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A New Generalized Weighted Exponential Distribution: Properties and Applications

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Abstract

This paper presents a new generalization of the weighted Exponential distribution using exponentiated generalized class of distributions. The proposed model has the influence as special cases the exponential, exponentiated exponential, exponentiated generalized exponential, and among other distributions. An account of statistical characteristics for the generalized weighted exponential distribution including; quantile function, moments, moment generating function, order statistics and entropy is presented. The model parameters are estimated through method of maximum likelihood and simulation study is performed to assess the stability of maximum likelihood estimates. Two data sets from the field of reliability and medical sciences are considered to analyze the impact of generalized weighted Exponential distribution in real life phenomena and results of selected criterion are compared with some existing models which indicate the efficiency of generalized weighted Exponential distribution.

Keywords: Maximum likelihood, moment, order statistics, quantile function, reliability analysis.

1. Introduction

Modeling the data is a delicate task in distribution theory. Classical distributions are extensively used for this purpose. In some situations, the modified forms of these distributions are required. Researchers are gradually developing the new method for generalizations of the classical distributions by enhancing the numbers of parameters to make these models more flexible and adaptable in applied areas. The classical exponential (Ex) distribution is a very attractive model for modification due to its interesting property known as "Lack of memory property". Numerous extensions of the Ex distribution are available in the literature. Some note able are mentioned as, Exponentaited Ex distribution by Gupta and Kundu (1999), Extended Ex distribution by Afify et al. (2018), Extended Exponentiated Ex distribution Abu-Youssef et al. (2015), Moment Ex distribution Dara and Ahmad

(2012), and Generalized exponentiated moment Ex distribution Iqbal et al. (2014). Gupta and Kundu (1999) introduced a new method of adding parameter in the power of a function. The new family of distributions is given the name as exponentiated family. Many generalizations of the classical distributions are derived using this technique. Gupta and Kundu (2001) studied exponentiated exponential distribution. Nadarajah and Kotz (2006) presented generalizations of Gamma, Weibull, Gumbel and Frechet distributions through exponentiated family. Silva et al. (2010) defined exponentiated exponential-geometric distribution. Lemonte and Cordeiro (2011) proposed the exponentiated generalized inverse Gaussian distribution. Flaih et al. (2012) provided the exponentiated inverted Weibull distribution. Lemonte et al. (2013) obtained the exponentiated Kumaraswamy distribution. Elbatal and Muhammed (2014) constructed the exponentiated generalized inverse Weibull distribution. Chukwu (2014) derived statistical properties of the exponentiated Nakagami distribution. De Andrade et al. (2016)established exponentiated generalized extended distribution.

Cordeiro et al. (2013) proposed a new exponentiated generalized class of distributions. The distribution function (cdf) of the proposed class is given as

$$G(x) = \left(1 - (H'(x))^a\right)^b, \tag{1}$$

where a and b are two additional shape parameters. Here, $H'(x) = 1 - H(x)$ where $H(x)$ is the cdf of any baseline distribution. The cdf of the exponentiated generalized model has great advantage over the beta family of distributions because it does not contain any special function like incomplete beta function. The model in (1) has also attractive characteristics for simulation studies due to simple form of quantile function. The corresponding density function (pdf) for this class is given by

$$g(x) = abh(x)H'(x)^{a-1}[1 - H'(x)^a]^{b-1}. \tag{2}$$

Cordeiro et al. (2013) provided that if the baseline pdf $g(x)$ with a symmetric model, then the resulting model will not be a symmetric model because the two additional parameters a and b can control the tail weights and possibly add entropy to the center of the exponentiated generalized family of distributions. Several distributions, exponentiated generalized extended distribution by Cordeiro et al. (2017), exponentiated generalized Weibull-Gompertz distribution by El-Bassiouny et al. (2017), exponentiated generalized inverted exponential distribution by Oguntunde et al. (2014), and exponentiated generalized Weibull distribution by Oguntunde et al. (2015) are derived. In literature, many weighted distributions are available. For this see, Kim (2008) , Kersey (2010), Shahbaz et al. (2010), Essam and Mohamed (2013), Alqallaf et al. (2015), and Hussian (2013). Oguntunde et al. (2016) provided a new weighted exponential distribution motivated by Nasiru (2015). The CDF of the distribution is given by

$$G(x) = 1 - e^{-\alpha x(1+\lambda)}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{3}$$

and the pdf is given as

$$g(x) = (1 + \lambda)\alpha e^{-\alpha x(1+\lambda)}, \quad x > 0, \quad \alpha, \lambda > 0. \tag{4}$$

The survival function of the weighted exponential distribution is given by

$$S(x) = e^{-\alpha x(1+\lambda)}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{5}$$

where α is the shape parameter and λ is the scale parameter.

