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Stochastic Decomposition Results for Poisson Input Queue and Its Applications

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Abstract

In this paper, we derive some decomposition results for Poisson input queue with two types of general heterogeneous services and generalized vacations under optional repeated service policy. Our results are based on an embedded Markov chain technique which includes several kinds of generalizations of some well-known results for the vacation model that leads to remarkable simplification while solving similar types of complex models. Finally, we demonstrate some well-known existing results as particular cases of our decomposition result.

Keywords: Heterogeneous services, generalized vacation, embedded Markov chain technique.

1. Introduction

In classical queueing theory, it is assumed that a unit which cannot get service immediately after arrival, either joins the waiting line (and then is served according to some queueing discipline) or leave the system forever. However, there may be some situations where as soon as the busy period ends, the server may shut down the service facility (the server may be engaged in some secondary works). Thus, the server may not be providing service when the next unit arrives at an empty system. This type of queueing system is known as a queue with a vacation. The period in which the server is not providing service in the system is known as the server's vacation period. Presently queue with vacations has been the subject matter of many papers mainly due to their theoretical structure as well as their applications in Computer and Communication system e.g., see Doshi (1986) and Takagi (1991).

The decomposition property for M/G/1 queue has been studied, among others by Graver (1962), Cooper (1970), Furhman (1984) and for GI/G/1 system by Doshi (1986). However, for generalized vacations the decomposition result was first established by Fuhrman and Cooper (1986). Shantikumar (1988) relaxed some of the assumptions considered by Fuhrman and Cooper (1986). Harris and Marchal (1988) extended Fuhrman and Cooper's result to the state- dependent case. Choudhury and Borthakur (2000) extended this result for batch arrival Poisson input queueing system. Choudhury

(2008) established the decomposition result for a single server queue with two phases of service and generalized vacations. Apart from theoretical arguments, many real-life situations meet the requirements of the Poisson process justifying the assumption of Poisson input for the arrival of a queue.

Recently, there have been several contributions considering Poisson input queueing system in which the server provides two types of general heterogeneous services and an arriving customer has an option to choose either type of service. Madan et al. (2004) studied such a batch arrival queueing system, where they introduced the concept of re-service. For a more detailed study on such models we refer the readers to see some related papers by Choudhury and Kalita (2017), Kalita and Choudhury (2019), Begum and Choudhury (2021) etc.

The interest in such type of queueing system is further enhanced in recent years because of its theoretical structure as well as their applications in many real-life situations such as computer and telecommunication as well as in inventory systems. Some of such situations are described below:

1. Consider the model building of a computer and communication system where a server, besides being engaged in primary functions (such as receiving, processing and transmitting data), has to undertake secondary works such as preventive maintenance or has to scan for new work for an occasional period of time, which we may refer to as vacation period.

2. Consider a single product $(s; S)$ inventory system, which is supplied by a source of a single machine production facility. Assuming that the production process does not start until the number of orders on the product accumulates to say N, and it goes on producing items until the on-hand stock i.e., when the inventory is at the base stock level, the production source remain idle. On the other hand, as soon as it drops to the re-order level, say $s(\leq S)$ due to the demand of customers, setup of the facility begins. Once the setup is over, the machine starts producing items until the stock is raised to S.

In this work, we aim to derive main stochastic decomposition result for two types of general heterogeneous services and generalized vacation under optional repeated service policy in order to generalize some well-known results of the classical vacation models.

The remainder of this paper is organized as follows. In Section 2, we give a brief description of the mathematical model under study. Section 3 deals with the derivation of the stochastic decomposition result. Section 4 discusses applications of the stochastic decomposition result in some well-known vacation models. Section 5 gives the concluding remark.

2. Mathematical Model

We consider an M/G/1 queueing system with two types of heterogeneous services in which arrivals occur according to Poisson process with rate λ. There is a single server which provides both the two types of heterogeneous service to each customer on first come first service (FCFS) basis. Before its service starts, each unit has an option to select either type of services i.e., each unit has an option to select either of the k^{th} type of service denoted by S_k (service time random variable) with probability p_k for k =1, 2 (here k =1 denotes the first type of service, and k =2 represents the second type of service). Thus, total service time denoted by 'T' required by a unit to complete the service is given by

$$T=\begin{cases} S_1, \text{ with probability } p_1 \\ S_2, \text{with probability } p_2. \end{cases}$$

Now, denoting distribution functions (DF) of S_k by $S_k(x)$, Laplace Stieltjes Transform (LST) by $S_k^*(\theta) = E\left[e^{-\theta S_k}\right]$ and d^{th} finite moment by $s_k^{(d)}$. Thus, the LST of the total service time provided by a unit is given by

$$T^*(\theta) = p_1 S_1^*(\theta) + p_2\, S_2^*(\theta), \tag{1}$$

where $T^*(\theta) = E\left[e^{-\theta T}\right]$.

