Thailand Statistician
April 2022; 20(2): 308-324
http://statassoc.or.th
Contributed paper

Beta Poisson-G Family of Distributions: Its Properties and Application with Failure Time Data

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Received: 26 November 2019
Revised: 24 May 2020
Accepted: 10 July 2020

Abstract

Poisson-G family is extended to propose beta Poisson-G family of distribution. Useful expansions of the probability density function and the cumulative distribution function of the proposed family are derived as infinite mixtures of the Poisson-G distribution. Moment generating function, power moments, entropy, quantile function, skewness and kurtosis are investigated. Illustrative numerical computation of moments, skewness, kurtosis and entropy are tabulated for select parameter values. Estimation of model parameters by methods of maximum likelihood is discussed. A simulation experiment is carried out under varying sample size to assess the performance of estimation. Finally, suitability check of the proposed model in comparison to a few recently introduced ones is carried out by considering two life time data sets modeling.

Keywords: Beta generated family, Poisson-G family, maximum likelihood, AIC.

1. Introduction

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in (\alpha, \beta)$ for $-\infty \le \alpha < \beta < \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function (cdf) of a random variable X such that $W[G(x)]$ satisfies the following conditions

(i) $W[G(x)] \in [\alpha, \beta]$

(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and (1)

(iii) $W[G(x)] \to \alpha$ as $x \to -\infty$ and $W[G(x)] \to \beta$ as $x \to \infty$.

Alzaatreh et al. (2013) defined the T-X family cdf by

$$F(x) = \int_{\alpha}^{W[G(x)]} r(t)\,dt = R\{W[G(x)]\}, \tag{2}$$

where $W[G(x)]$ satisfies the conditions (1). The pdf corresponding to (2) is given by

$$f(x)=\left\{\frac{d}{dx}W[G(x)]\right\}r\{W[G(x)]\},$$

where $G(x)$ is the cdf of any baseline distribution.

The Poisson-G family of distribution with cdf is given by (see Kumaraswamy Poisson-G family, Chakraborty et al. 2022)

$$F^{\mathrm{PG}}(x;\lambda)=\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}},\ \lambda\in R-\{0\};\ \ n=1,2,...$$

The corresponding pdf and hrf Poisson-G family is given by

$$f^{\mathrm{PG}}(x;\lambda)=\frac{\lambda\, g(x)\, e^{-\lambda G(x)}}{1-e^{-\lambda}},\ \lambda\in R-\{0\};\ -\infty<x<\infty,$$

and

$$h^{\mathrm{PG}}(x)=\frac{\lambda\, g(x)\, e^{-\lambda G(x)}}{e^{-\lambda G(x)}-e^{-\lambda}}.$$

In this paper, we propose a new class of continuous distributions called the beta generalized Poisson-G (in short $BP-G(m,n,\lambda)$) family by taking $W(G(x))=(1-e^{-\lambda})^{-1}(1-e^{-\lambda G(x)})$ and $r(t)=(B(m,n))^{-1}t^{m-1}(1-t)^{n-1}$, $0<t<1$ and $m,n,\alpha>0$. The main importance of this family lies in the fact that it reduces to $Poisson-G(\lambda)$ distribution for $m=1,n=1$ and tend to $Beta-G(m,n)$ distribution when $\lambda\to 0$.

The sf and cdf of $BP-G(m,n,\lambda)$ family of distribution are respectively given by

$$\bar{F}^{BP-G}(x;m,n,\lambda)=1-I_{\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}}(m,n),\ \ F^{BP-G}(x;m,n,\lambda)=I_{\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}}(m,n), \tag{3}$$

where $B(m,n)$ is the beta function and $I_x(a,b)=B(m,n)^{-1}\int_0^x t^{m-1}(1-t)^{n-1}\,dt$ is the incomplete beta function ratio, $G(x)$ is the baseline cdf. The corresponding pdf and hrf is given by

$$f^{BP-G}(x;m,n,\lambda)=\frac{\lambda\, g(x)\, e^{-\lambda G(x)}[1-e^{-\lambda G(x)}]^{m-1}[e^{-\lambda G(x)}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}}, \tag{4}$$

and

$$h^{BP-G}(x;m,n,\lambda)=\frac{\lambda\, g(x)\, e^{-\lambda G(x)}\left[1-e^{-\lambda G(x)}\right]^{m-1}\left[e^{-\lambda G(x)}-e^{-\lambda}\right]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}\left[1-I_{\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}}(m,n)\right]},$$

where $m,n,\lambda>0, x>0$ and $g(x)=G'(x)$ is the pdf of the baseline distribution.

Some of the notable G family of distributions derived recently includes the beta Marshall-Olkin-G family (Alizadeh et al. 2015), beta generated Kumaraswamy-G family (Handique et al. 2017), beta Kumaraswamy Marshall-Olkin-G family (Handique and Chakraborty 2017a), beta generalized Marshall-Olkin Kumaraswamy-G family (Handique and Chakraborty 2017b) Marshall-Olkin Kumaraswamy-G family (Handique et al. 2017), generalized Marshall-Olkin Kumaraswamy-G family (Chakraborty and Handique 2017), beta Marshall-Olkin Kumaraswamy-G family (Chakraborty et al. 2018), Kumaraswamy generalized Marshall-Olkin-G family (Chakraborty et al. 2018), odd moment exponential-G family (Haq et al. 2018), Zografos-Balakrishnan-Burr-XII family (Altun et al. 2018), generalized Marshall-Olkin Burr-XII family (Handique and Chakraborty 2018), zero truncated Poisson family (Abouelmagd et al. 2018), odd log-logistic Burr-X family (Rana et al. 2019), Zografos-Balakrishnan-Frechet family (Chakraborty et al. 2019), exponentiated generalized Marshall-Olkin-G family (Handique et al. 2019), exponentiated generalized extended Gompertz family (Andrade et al. 2019), odd log-logistic Burr-III family (Handique et al. 2020), generalized Marshall-Olkin Burr-III distribution (Chakraborty et al. 2020), generalized modified exponential-G family (Handique et al. 2020), new T-X family (Handique et al. 2021), Xgamma Fréchet distribution (Ibrahim et al. 2021), Poisson transmuted-G family (Handique et al. 2021), McDonald Lindley-Poisson family (Percontini et al. 2021), beta generalized Marshall-Olkin-G family (Handique et al. 2021) and odd Half-Cauchy family (Chakraborty et al. 2021) among others.

