,	Explicit	**345.2064**	**322.5176**	**301.7242**	**265.0997**	**195.9261**	**115.802**	**51.5157**
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	344.4481	321.8221	301.0852	264.5583	195.5601	115.6201	51.4577
			(Hrs.)	(1.24)	(1.24)	(1.24)	(1.24)	(1.24)	(1.24)	(1.24)
			PE(%)	0.22	0.22	0.21	0.20	0.19	0.16	0.11
ARFIMAX (2, 0.25, 1, 1)	4.5305238	$\phi_1 = 0.1$,	Explicit	**344.3906**	**321.0149**	**299.6455**	**262.1422**	**191.8621**	**111.6666**	**48.8251**
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	343.635	320.3256	299.0156	261.6141	191.514	111.5017	48.777
		$\beta_1 = 0.5$	(Hrs.)	(1.57)	(1.57)	(1.57)	(1.58)	(1.58)	(1.58)	(1.58)
			PE(%)	0.22	0.21	0.21	0.20	0.18	0.15	0.10
	4.262877	$\phi_1 = -0.1$,	Explicit	345.3404	322.7647	302.0664	265.5878	196.6009	116.4962	51.9758
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	344.5829	322.0694	301.4271	265.0453	196.2327	116.3119	51.9162
		$\beta_1 = 0.5$	(Hrs.)	(1.57)	(1.57)	(1.57)	(1.57)	(1.58)	(1.59)	(1.58)
			PE(%)	0.22	0.22	0.21	0.20	0.19	0.16	0.11
ARFIMAX (2, 0.35, 1, 1)	4.7842064	$\phi_1 = 0.1$,	Explicit	**343.2759**	**318.9664**	**296.8182**	**258.1376**	**186.418**	**106.2342**	**45.4075**
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	342.5413	318.3019	296.2161	257.6413	186.1043	106.098	45.3746
		$\beta_1 = 0.5$	(Hrs.)	(1.58)	(1.57)	(1.58)	(1.58)	(1.59)	(1.59)	(1.58)
			PE(%)	0.21	0.21	0.20	0.19	0.17	0.13	0.07
	4.51753	$\phi_1 = -0.1$,	Explicit	344.4419	321.1091	299.7754	262.3266	192.1142	111.9209	48.9882
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	343.6858	320.4190	299.1447	261.7973	191.7647	111.7548	48.9395
		$\beta_1 = 0.5$	(Hrs.)	(1.57)	(1.57)	(1.57)	(1.57)	(1.58)	(1.58)	(1.57)
			PE(%)	0.22	0.21	0.21	0.20	0.18	0.15	0.10

The results are expressed as percentage errors (PE%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (hours).

Table 2 ARL results for the explicit formulas and the NIE method in the out-of-control case on the CUSUM control chart for a long-memory ARFIMAX(p, d^*, q, r) process with $a = 3$ and $ARL_0 = 500$