In this article, we propose a new generalized weighted exponential (GWEx) distribution. The addition of two additional shape parameters make this model more compatible and flexible. These induction of shape parameters is also enhance its applicability to handle the complicated situations appeared during applied analysis. Many existing distributions are the special case of the proposed model. We hope this generalization will attract the researcher for modeling the real data problems.

The article is arranged as follow. We proposed the new model and studied its properties in Section 2. The Rényi entropy is given in Section 3. Estimation of the model parameters is derived in Section 4. Simulation study is performed in Section 5. Regression Model is discussed in Section 6. Application is provided in Section 7. Finally, conclusion is stated in Section 8.

2. GWEx Distribution and Its Properties

In this section, we derive a four parameter GWEx distribution. The cdf of the proposed model is obtained using (1) and (5) and given as

$$F(x) = [1 - e^{-a(1+\lambda)\alpha x}]^b, \;\; x > 0, \;\; \alpha, \lambda, a, b > 0. \tag{6}$$

The corresponding pdf is obtained by differentiating (6) and given as follow

$$f(x) = a\alpha b(1+\lambda)(e^{-\alpha x(1+\lambda)})(e^{-\alpha x(1+\lambda)})^{a-1}[1 - e^{-a(1+\lambda)\alpha x}]^{b-1},$$

alternatively, we can write

$$f_(x) = ab\alpha(1+\lambda)e^{-a(1+\lambda)\alpha x}[1 - e^{-a(1+\lambda)\alpha x}]^{b-1}, \; x > 0, \; \alpha, \lambda, a, b > 0. \tag{7}$$

Here $a, \; \alpha, \; b$ are shape parameters and λ is scale parameter.

It is worth seen that the proposed model reduces to exponentiated weighted exponential distribution when $a = 1$ and it reduces to exponentiated generalized exponential distribution when $\lambda = 0$. For $a = 1$ and $\lambda = 0$, it reduces to exponentiated exponential distribution and for $a = 1, b = 1$ and $\lambda = 0$, the distribution reduces to exponential distribution. If $a = 2$, the proposed model reduces to Topp-Leone weighted exponential distribution and if $a = 2$ and $\lambda = 0$, the proposed model reduces to Topp-Leone exponential distribution. Many statistical characteristics can be derived using exponentiated form of the pdf and the cdf. For this, we use the series representation of (1986) given as

$$(1-x)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k! \Gamma(b-1)} x^k.$$

Using (6), the exponentiated form of CDF of the proposed distribution is obtained as

$$F(x) = \sum_{j=0}^{\infty} \pi_j (e^{-\alpha x(1+\lambda)})^j, \tag{8}$$

where $\pi_j = \sum_{k,i=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(b+1)\Gamma(ka+1)\Gamma(j+1)}{i!j!k!\Gamma(b-k)\Gamma(j+1-i)} (e^{-a(1+\lambda)\alpha x})^i$.

Similarly, the pdf of the GWEx distribution is explained in weighted sum as

$$f(x) = \sum_{j=0}^{\infty} t_j (e^{-a(1+\lambda)\alpha x})^{j+1}, \tag{9}$$

where $t_j = \frac{abt_j^*(j+1)}{k+1}$,
and $t_j^* = \frac{(-1)^j \Gamma b}{j!} \sum_{k,i=0}^{\infty} \frac{(-1)^{i+k}\Gamma((k+1)a)(\Gamma(j+1))}{k!i!\Gamma(b-k)\Gamma((k+1)a-j)\Gamma(j+1-i)}$.

(9) expresses the pdf of the GWEx distribution in linear combination of Exponentiated G-Class by Gupta and Kundu (1999). Several mathematical characteristics are derived through this expression.