The rest of this paper is organized in five more Sections. In Section 2, a physical basis of new family is presented. In Section 3, we discuss some important mathematical and statistical characteristics of the family. Maximum likelihood method of estimation and simulation are presented in Section 4. Application of the proposed family is considered in Section 5. The paper ends with a conclusion in Section 6.

2. Physical Basis of BP-G

Theorem 1. *If m and n are both integers, then the probability distribution of $BP-G(m,n,\lambda)$ arises as distribution of the m^{th} order statistics of a random sample of size $m+n-1$ from $P-G(\lambda)$ distribution.*

Proof: Let $T_1, T_2, ..., T_{m+n-1}$ be a random sample of size $m+n-1$ from $P-G(\lambda)$ distribution with cdf, $\dfrac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}$. Then, the pdf of the m^{th} order statistics $T_{(m)}$ is given by

$$\frac{(m+n-1)!}{(m-1)![(m+n-1)-m]!}\left\{\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}\right\}^{m-1}\left\{\frac{e^{-\lambda G(x)}-e^{-\lambda}}{1-e^{-\lambda}}\right\}^{(m+n-1)-m}\frac{\lambda g(x)e^{-\lambda G(x)}}{1-e^{-\lambda}},$$

$$=\Gamma(m+n)/\{\Gamma(m)\ \Gamma(n)\}\frac{\lambda g(x)e^{-\lambda G(x)}[1-e^{-\lambda G(x)}]^{m-1}[e^{-\lambda G(x)}-e^{-\lambda}]^{n-1}}{(1-e^{-\lambda})^{m+n-1}},$$

$$=\frac{\lambda g(x)e^{-\lambda G(x)}[1-e^{-\lambda G(x)}]^{m-1}[e^{-\lambda G(x)}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}}.$$

3. General Results

In this section, we derive some general results for the proposed $BP-G(m,n,\lambda)$ family.

3.1. Expansions of pdf and cdf

By using binomial expansion from (4), we obtain

$$f^{BP-G}(x;m,n,\lambda)= f^{PG}(x;\lambda)\sum_{j=0}^{n-1}\mu_j\, F^{PG}(x;\lambda)^{j+m-1}, \tag{5}$$

$$=\sum_{j=0}^{n-1}\mu'_j\frac{d}{dx}[F^{PG}(x;\lambda)]^{j+m}, \tag{6}$$

where $\mu'_j=\dfrac{(-1)^j}{B(m,n)(j+m)}\dbinom{n-1}{j}$ and $\mu_j=\mu'_j(j+m)$. Thus, the pdf of $BP-G(m,n,\lambda)$ distribution can be seen as a mixture of $P-G(\lambda)$ pdf and cdf.

Now we can expand the cdf using "Incomplete Beta Function" from Math World-A Wolfram Web Resource (http://mathworld.Wolfram.com/Incomplete Beta Function.html).

$$B(z;a,b)=B_z(a,b)=z^a\sum_{i=0}^{\infty}\frac{(1-b)_i}{i!\,(a+i)}z^i=z^a\sum_{i=0}^{\infty}\binom{b-1}{i}\frac{(-1)^i}{a+i}z^i,$$

where $(x)_i$ is a Pochhammer symbol as

$$F^{BP-G}(x;m,n,\lambda)=\frac{1}{B(m,n)}\left(\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}\right)^m\sum_{p=0}^{\infty}\binom{n-1}{p}\frac{(-1)^p}{(m+p)}\left(\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}\right)^p,$$

$$=\sum_{p=q=0}^{\infty}\sum_{r=0}^{q}\frac{(-1)^{p+q+r}}{B(m,n)(m+p)}\binom{n-1}{p}\binom{m+p}{q}\binom{q}{r}F^{PG}(x;\lambda)^r.$$

Now in the summation exchanging the indices q and r in the sum symbol, we get

$$F^{BP-G}(x;m,n,\lambda)=\sum_{p=r=0}^{\infty}\sum_{q=r}^{\infty}\frac{(-1)^{p+q+r}}{B(m,n)(m+p)}\binom{n-1}{p}\binom{m+p}{q}\binom{q}{r}F^{PG}(x;\lambda)^r,$$

and then

$$F^{BP-G}(x;m,n,\lambda)=\sum_{r=0}^{\infty}\psi_r\,F^{PG}(x;\lambda)^r,$$

where $\psi_r=\sum_{p=0}^{\infty}\sum_{q=r}^{\infty}\frac{(-1)^{p+q+r}}{B(m,n)(m+p)}\binom{n-1}{p}\binom{m+p}{q}\binom{q}{r}.$

3.2. Probability weighted moments

The probability weighted moments (PWM), first proposed by Greenwood et al. (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{\text{th}}$ PWM of T is defined by $\Gamma_{p,q,r}=\int_{-\infty}^{\infty}x^pF(x)^q[1-F(x)]^rf(x)\,dx.$ From (5), the s^{th} moment of T can be written as

$$E(X^s)=\int_0^{\infty}x^sf^{BP-G}(x;m,n,\lambda)dx=\sum_{j=0}^{n-1}\mu_j\int_0^{\infty}x^sF^{PG}(x;\lambda)^{j+m-1}f^{PG}(x;\lambda)\,dx$$

$$=\sum_{j=0}^{n-1}\mu_j\int_0^{\infty}x^sF^{PG}(x;\lambda)^{j+m-1}f^{PG}(x;\lambda)\,dx=\sum_{j=0}^{n-1}\mu_j\,\Gamma_{s,\,j+m-1,0},$$

where $\Gamma_{p,q,r}=\int_0^{\infty}x^p\left(\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}\right)^q\left[1-\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}\right]^r\frac{\lambda\,g(x)\,e^{-\lambda G(x)}}{1-e^{-\lambda}}dx$ is the PWM of $P-G(\lambda)$ distribution. Therefore, the moments of the $BP-G(m,n,\lambda)$ distribution may be expressed in terms of the PWMs of $P-G(\lambda)$ distribution.