Further, it is assumed that as soon as either type of service is completed by a customer, such a customer has further option to repeat the same type of service denoted by B_k (repeated service time random variable) only once with probability q_k or leave the system with probability q_k^c where $q_k + q_k^c = 1,$ for $k = 1, 2.$

It should be noted that the same server performs the repeated service (RS) and total service time required by the server for k^{th} type of service to complete the service cycle for k = 1, 2 is given by,

$$T = \begin{cases} S_k + B_k, \text{ with probability } q_k \\ S_k, \text{with probability } q_k^c. \end{cases}$$

Further, it is assumed that services given in both the types of services are non-primitive i.e., once selected for service (first type of service (FTS) or second type of service (STS)); a unit is served to completion continuously. As soon as a busy period (attending FTS or STS) ends, the vacation period of the server begins. A vacation period may contain a number of vacations. The vacation periods begin and end according to well-defined rules, which may depend on either the current or past evaluation of the system.

Further, a vacation period can be terminated by a condition depending on the arrival process during that vacation period.

Moreover, it is assumed that service time random variables are independent of the arrival process, and each service time is independent of the sequence of vacation periods that precede that service time. A queueing system that satisfies these properties is called an M/G/1 queueing system with two types of heterogeneous services under repeated service policy and generalized vacations.

Denoting DF's for k^{th} type of repeated service by $B_k(x),$ LST by $B_k^*(\theta)$ and d^{th} finite moment by $b^{(d)},$ the LST of the total service time provided by a unit for k^{th} type of service is now given as,

$$T^*(\theta) = q_k^c S_k^*(\theta) + q_k\, S_k^*(\theta)\, B_k^*(\theta). \tag{2}$$

Thus, utilizing (1) in (2) we have,

$$T^*(\theta) = \sum_{k=1}^{2}\left[q_k^c + q_k B_k^*(\theta)\right] p_k\, S_k^*(\theta)$$

so that the utilization factor of the system 'ρ' is given by

$$\rho = \lambda\left[\sum_{k=1}^{2} p_k\left(s^{(1)} + q_k b^{(1)}\right)\right].$$

3. Stochastic Decomposition Result

Our main aim in this paper is to derive the stochastic decomposition result for our discussed model. Accordingly, we state the main result in the following Theorem 1.

Theorem 1. *Under the stability condition $\rho < 1$ and the following assumptions stated below:*

1. Arrivals occur in the system according to a Poisson process with parameter $\lambda > 0$, service time and repeated service time are given by general distribution functions $S_i(.)$ and $B_i(.)$, respectively for i =1, 2 and are independent of the arrival process and the sequence of vacation periods.

2. All units that arrive in the system are eventually served, i.e., the system has the infinite queueing capacity, and server utilization is less than unity. Moreover, units do not balk, defect, or renege from the system.

3. Units are served in an order that is independent of their service time and repeated service time respectively.

4. Service is non-preemptive i.e., once a unit is selected for service is served to completion in a continuous manner.

5. The rules that govern when the service or repeated service begins and ends is that vacations do not anticipate future jump of the Poisson arrival process, we have

$$\psi(z) = \frac{(1-\rho)\left[1-\alpha(z)\right]\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda-\lambda z)\right\} p_k S_k^*(\lambda-\lambda z)\right]}{\alpha'(1)\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda-\lambda z)\right\} p_k S_k^*(\lambda-\lambda z) - z\right]},$$

where $\alpha(z)$ is the probability generating function (PGF) of the queue size distribution at busy period initial epoch, $\psi(z)$ is the PGF of the queue size distribution at a departure epoch, $S_k^*(.)$ and $B_k^*(.)$ denotes the LST of the k^{th} service time distribution and repeated service time distribution, respectively and ρ is the utilization factor of the server.