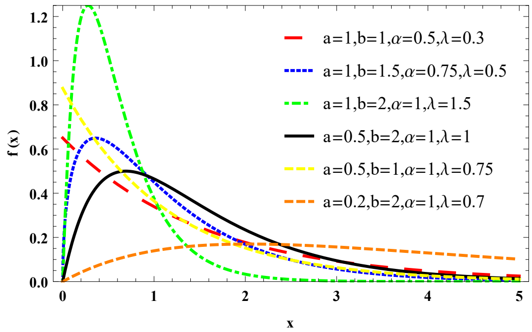

Figure 1 Density plot of GWEx for the several values of the parameters

The survival function of the GWEx distribution is given as

$$S(x) = 1 - [1 - e^{-a(1+\lambda)\alpha x}]^b,$$

and the hazard rate function (hrf) is provided as

$$h(x) = \frac{ab\alpha(1+\lambda)e^{-a(1+\lambda)\alpha x}[1 - e^{-a(1+\lambda)\alpha x}]^{b-1}}{1 - [1 - e^{-a(1+\lambda)\alpha x}]^b}.$$

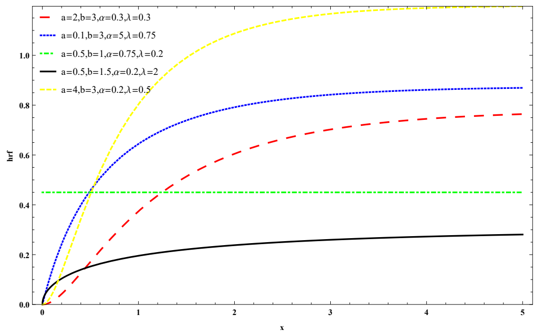

Figure 2 Hazard rate graph for GWEx distribution on several values of parameters

2.1. Quantile function

Theorem 1 *If a random variable X follows the GWEx distribution than its quantile function is obtained*

$$x = -\left(\frac{\ln(1 - q^{\frac{1}{b}})}{\alpha(1+\lambda)}\right). \tag{10}$$

Proof: Quantile function is generally explained as

$$Q_G(x) = F^{-1}(x).$$

Using (6), we have

$$q = \frac{1}{[1 - e^{-a(1+\lambda)\alpha x}]^b}.$$

After some simplifications, the quantile function of the GWEx distribution is derived as

$$x = -\left(\frac{\ln(1-q^{\frac{1}{b}})}{\alpha(1+\lambda)}\right).$$

The median of the proposed distribution is obtained by inserting $q = 0.5$ in (10).

2.2. Moments

Theorem 2 *If a random variable X follows the GWEx distribution than its moments are obtained*

$$\mu_r^{'} = \sum_{j=0} t_j \left(\frac{1}{\alpha(i+1)(1+\lambda)a}\right)^{r+1} \Gamma(r+1), \quad r = 1, 2, 3, \ldots. \tag{11}$$

Proof: The general method to find the ordinary moments for any distribution is given by

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r dF(x).$$

Using (9), the above express reduces to

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r \sum_{j=0}^{\infty} t_j (e^{-a(1+\lambda)\alpha x})^{j+1}.$$

Assuming $a(1+\lambda)\alpha x)(j+1) = z$, the moment of the GWEx distribution is derived as

$$\mu_r^{'} = \sum_{j=0} t_j \left(\frac{1}{\alpha(i+1)(1+\lambda)a}\right)^{r+1} \Gamma(r+1).$$

Corollary 1 *The coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the GWEx distribution are obtained as follows*

$$CV = \sqrt{\frac{\mu_2}{\mu_1} - 1},$$

$$CS = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{(\mu_2 - \mu_1)^{\frac{3}{2}}},$$

$$CK = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2}{(\mu_2 - \mu_1^2)^2}.$$

The incomplete moments are used to derive the mean deviation, Bonferroni, and Lorenz curves.

Theorem 3 *If a random variable X follows the GWEx distribution than its incomplete moments are obtained*

$$\delta_s(t) = \sum_{j=0} t_j \left(\frac{1}{\alpha(i+1)(1+\lambda)a}\right)^{r+1} \gamma(r+1, a(1+\lambda)ax). \tag{12}$$

Proof: The incomplete moment of any probability distribution is obtained as

$$\delta_s(t) = \int_0^t x^s dF(x).$$

Using (9), the above expression shapes as

$$\delta_s(t) = \int_0^t x^s \sum_{j=0}^{\infty} t_j (e^{-a(1+\lambda)\alpha x})^{j+1}.$$

After solving the above expression, the incomplete moment of GWEx distribution is given as

$$\delta_s(t) = \sum_{j=0} t_j \left(\frac{1}{\alpha(i+1)(1+\lambda)a}\right)^{r+1} \gamma(r+1, a(1+\lambda)ax), \;\; s = 1, 2, 3, \ldots.$$

Here γ is upper incomplete gamma function.