3.3. Moment generating function

The moment generating function of $BP-G(m,n,\lambda)$ family can be easily expressed in terms of those of the exponentiated $P-G(\lambda)$ distribution using the results of Section 3.1. For example using (6), it can be seen that

$$M_X(s)=E[e^{sX}]=\int_0^{\infty}e^{sx}f^{BP-G}(x;m,n,\lambda)\,dx=\int_0^{\infty}e^{sx}\sum_{j=0}^{n-1}\mu'_j\frac{d}{dx}[F^{PG}(x;\lambda)]^{j+m}\,dx,$$

$$= \sum_{j=0}^{n-1} \mu_j' \int_0^{\infty} e^{sx} \frac{d}{dx}[F^{PG}(x;\lambda)]^{j+m}\, dx = \sum_{j=0}^{n-1} \mu_j\, M_X(s),$$

where $M_X(s)$ is the mgf of a $P-G(\lambda)$ distribution. Mean, variance, skewness and kurtosis are computed for some values of parameters and tabulated in Table 1.

Table 1 Mean, variance, skewness and kurtosis of the $BP-E(m,n,\lambda,\beta)$ distribution with different values of m,n,λ and β

m	n	λ	β	Mean	Variance	Skewness	Kurtosis
5	1	2	2	0.1712	0.1562	3.1602	16.1856
4	1	2	2	0.2054	0.1677	2.8544	14.1588
3	1	2	2	0.2366	0.1683	2.6804	13.3436
5	2	2	2	0.0715	0.0350	3.3612	17.3340
5	3	2	2	0.0378	0.0130	3.7480	20.2250
5	4	2	2	0.0220	0.0059	4.2707	119.7388
5	4	3	2	0.0724	0.0101	1.5336	0.8385
5	4	4	2	0.0925	0.0062	0.9379	21.4520
5	4	5	2	0.0884	0.0032	0.9365	104.2939
8	4	5	1	0.2354	0.0213	0.5179	4.3436
8	4	5	2	0.4708	0.0855	0.5179	4.3436
2	2	1.5	2	0.0566	0.0216	3.8989	24.2507
2	2	3	3	0.0784	0.0087	2.4670	14.3546
2	3	5	5	0.0241	0.0004	0.1225	46.4268
2	3	5	6	0.0201	0.0002	2.9764	27.5143
3	3	5	6	0.0271	0.0004	−0.7494	49.5413
4	3	5	6	0.0330	0.0005	−2.0558	60.9605
4	4	5	6	0.0253	0.0002	−1.2278	30.0848
4	4	7.5	7.5	0.0162	0.0001	−0.0130	33.9609

3.4. Rényi entropy

The Rényi entropy is defined by $I_R(\delta) = (1-\delta)^{-1} \log\left(\int_{-\infty}^{\infty} f(x)^{\delta}\, dx \right)$, where $\delta > 0$ and $\delta \neq 1.$ Using binomial expansion in (4), we can write

$$f^{\mathrm{BPG}}(x;m,n,\lambda)^{\delta} = f^{PG}(x;\lambda)^{\delta} \sum_{i=0}^{\delta(n-1)} \zeta_i\, F^{PG}(x;\lambda)^{i+\delta(m-1)}.$$

Thus,

$$I_R(\delta) = (1-\delta)^{-1} \log\left(\int_0^{\infty} f^{\mathrm{PG}}(x;\lambda)^{\delta} \sum_{i=0}^{\delta(n-1)} \zeta_i\, [F^{\mathrm{PG}}(x;\lambda)]^{i+\delta(m-1)}\, dx \right)$$

$$= (1-\delta)^{-1} \log\left(\sum_{i=0}^{\delta(n-1)} \zeta_i \int_0^{\infty} f^{\mathrm{PG}}(x;\lambda)^{\delta} \left[F^{\mathrm{PG}}(x;\lambda)\right]^{i+\delta(m-1)} dx \right),$$

where $\zeta_i = \left(\frac{1}{B(m,n)} \right)^{\delta} \binom{\delta(n-1)}{i} (-1)^m$. Numerical computation of this measure is done for some select values of the parameters and reported in Table 2.

Table 2 Rényi entropy $BP-E(m,n,\lambda,\beta)$ distribution with different parameters values of m,n,λ and β

Parameters				δ					
m	n	λ	β	0.2	0.5	1.5	2	3	5
5	3	3	2	−0.1279	−1.2731	1.1414	0.3480	−0.1008	−0.3683
4	2	3	2	0.3367	−0.7125	0.8227	0.2245	−0.1310	−0.3546
2	2	3	2	0.2077	−0.7978	−0.1493	−0.5517	−0.8136	−0.9927
2	2	2	2	0.2567	−1.0798	1.8670	0.9004	0.3561	0.0354
2	2	1	1	0.6491	−1.9981	8.5351	5.6962	4.2193	3.4339
1.5	2	0.5	0.5	0.8716	−3.2126	14.9539	10.1996	7.7623	6.4947

3.5. Distribution of order statistic

Suppose $X_1, X_2, ..., X_\vartheta$ be a random sample from any $BP-G(m,n,\lambda)$ distribution. Let $X_{u:\vartheta}$ denote the u^{th} order statistic. The pdf of $X_{u:\vartheta}$ can be expressed as

$$f_{u:\vartheta}(x;m,n,\lambda)=\frac{\vartheta!}{(u-1)!\,(\vartheta-u)!}\sum_{j=0}^{\vartheta-u}(-1)^j\binom{\vartheta-u}{j}f^{\text{BPG}}(x)F^{\text{BPG}}(x)^{j+u-1}.$$

Now using the general expansion of the pdf and cdf of the $BP-G(m,n,\lambda)$ distribution, the pdf of the u^{th} order statistic for the $BP-G(m,n,\lambda)$ distribution is given as

$$f_{u:\vartheta}(x;m,n,\lambda)=\frac{\vartheta!}{(u-1)!\,(\vartheta-u)!}\sum_{j=0}^{\vartheta-u}(-1)^j\binom{\vartheta-u}{j}f^{PG}(x;\lambda)\sum_{l=0}^{n-1}\mu_l\,F^{PG}(x;\lambda)^{l+m-1}\times\left[\sum_{k=0}^{\infty}\psi_k\,F^{PG}(x;\lambda)^k\right]^{j+u-1},$$

where μ_l and ψ_k are defined in Section 3.1. Now,

$$\left[\sum_{k=0}^{\infty}\psi_k F^{PG}(x;\lambda)^k\right]^{j+u-1}=\sum_{k=0}^{\infty}d_{j+u-1,k}\,F^{PG}(x;\lambda)^k,$$

where $d_{j+u-1,k}=\frac{1}{k\,\mu_0}\sum_{c=1}^{k}[c\,(j+u)-k]\,\mu_c\,d_{j+u-1,\,k-c}$. (Nadarajah et al. 2015).