Proof: Let t_m be the time of m^{th} service completion epoch i.e., we are considering the epochs at which total service required by customer expires. Then the sequence $X_n = N(t_n + 0)$ (where $N(t_n)$ represents the number of units in the system at time instant t_n) form a discrete-time Markov chain (DTMC) which is an embedded Markov renewal process of a continuous time Markov process.

The sequence $\{X_n; n \geq 0\}$ is a homogeneous DTMC, and it is owing to the following transition,

$$X_{n+1} = \begin{cases} L_{n+1} + V_{n+1} - 1, & \text{if } X_n = 0 \\ X_n + L_{n+1} - 1, & \text{if } X_n > 0, \end{cases}$$

where L_n is the number of units that arrived during n^{th} total service period and V_n is the number of units that arrived during n^{th} vacation period.

Next, we define $\Delta_{m,n}$ matrix introduced and studied by Abolnikov and Dukhovny (1991), which is associated with our transition probability matrix (TPM) $P = (p_{ij})$.

$\Delta_{m,n}$ matrix: A finite or infinite stochastic matrix $P = (p_{ij})$ is called a $\Delta_{m,n}$ matrix, $n \geq m \geq 1$ if $p_{i,j}$, for $i > n$ and $i - j > m$, more specifically the stochastic matrix P is of the form:

$$P=\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,n-m} & p_{0,n-m+1} & \cdots \\ p_{10} & p_{11} & \cdot\cdot & p_{1,n-m} & p_{1,n-m+1} & \cdots \\ \vdots & \vdots & \ddots & & & \cdot \\ p_{n,0} & p_{21} & \cdots & p_{n,n-m} & p_{n,n-m+1} & \cdots \\ 0 & 0 & \cdots & 0 & p_{n+1,n-m+1} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

It should be noted here that when $m=n$ then $\Delta_{m,n}$ matrix reduces to Δ_n matrix, which in fact is a special case of $\Delta_{m,n}$ matrix. Thus, the TPM $P=(p_{ij})$ associated with the DTMC $\{X_n;n\geq 1\}$ considered in this paper is of the form:

$$P=\begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ 0 & p_{21} & p_{22} & \cdots \\ 0 & 0 & p_{32} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where, $p_{ij}=P\left(X_{n+1}=j\mid X_n=i\right)$ is the transition probability associated with the DTMC $\{X_n;n\geq 0\}$.

Clearly, our TPM $P=(p_{ij})$ is readily seen to be a Δ_2 matrix (see Abolnikov and Dukhovny (1991)) which is a special case of $\Delta_{m,n}$ matrix. Our Δ_2 matrix differs from that of classical M/G/1 queue in the first row only i.e., TPM $P=(p_{ij})$ associated with DTMC $\{X_n;n\geq 0\}$ has the elements.

$$p_{i.j}=\begin{cases} \sum_{m=1}^{j+1}\alpha_m\sum_{k=1}^{2}p_k\left(q_k^c\beta_{k,j-m+1}+q_k\Gamma_{k,j-m+1}\right), \text{ if } i=0, j\geq 0 \\ \sum_{k=1}^{2}p_k\left(q_k^c\beta_{k,j-i+1}+q_k\Gamma_{k,j-i+1}\right), \text{ if } i\geq j-1, j\geq 1 \\ 0, \text{ if } i\geq 1\text{ , } 0\leq j\leq i-1, \end{cases}$$

where $\alpha_m=\lim_{n\to\infty}\mathrm{P}(V_{n+1}=m)$ is the limiting probability that m units find the server to start a busy period, $\beta_{k,j}=\int_0^\infty \frac{e^{-\lambda x}\left(\lambda x\right)^j}{j!}dS_k(x)$ is the probability that j units arrive during k^{th} type of service, $\gamma_{k,j}=\int_0^\infty \frac{e^{-\lambda x}\left(\lambda x\right)^j}{j!}dB_k\left(x\right)$ is the probability that j units arrive during k^{th} type of repeated service, $\Gamma_{k,j}=\sum_{i=0}^{j}\beta_{k,j}\gamma_{k,j-1};j\geq 0$ is the probability that j units arrive during k^{th} type of total service period.