Theorem 4 *If a random variable X follows the GWEx distribution than its moment generating function is obtained*

$$M_X(t) = ab \sum_{j,i,p=0}^{\infty} t_j (-1)^i \frac{t^p \Gamma(j+1)}{i! p! \Gamma(j+1-i)} \left(\frac{1}{(1+\lambda)\alpha}\right)^p \left(\frac{1}{i+1}\right)^{p+1} \Gamma(p+1), \; p = 1, 2, 3, \ldots. \quad (13)$$

Proof: Generally, the moment generating function (mgf) of any probability distribution is explained as

$$M_X(t) = \int_0^t e^{tx} dF(x).$$

Using (9), the above expression shapes as

$$M_X(t) = \int_0^t e^{tx} \sum_{j=0}^{\infty} t_j (e^{-a(1+\lambda)\alpha x})^{j+1}.$$

Since $e^{tx} = \sum_{p=0}^{n} \frac{(tx)^p}{p!}$, the above expression converts to

$$M_X(t) = \int_0^t \sum_{p=0}^{n} \frac{(tx)^p}{p!} \sum_{j=0}^{\infty} t_j (e^{-a(1+\lambda)\alpha x})^{j+1}.$$

Solution of above expression lead us to mgf of GWEx distribution as

$$M_X(t) = ab \sum_{j,i,p=0}^{\infty} t_j (-1)^i \frac{t^p \Gamma(j+1)}{i! p! \Gamma(j+1-i)} \left(\frac{1}{(1+\lambda)\alpha}\right)^p \left(\frac{1}{i+1}\right)^{p+1} \Gamma(p+1).$$

2.3. Order Statistics

Order statistics is widely use in both reliability and life testing. In reliability, $X_{(i:n)}$ is used to model the lifetime of an $(ni+1)$-out-of-n system which consists of n independent and identically distributed components.

Theorem 5 *If a random variable X follows the GWEx distribution than the expression of the order statistics is given as*

$$f_{i,n}(x) = \frac{ab}{B(i, n-i)} (1-\lambda)\alpha \{e^{-(1+\lambda)\alpha x}\}^a \sum_{k=0}^{n-i} (-1)^k \frac{\Gamma(n-i+1)}{k!(n-i+1-k)} [1 - \{e^{-(1+\lambda)\alpha x}\}^a]^{b(i+k)-1}. \quad (14)$$

Proof: Let $X_1, X_2, \ldots, X_n$ be a simple random sample from GWEX distribution (a, b, α, λ) with distribution and density functions given in (6) and (7) respectively. Let $X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(n:n)}$ denote the order statistics obtained from this sample. Than The density function of $X_{(i:n)}$, $1 \leq k \leq n$ is given as follows:

$$f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)$$

Using (6) and (7), the density of the order statistics of GWEx distribution is extracted as

$$\begin{aligned} f_{i,n}(x) =& \frac{ab}{B(i,n-i)}(1-\lambda)\alpha\{e^{-(1+\lambda)\alpha x}\}^a \sum_{k=0}^{n-i}(-1)^k \frac{\Gamma(n-i+1)}{k!(n-i+1-k)} \\ & [1-\{e^{-(1+\lambda)\alpha x}\}^a]^{b(i+k)-1}. \end{aligned}$$

The first order statistic is given by $X_{(1)} = min(X_1, X_2, \ldots, X_n)$ and the last order statistics is given by $X_{(n)} = max(X_1, X_2, \ldots, X_n)$.

3. Rènyi Entropy

The amount of uncertainty in a random variable shows through entropy. The Rènyi entropy has broader application in the field of statistics, mathematics, computer science, and economics.