Therefore, the density function of the u^{th} order statistics of $BP-G(m,n,\lambda)$ distribution can be expressed as

$$\begin{aligned}
f_{u:\vartheta}(x;m,n,\lambda)&=\frac{\vartheta!}{(u-1)!\,(\vartheta-u)!}\sum_{j=0}^{\vartheta-u}(-1)^j\binom{\vartheta-u}{j}f^{PG}(x;\lambda)\sum_{l=0}^{n-1}\mu_l F^{PG}(x;\lambda)^{l+m-1}\times\sum_{k=0}^{\infty}d_{j+u-1,k}\,F^{PG}(x;\lambda)^k\\
&=f^{PG}(x;\lambda)\sum_{l=0}^{n-1}\sum_{k=0}^{\infty}\gamma_{l,k}F^{PG}(x;\lambda)^{k+l+m-1}\\
&=f^{PG}(x;\lambda)\sum_{l=0}^{n-1}\sum_{k=0}^{\infty}\gamma_{l,k}\sum_{z=0}^{k+l}\binom{k+l}{z}(-1)^z\left[\bar{F}^{PG}(x;\lambda)\right]^z\\
&=f^{PG}(x;\lambda)\sum_{z=0}^{k+l}\chi_z\left[\bar{F}^{\text{KwG}}(t;a,b,\xi)\right]^z\\
&=-\sum_{z=0}^{k+l}\frac{\chi_z}{z+1}\frac{d}{dx}\left[\bar{F}^{PG}(x;\lambda)\right]^{z+1}\\
&=\sum_{z=0}^{k+l}\chi'_z f^{PG}(x;\lambda(z+1)),
\end{aligned}$$

where

$$\chi'_z = \sum_{l=0}^{n-1}\sum_{k=0}^{\infty} \frac{(-1)^{z+1}\gamma_{l,k}}{z+1}\binom{k+l}{z}, \; \chi_z = \chi'_z\,(z+1), \text{ and}$$

$$\gamma_{l,k} = \frac{\vartheta!}{(u-1)!(\vartheta-u)!}\sum_{j=0}^{\vartheta-u}(-1)^j\binom{\vartheta-u}{j}\mu_l d_{j+u-1,k}.$$

3.6. Quantile function and related results

The quantile function X, let $x = Q(u) = F^{-1}(u)$, can be obtained by inverting (3). Let $z = Q_{m,n}(u)$ be the beta quantile function. Then,

$$x = Q(u) = Q_G\left[-\frac{1}{\lambda}\log[1 - Q_{m,n}(u)(1-e^{-\lambda})]\right].$$

For example, let the G be exponential distribution with parameter $\beta > 0$, having pdf and cdf as $g(x;\beta) = \beta e^{-\beta x}$, $x > 0$ and $G(x;\beta) = 1 - e^{-\beta x}$ respectively. Then the p^{th} quantile is obtained as $-(1/\beta)\log(1-p)$. Therefore, the p^{th} quantile x_p, of $BP-E$ distribution is given by

$$x_p = -\frac{1}{\beta}\log\left[1 + \frac{1}{\lambda}\log\left(1 - Q_{m,n}(u)(1-e^{-\lambda})\right)\right].$$

It is possible to obtain an expansion for $Q_{m,n}(u)$ as $z = Q_{m,n}(u) = \sum_{i=0}^{\infty} e_i\, u^{i/m}$ (see "Power Series" from MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/PowerSeries.html), where

$$e_i = [m\,B(m,n)]^{1/m}\, d_i \text{ and } d_0 = 0,\, d_1 = 1,\, d_2 = (n-1)/(m+1),$$

$$d_3 = \{(n-1)(m^2+3mn-m+5n-4)\}/\{2(m+1)^2(m+2)\},$$

$$d_4 = \frac{(n-1)\left[m^4+(6n-1)m^3+(n+2)(8n-5)m^2+(33n^2-30n+4)m+n(31n-47)+18\right]}{[3(m+1)^3(m+2)(m+3)]}\ldots$$

3.7. Plots of the skewness and kurtosis

Here the flexibility of skewness and kurtosis of $\text{BP}-\text{G}(m,n,\lambda)$ distribution is checked by plotting Galton skewness (S) that measures the degree of the long tail and Morrs (1988) kurtosis (K) that measures the degree of tail heaviness. These are respectively defined by

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \text{ and } K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

3.8. Plots of the pdf and hrf

In this section, we have plotted the pdf and hrf of the $BP-G(m,n,\lambda)$ distribution taking G to be Weibull (W) and exponential (E) distributions for some chosen values of the parameters to show the variety of shapes assumed by the family. The pdf and hrf of these distributions are obtained from $BP-G(m,n,\lambda)$ distribution as follows

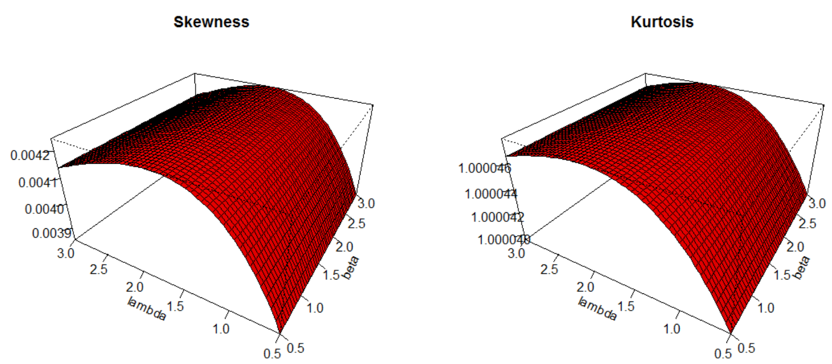

Figure 1 Plots of the Galton skewness *S* and Moor kurtosis *K* for the BP-E distribution with parameters $m=2,\ n=3,\ \lambda\in(0.5,3)$ and $\beta\in(0.5,3)$