Next, we assume that $\rho=\lambda\left[\sum_{k=1}^{2}p_k\left(s^{(1)}+q_kb^{(1)}\right)\right]<1$ to guarantee that $\{X_n;n\geq 0\}$ is positive recurrent. Thus, limiting probability, $\psi_j=\lim_{n\to\infty}\mathrm{P}\left(X_n=j\right)$ exist and is positive. Then the Kolmogorov equations associated with DTMC $\{X_n;n\geq 0\}$ can be written as $\psi_j=\sum_{i=0}^{\infty}\psi_i P_{ij}$.

This implies that, for $j \geq 0,$ we have

$$\psi_j = \sum_{n=1}^{j+1} \{ \psi_0 \alpha_n + \psi_n \} \left[\sum_{k=1}^{2} p_k \left(q_k^c \beta_{k,j-n+1} + q_k \Gamma_{k,j-n+1} \right) \right]; j \geq 0 \tag{3}$$

Let $\psi(z) = \sum_{j=0}^{\infty} z^j \psi_j$, $\alpha(z) = \sum_{j=0}^{\infty} z^j \alpha_j$, $\beta_k(z) = \sum_{j=0}^{\infty} z^j \beta_{k,j}$ and $\Gamma_k(z) = \sum_{j=0}^{\infty} z^j \Gamma_{k,j}$ $|z| \leq 1$ be PGFs of ψ_j, α_j, $\beta_{k,j}$ and $\Gamma_{k,j}$, respectively for k =1, 2, then from (3) we have

$$\psi(z) = \psi_0 \alpha(z) \left[\sum_{k=1}^{2} p_k \left(\beta_k(z) q_k^c + q_k \Gamma_k(z) \right) z^{-1} + \{ \psi(z) - \psi_0 \} \left\{ \sum_{k=1}^{2} p_k \left\{ q_k^c \beta_k(z) + q_k \Gamma_k(z) \right\} z^{-1} \right\} \right] \tag{4}$$

Now, because of the presence of convolution, (4) can be transformed with the help of the following PGFs $\beta_k(z) = S_k^*(\lambda - \lambda z)$, $\Gamma_k(z) = \beta_k(z)\gamma_k(z)$ for k =1, 2. Note that $\gamma_k(z) = B_k^*(\lambda - \lambda z)$ for k =1, 2 and therefore from (4), we have

$$\psi(z) = \frac{\psi_0 [1 - \alpha(z)] \left[\sum_{k=1}^{2} \left(q_k^c + q_k B_k^* (\lambda - \lambda z) \right) p_k S_k^* (\lambda - \lambda z) \right]}{\left[\sum_{k=1}^{2} \left(q_k^c + q_k B^* (\lambda - \lambda z) \right) p_k S_k^* (\lambda - \lambda z) - z \right]}. \tag{5}$$

Now since $\sum_{j=0}^{\infty} \psi_j = \psi(1) = 1,$ Equation (5) yields,

$$\psi_0 = \frac{1 - \rho}{\alpha'(1)}. \tag{6}$$

Now from the above expression, we observe that $\rho < 1,$ which is the necessary and sufficient condition for the existence of a steady state solution of our model. Since $\rho < 1,$ therefore equation $\sum_{k=1}^{2} \left(q_k^c + q_k B^* (\lambda - \lambda z) \right) p_k S_k^* \ (\lambda - \lambda z) - z = 0$ never vanishes inside the region $|z| \leq 1,$ by Rouche's theorem. Hence, utilizing (6) in (5), we finally get the required result.

Remark 1.

If we take $\alpha_1 = 1$ and $\alpha_n = 0$ if $n \neq 1,$ then $\alpha(z) = z$ and $\alpha'(1) = 1.$ Hence, from Theorem 1, we have

$$\psi(z) = \frac{(1 - \rho)(1 - z) \left[\sum_{k=1}^{2} \left(q_k^c + q_k B_k^* (\lambda - \lambda z) \right) p_k S_k^* (\lambda - \lambda z) \right]}{\left[\sum_{k=1}^{2} \left(q_k^c + q_k B_k^* (\lambda - \lambda z) \right) p_k S_k^* (\lambda - \lambda z) - z \right]},$$

which is consistent with the result obtained by Madan et al. (2004) for a single unit arrival case.

Remark 2.