Theorem 6 *If a random variable X follows the GWEx distribution than the expression of Rènyi entropy is given as*

$$\begin{aligned} I-\rho)(x) =& \frac{1}{1-\rho}\Bigg(\rho\{log(a)+log(b)+log\alpha+log(1+\lambda)\} \\ & + \sum_{j=0}^{\infty} log\frac{(-1)^j\Gamma(\rho(b-1)+1)}{j!\Gamma(\rho(b-1)+1)}\frac{1}{a\alpha(1+\lambda)(\rho+j)}\Bigg). \end{aligned} \tag{15}$$

Proof: We can obtain Rènyi entropy as

$$I_R(\rho) = \frac{1}{1-\rho}[logI(\rho)],$$

where

$$I(\rho) = \int \left(ab\alpha(1+\lambda)e^{-a(1+\lambda)\alpha x}[1-e^{-a(1+\lambda)\alpha x}]^{b-1}\right)^{\rho} dx, \quad \rho > 0.$$

Using (7), we have

$$I(\rho) = \int (f(x))^{\rho} dx,$$

$$I(\rho) = a^{\rho}b^{\rho}\alpha^{\rho}(1+\lambda)^{\rho}\int_0^{\infty}(e^{-a(1+\lambda)\alpha x})^{\rho}[1-e^{-a(1+\lambda)\alpha x}]^{\rho(b-1)}dx.$$

Therefore, the Rè'nyi entropy of the GWEx distribution is obtained as

$$\begin{aligned} I-\rho)(x) =& \frac{1}{1-\rho}\Bigg(\rho\{log(a)+log(b)+log\alpha+log(1+\lambda)\} \\ & + \sum_{j=0}^{\infty} log\frac{(-1)^j\Gamma(\rho(b-1)+1)}{j!\Gamma(\rho(b-1)+1)}\frac{1}{a\alpha(1+\lambda)(\rho+j)}\Bigg). \end{aligned}$$

4. Estimation

Let $X_1, X_2, \ldots, X_n$ be the random samples drawn from the GWEx distribution with parameter a, b, α, and λ. Using the method of Maximum Likelihood (ML), likelihood function L is given by

$$L(a, b, \alpha, \lambda : x_i) = \prod_{i=1}^{n} f(x_i).$$

The log-likelihood l of above expression is given by

$$\begin{aligned} l =& n\log(a) + n\log(b) + n\log(\alpha) + n\log(1+\lambda) - a(1+\lambda)\alpha\sum_{i=1}^{n} x_i \\ &+ (b-1)\sum_{i=1}^{n}\log\left(1 - e^{-a(1+\lambda)\alpha x_i}\right). \end{aligned}$$

Differentiating l w.r.t all parameters ($a, b, \alpha,\ \&\ \lambda$) and equating them zero, we have

$$\frac{n}{a} - \alpha(1+\lambda)\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\frac{e^{-a(1+\lambda)\alpha x_i}\alpha x_i(1+\lambda)}{1-e^{-a(1+\lambda)\alpha x_i}} = 0 \tag{16}$$

$$\frac{n}{b} + \sum_{i=1}^{n}\log\left(1 - e^{-a(1+\lambda)\alpha x_i}\right) = 0 \tag{17}$$

$$\frac{n}{\alpha} - a(1+\lambda)\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\frac{e^{-a(1+\lambda)\alpha x_i} a x_i(1+\lambda)}{1-e^{-a(1+\lambda)\alpha x_i}} = 0 \tag{18}$$

$$\frac{n}{(1+\lambda)} - a\alpha\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\frac{e^{-a(1+\lambda)\alpha x_i} a\alpha x_i}{1-e^{-a(1+\lambda)\alpha x_i}} = 0. \tag{19}$$

The estimates are obtained by solving (16), (17), (18), and (19) numerically. The fisher information matrix can be obtained by taking the minus expectation of the second derivatives of the estimates. The variance and covariance of the maximum likelihood estimates can be obtained by taking the inverse of fisher information matrix.