The $BP-Weibull$ ($BP-W$) distribution

Considering the Weibull distribution (Weibull 1951) with parameters $\beta>0$ and $\delta>0$ having pdf and cdf, $g(x)=\delta\beta x^{\delta-1}e^{-\beta x^{\delta}}$ and $G(x)=1-e^{-\beta x^{\delta}}$, respectively we get the pdf and hrf of $BP\text{-}W\,(m,n,\lambda,\beta,\delta)$ distribution as

$$f^{BP-W}(x)=\frac{\lambda\delta\beta x^{\delta-1}\,e^{-\beta x^{\delta}}\;e^{-\lambda(1-e^{-\beta x^{\delta}})}[1-e^{-\lambda(1-e^{-\beta x^{\delta}})}]^{m-1}[e^{-\lambda(1-e^{-\beta x^{\delta}})}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}},$$

$$h^{BP-W}(x)=\frac{\lambda\delta\beta x^{\delta-1}\,e^{-\beta x^{\delta}}\;e^{-\lambda(1-e^{-\beta x^{\delta}})}[1-e^{-\lambda(1-e^{-\beta x^{\delta}})}]^{m-1}[e^{-\lambda(1-e^{-\beta x^{\delta}})}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}\left[1-I_{\frac{1-e^{-\lambda(1-e^{-\beta x^{\delta}})}}{1-e^{-\lambda}}}(m,n)\right]}.$$

Taking $\delta=1$ in $BP-W(m,n,\lambda,\beta,\delta)$ distribution, we get the $BP-E(m,n,\lambda,\beta)$ distribution with pdf and hrf is given by

$$f^{BP-E}(x)=\frac{\lambda\,\beta\,e^{-\beta x}\,e^{-\lambda(1-e^{-\beta x})}[1-e^{-\lambda(1-e^{-\beta x})}]^{m-1}[e^{-\lambda(1-e^{-\beta x})}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}},$$

$$h^{BP-E}(x)=\frac{\lambda\,\beta\,e^{-\beta x}\,e^{-\lambda(1-e^{-\beta x})}[1-e^{-\lambda(1-e^{-\beta x})}]^{m-1}[e^{-\lambda(1-e^{-\beta x})}-e^{-\lambda}]^{n-1}}{B(m,n)(1-e^{-\lambda})^{m+n-1}\left[1-I_{\frac{1-e^{-\lambda(1-e^{-\beta x})}}{1-e^{-\lambda}}}(m,n)\right]}.$$

From the plots in Figure 2, it can be seen that the family is very flexible and can offer many different types of shapes increasing and decreasing failure rate.

4. Maximum Likelihood Estimation

Let $x=(x_1,x_2,...,x_w)^X$ be a random sample of size w from $BP-E(m,n,\lambda,\beta)$ distribution with parameter vector $\boldsymbol{\rho}=(m,n,\ \lambda,\boldsymbol{\beta}^T)^T$, where $\boldsymbol{\beta}=(\beta_1,\beta_2,...,\beta_q)^T$ corresponds to the parameter vector of the baseline distribution *G*. Then the log-likelihood function for $\boldsymbol{\rho}$ is given by

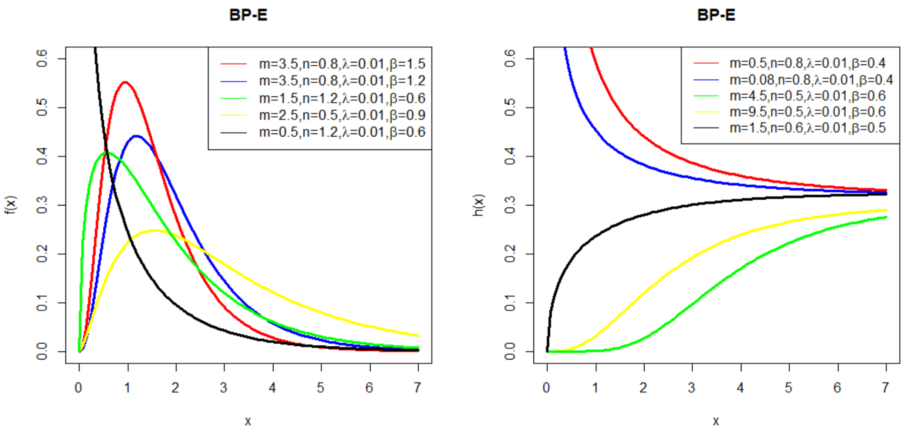

Figure 2 Density (left) and hazard (right) plots of BP-E distribution

$$\ell = \ell(\boldsymbol{\rho}) = r\log(\lambda) + \sum_{i=1}^{w}\log[g(x_i,\boldsymbol{\beta})] - \lambda\sum_{i=1}^{w}G(x_i,\boldsymbol{\beta}) + (m-1)\sum_{i=1}^{w}\log\left(1-e^{-\lambda G(x_i,\boldsymbol{\beta})}\right)$$
$$+(n-1)\sum_{i=1}^{w}\log(e^{-\lambda G(x_i,\boldsymbol{\beta})} - e^{-\lambda}) - r\log[B(m,n)] - (m+n-1)\sum_{i=1}^{w}\log(1-e^{-\lambda}).$$

The maximum likelihood estimations (MLE) are obtained by maximizing the log-likelihood function numerically by using available function from R. The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\boldsymbol{\rho})$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The ij^{th} elements of $\mathrm{I}_n(\boldsymbol{\rho})$ are given by

$$\mathrm{I}_{ij} = -E[\partial^2 l(\boldsymbol{\rho})/\partial\rho_i\partial\rho_j], \quad i,j = 1,2,\ldots,3+q.$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\boldsymbol{\rho})$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\boldsymbol{\rho}}) = (\hat{\mathrm{I}}_{ij})$ defined as

$$\hat{\mathrm{I}}_{ij} \approx \left(-\partial^2 l(\boldsymbol{\rho})/\partial\rho_i\,\partial\rho_j\right)_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}}, \quad i,j = 1,2,\ldots,3+q.$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$ is $N_k(0,V_n)$ where $V_n = (v_{jj}) = \mathrm{I}_n^{-1}(\boldsymbol{\rho})$. The asymptotic behavior remains valid if V_n is replaced by $\hat{V}_n = \hat{\mathrm{I}}^{-1}(\hat{\boldsymbol{\rho}})$. Using this result large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$.