If we take $q_k = 0$ and $p_2 = 0$ (i.e., no second choice of service and no repetition of service) then $\rho = \lambda s^{(1)} < 1$ and form Theorem 1, we get

$$\psi(z) = \frac{(1 - \rho)[1 - \alpha(z)] S_1^* (\lambda - \lambda z)}{\alpha'(1) \left[S_1^* (\lambda - \lambda Z) - Z \right]},$$

which is consistent with the result obtained by Fuhrmann and Cooper (1986).

Remark 3.

Further, if we consider the case of non-exhaustive service discipline, where vacation may start even when some units are present in the system (tagged vacation), then corresponding PGF of the queue size distribution at a departure epoch for our model is found to be

$$\psi(z)=\xi(z)\frac{(1-\rho)[1-\alpha(z)]\left[\sum_{k=1}^{2}\left(q_k^c+q_kB_k^*(\lambda-\lambda x)\right)p_kS_k^*(\lambda-\lambda x)\right]}{\alpha'(1)\left[\sum_{k=1}^{2}\left(q_k^c+q_kB_k^*(\lambda-\lambda x)\right)p_kS_k^*(\lambda-\lambda x)-z\right]},$$

where $\xi(z)$ is the PGF of the customers that arrive during the tagged vacation. Note that the gated service, limited service and decrementing service disciplines are examples of non-exhaustive service discipline.

4. Some Applications of the Stochastic Decomposition Result

Here we briefly discuss some well-known vacation models for our type of queueing system as an application of the decomposition result established in Theorem 1.

4.1. Randomized vacation policy queueing model

In randomized vacation policy queueing model, as soon as the system becomes empty, the server deactivates and leaves for a vacation. Upon returning from the vacation, if at least one unit is found in the queue for any one type of service, the server starts providing service to the unit. Otherwise, if there are no units found waiting in the queue, the server either remains idle in the system or takes another vacation. This pattern continues until the number of vacations taken reaches M. If the system is still empty by the end of the M^{th} vacation, the server becomes idle in the system until at least one unit waiting in the queue (dormant period). Thus, in this system, a random vacation period, a dormant period and a busy period constitute a cycle. Moreover, the system remains idle during a random vacation period and a dormant period, and these two periods together constitute an idle period.

This type of model was first investigated by Takagi (1991) (see p.127) for M/G/1 queueing system. Next, we define: V -vacation time random variable, $V(x)$ - probability distribution function of V and $V^*(\theta)=E\left[e^{-\theta V}\right]$ LST of V. Now, if we denote h_j by the probability that j units arrive during a vacation time V, then for $j=0, 1, 2,\ldots,$ we have

$$h_j=\int_0^\infty\frac{e^{-\lambda x}(\lambda x)^j}{j!}dV(x);\ j\geq 0.$$

So that the probability that j units arrived and accepted to start a busy period is given by

$$\alpha_j=h_j\Pi_M(h_0)+(h_0)^M\delta_{j,M};\ j\geq 1 \tag{7}$$

where $\Pi_M(h_0)=\sum_{n=0}^{M-1}(h_0)^n=\frac{\left(1-(h_0)^M\right)}{(1-h_0)}$, $h_0=V^*(\lambda)$ and $\delta_{i',j}=\begin{cases}1, \text{ if } i=j\\ 0, \text{ otherwise.}\end{cases}$

Denotes Kronecker's delta function. Then, from (7), we have

$$\alpha(z)=\sum_{j=1}^{\infty}z^j\alpha_j=\Pi_M(h_0)\left[V^*(\lambda-\lambda z)-h_0\right]+\left[h_0\right]^M z \tag{8}$$

and

$$\alpha'(1)=\lambda\Pi_M(h_0)E(V)+h_0^M \tag{9}$$

Utilizing (8), (9) and (7) in Theorem 1, we have

$$\psi(z)=\frac{(1-\rho)\left[\Pi_M(h_0)\left(1-V^*(\lambda-\lambda z)+h_0^M(1-z)\right)\right]\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)\right]}{\left[\lambda\Pi_M(h_0)E(V)+h_0^M\right]\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)-z\right]}, \tag{10}$$

which is the required expression for PGF of queue size distribution at departure epoch of our model.