Models	h	Parameters	ARL	λ_1						
				1.01	1.02	1.03	1.05	1.10	1.20	1.40
ARFIMAX (1, 0.25, 1, 1)	4.635784	$\phi_1 = 0.1$,	Explicit	464.4094	431.9796	402.3839	350.5727	254.0075	144.9958	61.1589
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	463.2989	430.9671	401.4592	349.7979	253.4975	144.7543	61.0880
			(Hrs.)	(1.39)	(1.40)	(1.39)	(1.39)	(1.39)	(1.40)	(1.40)
			PE(%)	0.24	0.23	0.23	0.22	0.20	0.17	0.12
	4.394693	$\phi_1 = -0.1$,	Explicit	465.5070	433.9997	405.1757	354.5372	259.4316	150.4777	64.6975
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	464.4093	432.9938	404.2526	353.7563	258.9057	150.2177	64.6150
			(Hrs.)	(1.42)	(1.42)	(1.42)	(1.42)	(1.42)	(1.42)	(1.43)
			PE(%)	0.24	0.23	0.23	0.22	0.20	0.17	0.13
ARFIMAX (1, 0.35, 1, 1)	4.921236	$\phi_1 = 0.1$,	Explicit	462.7985	429.0229	398.3079	344.8123	246.2152	137.2809	56.3496
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	461.6955	428.0255	397.4047	344.0678	245.7445	137.0750	56.2982
			(Hrs.)	(1.43)	(1.42)	(1.42)	(1.43)	(1.42)	(1.43)	(1.43)
			PE(%)	0.24	0.23	0.23	0.22	0.19	0.15	0.09
	4.681308	$\phi_1 = -0.1$,	Explicit	464.1772	431.5529	401.7949	349.7384	252.8727	143.8612	60.4398
		$\theta_1 = 0.1$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\beta_1 = 0.5$	NIE	463.0661	430.5409	400.8718	348.9666	252.3674	143.6243	60.3715
			(Hrs.)	(1.43)	(1.42)	(1.42)	(1.42)	(1.42)	(1.46)	(1.43)
			PE(%)	0.24	0.23	0.23	0.22	0.20	0.16	0.11
ARFIMAX (2, 0.25, 1, 1)	4.936225	$\phi_1 = 0.1$,	Explicit	462.7028	428.8476	398.0666	344.4721	245.7578	136.8332	56.0760
		$\phi_2 = 0.2$	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	461.6010	427.8518	397.1652	343.7299	245.2899	136.6297	56.0258
		$\beta_1 = 0.5$	(Hrs.)	(1.99)	(2.02)	(1.99)	(1.99)	(1.99)	(1.99)	(1.99)
			PE(%)	0.24	0.23	0.23	0.22	0.19	0.15	0.09
	4.635785	$\phi_1 = -0.1$,	Explicit	464.4097	431.9800	402.3842	350.5729	254.0076	144.9959	61.1589
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	463.2992	430.9674	401.4595	349.7982	253.4977	144.7544	61.0880
		$\beta_1 = 0.5$	(Hrs.)	(1.99)	(1.99)	(1.99)	(1.99)	(1.99)	(1.99)	(1.98)
			PE(%)	0.24	0.23	0.23	0.22	0.20	0.17	0.12
ARFIMAX (2, 0.35, 1, 1)	5.242483	$\phi_1 = 0.1$,	Explicit	460.4471	424.7189	392.3917	336.4982	235.1194	126.5613	49.9393
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	459.3926	423.7797	391.5538	335.8286	234.7285	126.4189	49.9195
		$\beta_1 = 0.5$	(Hrs.)	(1.98)	(1.99)	(1.99)	(1.99)	(1.98)	(1.98)	(1.98)
			PE(%)	0.23	0.22	0.21	0.20	0.17	0.11	0.04
	4.921236	$\phi_1 = -0.1$,	Explicit	462.7985	429.0229	398.3079	344.8123	246.2152	137.2809	56.3496
		$\phi_2 = 0.2$,	(Sec.)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
		$\theta_1 = 0.1$,	NIE	461.6955	428.0255	397.4047	344.0678	245.7445	137.0750	56.2982
		$\beta_1 = 0.5$	(Hrs.)	(1.99)	(1.99)	(2.00)	(1.99)	(2.00)	(2.00)	(2.01)
			PE(%)	0.24	0.23	0.23	0.22	0.19	0.15	0.09

The results are expressed as percentage errors (PE%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (hours).

Table 3 ARL results for the explicit formulas and the NIE method in the out-of-control case for a long-memory ARFIMAX(1, 0.499999,1,1) process on a CUSUM control chart under data on USD exchange rate with in-control case λ_0 = 1.3919

ARL_0	a	h	ARL	λ_1							
				1.4019	1.4119	1.4219	1.4419	1.4919	1.5919	1.7919	2.3919
370	1.5	1.304021	Explicit	354.6527	340.1446	326.4205	301.1234	248.4526	175.3344	98.0433	30.5826
			(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
			NIE	354.4193	339.9224	326.2084	300.9305	248.2989	175.233	97.9933	30.5714
			(Hrs.)	(1.35)	(1.35)	(1.35)	(1.35)	(1.35)	(1.35)	(1.33)	(1.33)
			PE(%)	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.04
500	1.5	1.724015	Explicit	478.2225	457.6806	438.2898	402.6578	328.9754	228.0561	123.7877	36.3935
			(Sec.)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
			NIE	477.8065	457.2843	437.9130	402.3174	328.7071	227.8820	123.7042	36.3658
			(Hrs.)	(1.37)	(1.37)	(1.37)	(1.36)	(1.36)	(1.36)	(1.36)	(1.36)
			PE(%)	0.09	0.09	0.09	0.08	0.08	0.08	0.07	0.07

The results are expressed as percentage errors (PE%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (hours). Coefficients ϕ_1 = 0.857998, θ_1= −0.658997, and β_1 = −7.048698.

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