4.1. Simulation

In this section, simulation experiments are conducted to evaluate performance of the MLE for the $BP-E(m,n,\lambda,\beta)$ distribution with respect to their mean square errors (MSE) for sample sizes 20, 50, 100, 200 and 300. The experiment is repeated 3,000 times, where in each replication, the MLEs of the parameters is computed. The average and the MSEs of the estimators is reported in Table 3. From Table 3 its obvious that the estimates are quite stable and more importantly, are close to the true values. More over it is observed that in general the MSEs decreases as n increases.The simulation study therefore shows that the maximum likelihood method is appropriate for estimating the BP-E parameters.

Table 3 The average MLEs and MSEs of $BP-E(m,n,\lambda,\beta)$ distribution

Sample size (w)	True Parameter values	MLE	MSE	True Parameter values	MLE	MSE
	$m=2.2$	2.0015	0.1013	$m=2.0$	2.5118	0.0532
	$n=2.8$	2.1198	0.1935	$n=1.8$	1.4983	0.1839
20	$\lambda=0.5$	0.4889	0.1190	$\lambda=1.5$	1.4449	0.0045
	$\beta=2.0$	1.0028	1.7128	$\beta=2.0$	2.6849	1.1939
	$m=2.2$	2.0450	0.0598	$m=2.0$	2.0395	0.0354
50	$n=2.8$	2.2570	0.1216	$n=1.8$	1.5368	0.1119
	$\lambda=0.5$	0.4986	0.0713	$\lambda=1.5$	1.5068	0.0024
	$\beta=2.0$	1.0801	1.5307	$\beta=2.0$	2.5507	1.0447
	$m=2.2$	2.0915	0.0418	$m=2.0$	2.0160	0.0195
100	$n=2.8$	2.4303	0.1078	$n=1.8$	1.6170	0.0865
	$\lambda=0.5$	0.5088	0.0405	$\lambda=1.5$	1.5023	0.0018
	$\beta=2.0$	1.7711	1.0606	$\beta=2.0$	2.1509	1.0043
	$m=2.2$	2.1903	0.0247	$m=2.0$	2.0057	0.099
200	$n=2.8$	2.7899	0.0902	$n=1.8$	1.8086	0.0511
	$\lambda=0.5$	0.5019	0.0287	$\lambda=1.5$	1.5010	0.0007
	$\beta=2.0$	1.9017	0.8983	$\beta=2.0$	2.0246	0.9005
	$m=2.2$	2.2103	0.0147	$m=2.0$	1.9998	0.0040
300	$n=2.8$	2.8019	0.0512	$n=1.8$	1.8000	0.0301
	$\lambda=0.5$	0.5002	0.0120	$\lambda=1.5$	1.5000	0.0004
	$\beta=2.0$	2.0017	0.4903	$\beta=2.0$	2.0046	0.1015

Additionally, we graphical display of the simulation results in Figures 3 and 4. For this we generated for samples of size w = 10 to 100 from $BP-E(m,n,\lambda,\beta)$ distribution with true parameters values $m=2.4, n=2.5, \lambda=1.5$ and $\beta=1.4$, and calculated the bias and mean square error (MSE) of the MLEs by repeating the experiment 3,000 times. We displayed the curves up to $w=60$ as after that the values stabilized. From Figures 3 and 4, we observe that when the sample sizes increases, the empirical biases and MSEs approach to zero in all cases justifying the asymptotic normal distribution as an adequate approximation to the finite sample distribution of the MLEs.

5. Real Life Applications

Here we consider fitting of two failure time data sets to show that the distributions from the proposed $BP-E(m,n,\lambda,\beta)$ family can provide better model than the corresponding distributions exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E), generalized Marshall-Olkin exponential (GMO-E), Kumaraswamy exponential (Kw-E), Beta exponential (BE), Marshall-Olkin Kumaraswamy exponential (MOKw-E) and Kumaraswamy Marshall-Olkin exponential (KwMO-E) distributions.

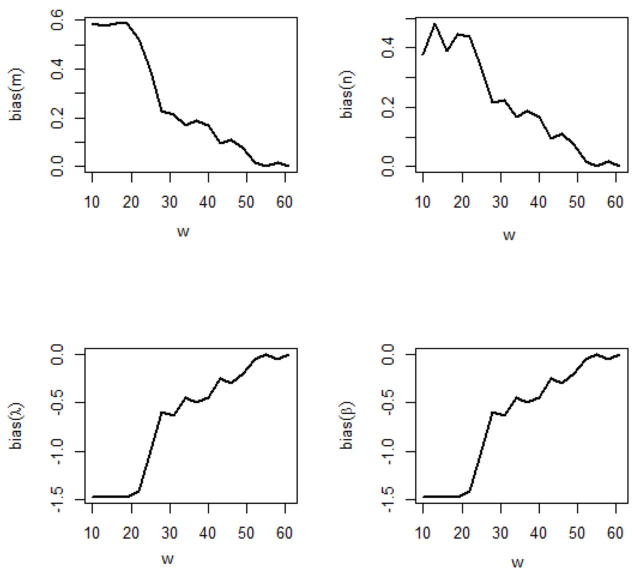

Figure 3 The biases for the parameter values $m = 2.4, n = 2.5, \lambda = 1.5$ and $\beta = 1.4$

Data I: This data is about survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960).

Data II: The data represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data was reported by Gross and Clark (1975) and it has 20 observations.

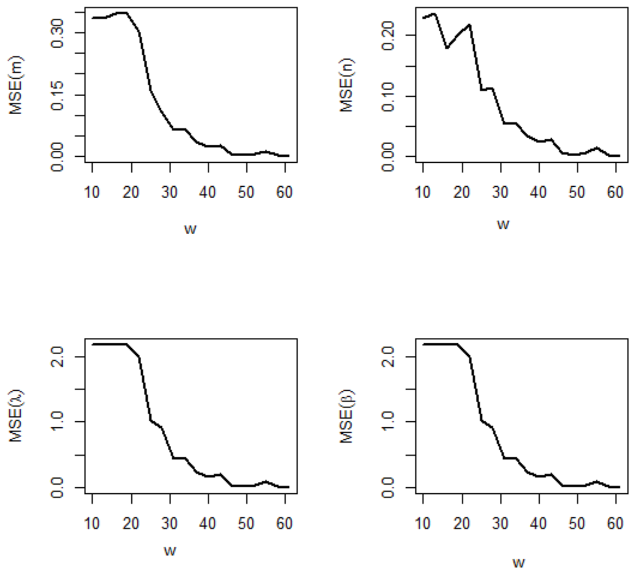

Figure 4 The MSEs the parameter values $m = 2.4, n = 2.5, \lambda = 1.5$ and $\beta = 1.4$

TTT plots and Descriptive Statistics for the Data Sets

The TTT plots (see Aarset 1987) for the data sets Figure 5 indicate that the both data sets have increasing hazard rate. Descriptive statistics for the data sets I and II are reported in Table 4.