Now, putting $M=1$ in the above expression (10), we get

$$\psi(z)=\frac{(1-\rho)\left[\left(1-V^*(\lambda-\lambda z)\right)+V(\lambda)(1-z)\right]\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)\right]}{\left[\lambda E(V)+V^*(\lambda)\right]\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)-z\right]}.$$

which is the PGF of the queue size distribution for our model under single vacation policy, where the server takes exactly one vacation between two successive busy periods. Similarly, if we take $M\to\infty$, we have

$$\lim_{M\to\infty}(h_0)^M\to 0 \ as \ |h_0|\le 1,$$

and

$$\lim_{M\to\infty}\Pi_M(h_0)=\frac{1}{(1-h_0)}.$$

Consequently, from above, we have

$$\psi(z)=\frac{(1-\rho)\left[\left(1-V^*(\lambda-\lambda z)\right)\right]\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)\right]}{\lambda E(V)\left[\sum_{k=1}^{2}\left\{q_k^c+q_kB_k^*(\lambda-\lambda z)\right\}p_kS_k^*(\lambda-\lambda z)-z\right]},$$

which is the PFG of the queue size distribution at a departure epoch for our model under multiple vacation policy, where the server keeps on taking a sequence of vacations until it finds some units in the system after returning from a vacation. Note that these two basic models were first investigated by Levy and Yechiali (1975) for M/G/1 queueing system.

4.2. Queueing system with set-up time under N-policy

Takagi (1991) and Medhi and Templeton (1992) studied such a model for M/G/1 queueing system; where the server is turned off each time as soon as the system becomes empty. The server remains idle until the queue size builds up to a pre-assigned threshold level N (build-up period). As soon as the queue size becomes $N(\ge 1)$ the server has to undertake a gear up time called setup time (SET) in order to setup the system into operative mode (set up period), on completion of which service starts (busy period).

Let us define the following notations, U is the SET random variable, $U(x)$ is the probability distribution function of U, and $U^*(\theta)$ is LST of U. Now, if we define g_j by the probability that j units arrived during a SET, then for $j\ge 0$, we have

$$g_j = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^j}{j!} dU(x);\ j \ge 0$$

so that the probability that j units arrived and accepted to start a busy period is given by

$$\alpha_j = \begin{cases} 0, \text{ if } j = 1,2,3,...,N-1 \\ g_j - N, \text{ if } j \ge N, \end{cases}$$

and therefore, we have

$$\alpha(z) = \sum_{j=1}^{\infty} z^j \alpha_j = \sum_{j=N}^{\infty} z^j g_j - N = z^N U^*(\lambda - \lambda z), \tag{11}$$

and

$$\alpha'(1) = N + \lambda E(U). \tag{12}$$

Now, utilizing (11) and (12) in Theorem 1 we get,

$$\psi(z) = \frac{(1-\rho)\left[\left(1 - z^N U^*(\lambda - \lambda z)\right)\right]\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda - \lambda z)\right\} p_k S_k^*(\lambda - \lambda z)\right]}{\left[N + \lambda E(U)\right]\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda - \lambda z)\right\} p_k S_k^*(\lambda - \lambda z) - z\right]} \tag{13}$$

which is the required expression for PGF of departure queue size distribution for our type of model.

Further, if we take $N = 1$ in the above expression (13); then we have,

$$\psi(z) = \frac{(1-\rho)\left[\left(1 - z S^*(\lambda - \lambda z)\right)\right]\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda - \lambda z)\right\} p_k S_k^*(\lambda - \lambda z)\right]}{\left[1 + \lambda E(s)\right]\left[\sum_{k=1}^{2}\left\{q_k^c + q_k B_k^*(\lambda - \lambda z)\right\} p_k S_k^*(\lambda - \lambda z) - z\right]}$$

which is the PGF of the queue size distribution at a departure epoch for random setup time model of our queueing system. Note that model of this nature has been studied by Levy and Khienrock (1986) extensively for M/G/1 type of vacation model.

5. Conclusions

This note demonstrates certain decomposition result of a class of two types of heterogeneous services and optional repeated service queueing system with generalized vacations. Our results cover several kinds of generalizations of some well-known results for the vacation model, which are demonstrated in section 4. The results can be further generalized to the case where the arrival process is a Compound Poisson. Also, we feel that a similar type of investigation for the unreliable queueing system might be possible. Further investigation in this direction, we feel, would be quite rewarding from theoretical as well as application viewpoint.

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