$$\tfrac{1}{V} = -E\begin{pmatrix} \hat{V}_{aa} & \hat{V}_{ab} & \hat{V}_{a\alpha} & \hat{V}_{a\lambda} \\ \hat{V}_{ba} & \hat{V}_{bb} & \hat{V}_{b\alpha} & \hat{V}_{b\lambda} \\ \hat{V}_{\alpha a} & \hat{V}_{\alpha b} & \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\lambda} \\ \hat{V}_{\lambda a} & \hat{V}_{\lambda b} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda} \end{pmatrix},$$

$$\begin{aligned} &V_{aa} = \frac{\partial^2 l}{\partial a^2},\ V_{bb} = \frac{\partial^2 l}{\partial b^2},\ V_{\alpha\alpha} = \frac{\partial^2 l}{\partial \alpha^2},\ V_{\lambda\lambda} = \frac{\partial^2 l}{\partial \lambda^2},\ V_{ab} = \frac{\partial^2 l}{\partial a\partial b},\ V_{a\alpha} = \frac{\partial^2 l}{\partial a\partial \alpha}, \\ &V_{a\lambda} = \frac{\partial^2 l}{\partial a\partial \lambda},\ V_{b\alpha} = \frac{\partial^2 l}{\partial a\partial \alpha}\ V_{b\lambda} = \frac{\partial^2 l}{\partial b\partial \lambda}\ V_{\alpha\lambda} = \frac{\partial^2 l}{\partial \alpha\partial \lambda}. \end{aligned}$$

The entries of the fisher information matrix are given follows

$$\hat{V}_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n}\left(\frac{e^{-2ax_i\alpha(1+\lambda)x_i^2\alpha^2}(1+\lambda)}{(1-e^{-ax_i\alpha(1+\lambda)})^2} + \frac{e^{-ax_i\alpha(1+\lambda)x_i^2\alpha^2}(1+\lambda)}{1-e^{-ax_i\alpha(1+\lambda)}}\right),$$

$$\hat{V}_{ab} = \sum_{i=1}^{n}\frac{e^{-ax_i\alpha(1+\lambda)}x_i a(1+\lambda)}{1-e^{-ax\alpha(1+\lambda)}},$$

$$\hat{V}_{a\alpha} = -(1+\lambda)\sum_{i=1}^{n}x_i + (b-1)\sum_{i=1}^{n}\{\frac{e^{-ax_i\alpha(1+\lambda)}x_i(1+\lambda)}{1-e^{-ax_i\alpha(1+\lambda)}} - \frac{e^{-2ax\alpha(1+\lambda)}ax_i^2\alpha(1+\lambda)^2}{(1-e^{-ax_i\alpha(1+\lambda)})^2}$$
$$- \frac{e^{-ax_i\alpha(1+\lambda)}ax_i^2\alpha(1+\lambda)^2}{1-e^{-ax_i\alpha(1+\lambda)}}\},$$

$$\hat{V}_{a\lambda} = -\alpha\sum_{i=1}^{n}x_i + (b-1)\sum_{i=1}^{n}\{\frac{e^{-ax_i\alpha(1+\lambda)}x_i\alpha}{1-e^{-ax_i\alpha(1+\lambda)}} - \frac{e^{-ax_i\alpha(1+\lambda)}ax_i^2\alpha^2(1+\lambda)}{(1-e^{-ax_i\alpha(1+\lambda)})^2}$$
$$- \frac{e^{-ax_i\alpha(1+\lambda)}ax_i^2\alpha^2(1+\lambda)}{1-e^{-ax_i\alpha(1+\lambda)}}\},$$

$$\hat{V}_{bb} = -\frac{n}{b^2},$$

$$\hat{V}_{b\alpha} = \sum_{i=1}^{n}\frac{e^{-ax_i\alpha(1+\lambda)}ax_i(1+\lambda)}{1-e^{-ax_i\alpha(1+\lambda)}},$$

$$\hat{V}_{b\lambda} = \sum_{i=1}^{n}\frac{e^{-ax_i\alpha(1+\lambda)}ax_i\alpha}{1-e^{-ax_i\alpha(1+\lambda)}},$$

5. Simulation

In the part, we evaluate the performance of the MLEs of the GWEx distribution through simulation study. For this purpose, we generate 1000 samples for different sizes. The examination of estimates is done by the mean and mean square error (MSE) of the MLEs of the GWEx distribution. We use R programing for this study and obtained results are given in table 1. The results in table 1 clearly indicating that the values of the mean and MSE decreases as n increases which justifies the fact that the method of maximum likelihood is suitable for the estimation of model parameters. Further, we see that the MSE tend to be closest to the true values of the parameters. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The adjustment of bias can be used for the improvement of the normal approximation. We use the small values of the parameters because the values of MSE are higher for large values and the program could not generate suitable random data.