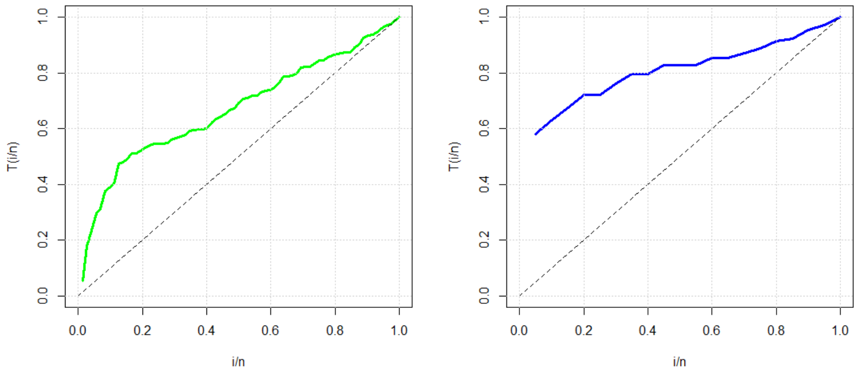

Figure 5 TTT-plots for the data set I (left) and data set II (right)

We have considered model selection criteria namely the AIC, BIC, CAIC and HQIC and the Kolmogorov-Smirnov (K-S) statistics, Anderson-Darling (A) and Cramér-Von Mises (W) for goodness of fit to compare the fitted models. We have also provided the asymptotic standard errors and confidence intervals of the MLEs of the parameters for each competing models. Comparison fitted densities and the fitted cdf's with observed ones are presented in the form of a histograms and Ogives of the data in Figures 6 and 7.

Table 4 Descriptive statistics for the data sets I and II

Data Sets	n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1^{st} Qu.	3^{rd} Qu.	Max.
I	72	0.100	1.851	1.560	1.200	1.788	4.157	1.080	2.303	7.000
II	20	1.100	1.900	1.700	0.704	1.592	2.346	1.475	2.050	4.100

In the Tables 5, 6, 7 and 8 the MLEs with standard errors of the parameters for all the fitted models along with their AIC, BIC, CAIC, HQIC, A, W and KS statistic with p-value from the fitting results of the data sets I and II are presented respectively.

From the findings presented in the Tables 6 and 8 on the basis of the lowest value different criteria like AIC, BIC, CAIC, HQIC, A, W and highest p-value of the KS statistics the $\mathrm{BP-E}$ is found to be a better model than its recently introduced model Exp, ME, MO-E, GMO-E, Kw-E, B-E, MOKw-E and KwMO-E for all the data sets considered here. A visual comparison of the closeness of the fitted densities with the observed histogram and fitted cdf's with the observed ogive of the data sets I and II are presented in the Figures 6 and 7 respectively. These plots also indicate that the proposed distributions provide comparatively closer fit to these data sets.

Table 5 MLEs, standard errors, confidence intervals (in parentheses) values for the guinea pigs survival time's data set I

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{m}$	$\hat{n}$	$\hat{\beta}$
Exp	-	-	-	-	0.540
(β)					(0.063)
					(0.42, 0.66)
ME	-	-	-	-	0.925
(β)					(0.077)
					(0.62, 1.08)
MO-E	-	8.778	-	-	1.379
(α,β)		(3.555)			(0.193)
		(1.81, 15.74)			(1.00,1.75)
GMO-E	0.179	47.635	-	-	4.465
(λ,α,β)	(0.070)	(44.901)			(1.327)
	(0.04, 0.32)	(0, 135.64)			(1.86. 7.07)
Kw-E	-	-	3.304	1.100	1.037
(m,n,β)			(1.106)	(0.764)	(0.614)
			(1.13,5.47)	(0, 2.59)	(0, 2.24)
B-E	-	-	0.807	3.461	1.331
(m,n,β)			(0.696)	(1.003)	(0.855)
			(0, 2.17)	(1.49, 5.42)	(0, 3.01)
MOKw-E	-	0.008	2.716	1.986	0.099
(α,m,n,β)		(0.002)	(1.316)	(0.784)	(0.048)
		(0.004, 0.01)	(0.14, 5.29)	(0.449, 3.52)	(0, 0.19)
KwMO-E	-	0.373	3.478	3.306	0.299
(α,m,n,β)		(0.136)	(0.861)	(0.779)	(1.112)
		(0.11, 0.64)	(1.79, 5.17)	(1.78, 4.83)	(0, 2.48)
BP-E	0.014	-	3.595	0.724	1.482
(λ,m,n,β)	(0.010)		(1.031)	(1.590)	(0.516)
	(0, 0.03)		(1.57, 5.62)	(0, 3.84)	(0.47, 2.49)

Table 6 Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the guinea pigs survival times data set I

Models	AIC	BIC	CALC	HQIC	A	W	KS (p-value)
Exp	234.63	236.91	234.68	235.54	6.53	1.25	0.27
(β)							(0.06)
ME	210.40	212.68	210.45	211.30	1.52	0.25	0.14
(β)							(0.13)
MO-E	210.36	214.92	210.53	212.16	1.18	0.17	0.10
(α,β)							(0.43)
GMO-E	210.54	217.38	210.89	213.24	1.02	0.16	0.09
(λ,α,β)							(0.51)
Kw-E	209.42	216.24	209.77	212.12	0.74	0.11	0.08
(m,n,β)							(0.50)

Table 6 (continued)

Models	AIC	BIC	CALC	HQIC	A	W	KS (p-value)
B-E (m,n,β)	207.38	214.22	207.73	210.08	0.98	0.15	0.11 (0.34)
MOKw-E (α,m,n,β)	209.44	218.56	210.04	213.04	0.79	0.12	0.10 (0.44)
KwMo-E (α,m,n,β)	207.82	216.94	208.42	211.42	0.61	0.11	0.08 (0.73)
BP-E (λ,m,n,β)	205.42	214.50	206.02	209.02	0.55	0.08	0.09 (0.81)

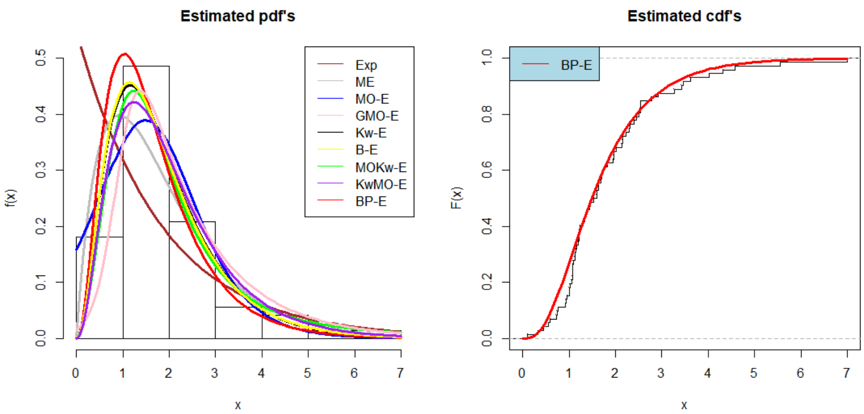

Figure 6 Plots of the observed histogram and estimated pdf's on left for the Exp, ME, MO-E, GMO-E, Kw-E, B-E, MOKw-E, KwMO-E, BP-E models and observed ogive and estimated cdf on right for the BP-E data set I

Table 7 MLEs, standard errors, confidence intervals (in parentheses) values for the relief times of patients receiving an analgesic failure time data set II

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{m}$	$\hat{n}$	$\hat{\beta}$
Exp (β)	-	-	-	-	0.526 (0.117) (0.29, 0.75)
ME (β)	-	-	-	-	0.950 (0.150) (0.66, 1.24)
MO-E (α,β)	-	54.474 (35.582) (0, 124.21)	-	-	2.316 (0.374) (1.58,3.04)
GMO-E (λ,α,β)	0.519 (0.256) (0.02, 1.02)	89.462 (66.278) (0, 219.37)	-	-	3.169 (0.772) (1.66, 4.68)
Kw-E (m,n,β)	-	-	83.756 (42.361) (0.73, 166.78)	0.568 (0.326) (0, 1.21)	3.330 (1.188) (1.00, 5.66)

Table 7 (continued)

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{m}$	$\hat{n}$	$\hat{\beta}$
B-E (m,n,β)	-	-	81.633 (120.41) (0, 317.63)	0.542 (0.327) (0, 1.18)	3.514 (1.410) (0.75, 6.28)
MOKw-E (α,m,n,β)	-	0.133 (0.332) (0, 0.78)	33.232 (57.837) (0, 146.59)	0.571 (0.721) (0, 1.98)	1.669 (1.814) (0, 5.22)
KwMO-E (α,m,n,β)	-	28.868 (9.146) (10.94, 46.79)	34.826 (22.312) (0, 78.56)	0.299 (0.239) (0, 0.76)	4.899 (3.176) (0, 11.12)
BP-E (λ,m,n,β)	1.965 (0.341) (1.29, 2.63)	-	13.396 (1.494) (10.46, 6.32)	9.600 (1.091) (7.46, 11.73)	0.244 (0.037) (0.17, 0.32)

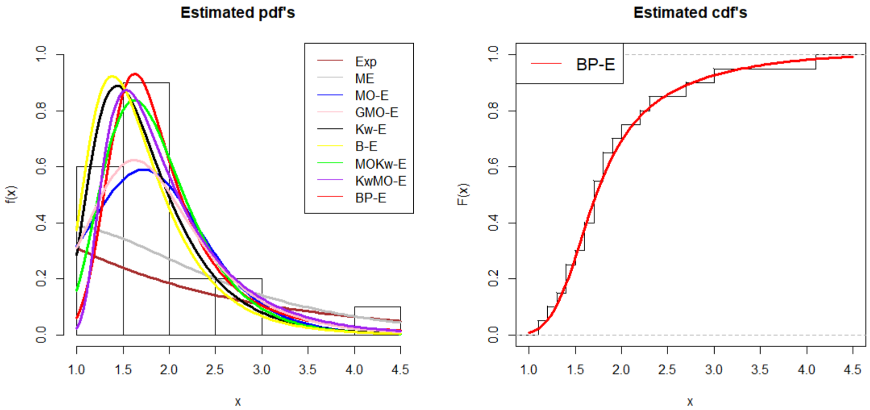

Figure 7 Plots of the observed histogram and estimated pdf's on left and observed ogive and estimated cdf's for the Exp, ME, MO-E, GMO-E, Kw-E, B-E, MOKw-E, KwMO-E and BP-E models for data set II

Table 8 Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the relief times of patients receiving an analgesic failure time data set II

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
Exp (β)	67.67	68.67	67.89	67.87	4.60	0.96	0.44 (0.004)
ME (β)	54.32	55.31	54.54	54.50	2.76	0.53	0.32 (0.07)
MO-E (α,β)	43.51	45.51	44.22	43.90	0.81	0.14	0.18 (0.55)
GMO-E (λ,α,β)	42.75	45.74	44.25	43.34	0.51	0.08	0.15 (0.78)
Kw-E (m,n,β)	41.78	44.75	43.28	42.32	0.45	0.07	0.14 (0.86)

Table 8 (continued)

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
B-E (m,n,β)	43.48	46.45	44.98	44.02	0.70	0.12	0.16 (0.80)
MOKw-E (α,m,n,β)	41.58	45.54	44.25	42.30	0.60	0.11	0.14 (0.87)
KwMO-E (α,m,n,β)	42.88	46.84	45.55	43.60	1.08	0.19	0.15 (0.86)
BP-E (λ,m,n,β)	38.07	42.02	40.73	38.78	0.39	0.06	0.14 (0.91)

6. Conclusions

A new extension of the Poisson-G family of distributions is introduced and some of its important properties are studied. The MLE for estimating the parameters is discussed. Two applications of life time data fitting shows good result in favour of the distributions from the proposed family. Thus we expect that the proposed family will be another meaningful contribution to the existing list of continuous distributions.

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