6. The Log-GWEx Regression Model

It is noted that lifetime models are affected by explanatory variable such as the cholesterol level, blood pressure and many others. So, it is very much important to explain the relationship between lifetime and independent variables. For this purpose, a new regression model is obtained from the log exponentiated generalized weighted Exponential (LEGWE) distribution.

Theorem 7 *If a random variable X follows the GWEx distribution than the log GWEx regression model is given as*

$$f(y) = ab\alpha(1+\lambda)exp\{-a(1+\lambda)\alpha exp\left(\frac{y-\mu}{\sigma}\right)\}\left[1-exp\{-a(1+\lambda)\alpha exp\left(\frac{y-\mu}{\sigma}\right)\}\right]^{b-1}. \quad (20)$$

Table 1 Mean and MSE for the of the MLEs of the parameters of the EGWE model

					Mean				MSE			
α	a	b	λ	n	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$
0.5	0.5	1	0.5	20	0.524	0.672	1,460	0.758	0.763	0.310	1,326	1,149
				50	0.516	0.648	1.218	0.643	0.740	0.264	1.318	1.136
				100	0.511	0.514	1.162	0.528	0.729	0.136	1.288	1.273
				500	0.508	0.501	0.093	0.501	0.711	0.122	1.147	0.969
1	0.5	0.5	1	20	1.370	0.578	0.428	1.046	0.572	0.259	0.807	0.562
				50	1.153	0.561	0.459	1.058	0.550	0.241	0.791	0.533
				100	1.142	0.554	0.467	1.007	0.519	0.227	0.667	0.529
				500	1.122	0.502	0.508	1.002	0.502	0.016	0.009	0.515
1	1	1	1	20	1.068	1.240	1.341	1.179	0.169	0.245	0.036	0.214
				50	1.044	1.171	1.292	1.136	0.148	0.122	0.548	0.208
				100	1.036	1.063	1.165	1.117	0.137	0.095	0.983	0.201
				500	1.009	1.009	1.002	1.003	0.018	0.027	0.427	0.200
1	0.5	1	3	20	1.028	1.028	1.068	3.514	0.012	0.215	0.554	0.119
				50	1.016	1.016	1.054	3.129	0.086	0.109	0.213	0.102
				100	1.008	1.008	1.032	3.095	0.004	0.045	0.109	0.097
				500	0.967	0.967	1.016	3.001	0.012	0.003	0.015	0.082
1	1	2	4.00	20	0.877	1.126	2.374	4.359	0.102	0.145	0.241	0.019
				50	0.772	1.099	2.565	4.213	0.105	0.112	0.080	0.008
				100	0.718	1.078	2.142	4.196	0.100	0.085	0.052	0.006
				500	0.656	0.996	2.046	4.005	0.010	0.055	0.026	0.005

Proof: If X has the GWEx distribution given in (7), the random variable $Y = \sigma \log(X, \gamma)$ defines the density function of Log-GWEx distribution, parameterized in term of $\gamma = exp(\mu)$. The density function of Y using transformation is given as

$$f(y) = ab\alpha(1+\lambda)exp\{-a(1+\lambda)\alpha exp\left(\frac{y-\mu}{\sigma}\right)\}\left[1 - exp\{-a(1+\lambda)\alpha exp\left(\frac{y-\mu}{\sigma}\right)\}\right]^{b-1}.$$

where $y \in \Re,\ \mu \in \Re,\ a > 0,\ b > 0, \alpha$ and $\lambda > 0$. Where μ is a location parameter, σ is a scale and $a,\ b,\ \alpha$ are shape parameters.

Corollary 2 *The corresponding survival function is given by*

$$S(y) = 1 - \left[1 - exp\{-a(1+\lambda)\alpha exp\left(\frac{y-\mu}{\sigma}\right)\}\right]^{b},$$

and the hazard rate function can be obtained using $f(y) = \frac{f(y)}{s(y)}$. *We represent the density in standard normal form using standard normal random variable* $Z = \frac{y-\mu}{\sigma}$ *as*

$$f(y) = ab\alpha(1+\lambda)exp\{-a(1+\lambda)\alpha exp(z)\}\left[1 - exp\{-a(1+\lambda)\alpha exp(z)\}\right]^{b-1}.$$

7. Application

In this section, we study the application of the GWEx distribution in real life. For this purpose, we consider two data sets. We estimate the model parameters, $LogLikelihood$ (LL), and Akaiki Information criterion (AIC). These results are compared with some well know existing model and they are listed below in Table 2.

Table 2 Fitted distributions and their abbreviations

Model	Abbrevation	Reference
Generalized Weighted Exponential	GWEx	Proposed
Nadarajah Haghighi	NH	Nadarajah and Kotz (2006),
Exponential	Ex	Nadarajah and Kotz (2006),
Momemt Exponential	MEx	Dara and Ahmad (2012),
Inverse Weibull	IW	Keller et al. (1982),
Weighted Exponential	WEx	Oguntunde et al. (2016),
Weighted Weibull	WW	Oguntunde et al. (2015),

7.1. Data I

First data set consist of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure by Andrews and Herzberg (2012).

Table 3 Estimated model parameters, LL, and AIC for Data I

Model	Parameters				LL	AIC
	a	b	α	λ		
GWEx	0.3972	1.7094	0.6882	1.5710	122.244	250.487
NH	0.1948	2.0078			124.738	253.475
Ex	0.5103				127.114	256.229
MEx	0.6248				163.101	328.203
IW	0.7321				154.278	310.556
WEx	0.3149	0.6210			127.114	258.229
WW	0.2987	0.3261	1.3256		122.525	251.049

Table 3 reports the values of model parameters, LL and AIC for first data set. It is observe from table 3 that the proposed model has lowest values of LL and AIC which clearly justifies the fitness of the GWEx distribution.

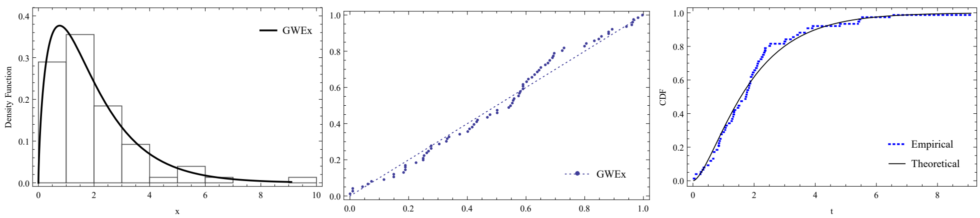

Figure 3 Fitted density, probability plot, and empirical CDF of GWEx distribution for Data I

Figure 3 demonstrates that the proposed model adequately explain the data through density plot, probability plot, and empirical CDF.

7.2. Data II

The second data set contains $n = 128$ measures on the remission times in months of bladder cancer patients Lee and Wang (2003).

Table 4 Estimated model parameters, $-2LL$, and AIC for Data set II

Model	Parameters				LL	AIC
	a	b	α	λ		
GWEx	0.0966	1.2347	0.4861	1.64579	410.75	827.508
NH	0.1233	0.9243			412.074	828.147
MEx	2.47892				457.826	959.687
IW	0.6762				478.844	959.687
WW	0.0537	0.7714	1.0512		411.892	829.785

From Table 4, it is worth seen that the proposed model is explaining the data in good manner on the lowest of LL and AIC.

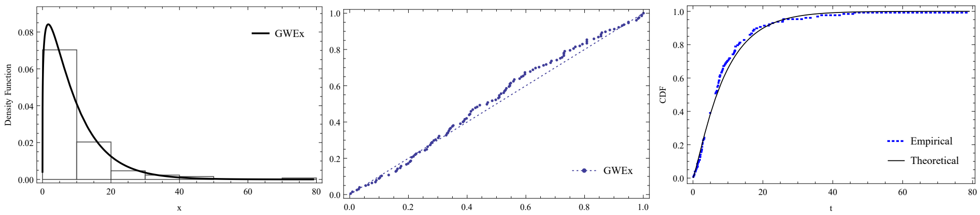

Figure 4 Fitted density, probability plot, and empirical CDF of GWEx distribution for Data II

Figure 4 exhibits that GWEx distribution is better for the fitting of considered data.

8. Conclusions

In this article, a new generalized weighted exponential distribution is developed. Several important mathematical characteristics of the new model are derived and discussed. Entropy measure is also provided. Estimation of the model parameters is done through method of maximum likelihood. The assessment of the model parameters are testified with the help of simulation study. We provide the log-GWEx regression model. The applicability of the proposed model is justified by mean of two real data sets. The discussion ends with some concluding remarks.

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