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Transmuted Quasi Akash Distribution Applicable to Survival Times Data

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Abstract

We formulated transmuted quasi Akash distribution by using quadratic rank transmutation map technique. Various necessary statistical characteristics of transmuted quasi Akash distribution are studied. The reliability measures of proposed model are also derived and model parameters are estimated by maximum likelihood estimation method. Simulation study is carried out to test the performance of maximum likelihood estimates and to compare the proposed model with base model. The significance of transmuted parameter has been tested. Finally proposed model and its related models are fitted to two real life data sets to examine the significance of newly introduced model.

Keywords: Quadratic rank transmutation map technique, transmuted parameter, structural properties, hazard rate, simulation, maximum likelihood estimation.

1. Introduction

Researchers over the years have fitted various probability models to the lifetime data of animals, human beings, plants, insects and birds. Modeling and analysis of lifetime data is important as it helps administration and people to take precautionary measures and increase life span. There are many occasions where existing models do not provide appropriate fit to life time data. Some situations while dealing with life time data may demand need of applying generalized models. One of the techniques for adding extra parameter to the existing models for capturing more variation and for bringing extra flexibility in applying probability models to life time data is quadratic rank transmutation map (QRTM) technique. For analyzing the complex data analysts make use of QRTM technique. Shaw and Buckley (2007) formulated QRTM technique for generalization of classical probability models. Mudasir and Ahmad (2015) studied structural properties of length biased Nakagami distribution. Aryal and Tsokos (2009) introduced transmuted generalized extreme value distribution and obtained its various vital properties. Merovci (2013) obtained transmuted Rayleigh distribution and studied its necessary properties. Para and Jan (2018) introduced transmuted inverse log-logistic distribution and obtained its various characteristic properties. Elbatal and Elgarhy (2013) introduced transmuted quasi Lindley distribution as a generalization of quasi Lindley distribution and obtained its important properties. Haq (2016) studied transmuted exponentiated inverse Rayleigh distribution and obtained its various properties. R Core Team (2019) developed R software version 3.5.3 which we have used for analyzing data in this paper. Here, we have incorporated an extra parameter known as transmuted parameter to two parameter quasi Akash distribution which is useful life time model introduced Rama Shankar (2016).

A continuous random variable X is said to follow quasi Akash distribution (QAD) if its probability density function is of the form

$$g_b\left(x,\alpha,\theta\right)=\frac{\theta^2}{(\alpha\theta+2)}(\alpha+\theta x^2)e^{-\theta x},\ \ \alpha>0,x>0,\theta>0. \tag{1}$$

and is denoted by $X\sim QAD(\alpha,\theta)$. The cumulative distribution function of quasi Akash distribution is given by

$$G_b\left(x,\alpha,\theta\right)=1-\left[1+\frac{\theta x(\theta x+2)}{(\alpha\theta+2)}\right]e^{-\theta x}. \tag{2}$$

By using quadratic rank transmutation map technique. The c.d.f. of transmuted model $F_\tau(x)$ is of the below form

$$F_\tau(x)=(1+\lambda)G_b(x)-\lambda\left(G_b(x)\right)^2,\ \ -1\le\lambda\le 1.$$

Which on differentiation yields the p.d.f. $f_\tau(x)$ of transmuted model as

$$f_\tau(x)=g_b(x)\left(1+\lambda-2\lambda G_b(x)\right),$$

where $g_b(x)$ and $G_b(x)$ are p.d.f. and c.d.f. of base model, respectively.

The necessity and objective of this paper is to introduce the size biased version of Lindley-quasi Xgamma distribution for modelling of unequally recorded observations. Proposed model is developed for increasing flexibility in respect of skewness, kurtosis, etc., and for better fitting of complex data than base model and related models. The proposed model is described in Section 2 of paper. In Section 3, need of proposed model is discussed. Reliability analysis of proposed transmuted quasi Akash distribution is introduced in Section 4. Statistical properties of proposed model are obtained in Section 5. Expressions for order statistics are obtained in Section 6. In Section 7, Bonferroni and Lorenz curves and indices are discussed. Section 8 deals with estimation of unknown parameters of proposed model. Simulation analysis is provided in Section 9 and model comparison on simulated data is provided in Section 10. Real life applications are presented in Section 11. Finally, conclusion is presented in Section 12.

2. Transmuted Quasi Akash Distribution

A non-negative random variable X is said to follow a transmuted quasi Akash distribution (TQAD) if its cumulative distribution function $F_\tau(x,\alpha,\theta,\lambda)$ is obtained as

$$F_\tau\left(x,\alpha,\theta,\lambda\right)=(1+\lambda)G_b(x,\alpha,\theta)-\lambda\left(G_b(x,\alpha,\theta)\right)^2,-1\le\lambda\le 1,$$

$$F_\tau(x,\alpha,\theta,\lambda)=G_b(x,\alpha,\theta)\left(1+\lambda\left(1-G_b(x,\alpha,\theta)\right)\right). \tag{3}$$

Putting the value of $G_b(x,\alpha,\theta)$ from Equation (2) in Equation (3) we get

$$F_\tau\left(x,\alpha,\theta,\lambda\right)=\left(\frac{1}{\left(\alpha\theta+2\right)^2}\left(\begin{array}{c}\left(\left(\alpha\theta+2\right)-\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)\\ \left(\left(\alpha\theta+2\right)+\lambda\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)\end{array}\right)\right), \tag{4}$$

where $G_b(x,\alpha,\theta)$ is the c.d.f. of base distribution (quasi Akash distribution). For $\lambda = 0$ in (4), we get the base distribution.

The corresponding probability density function of transmuted quasi Akash distribution $f_\tau(x,\alpha,\theta,\lambda)$ is obtained by differentiating (3) as

$$f_\tau(x,\alpha,\theta,\lambda) = g_b(x,\alpha,\theta)\left(1+\lambda-2\lambda G_b(x,\alpha,\theta)\right). \tag{5}$$

Putting the value of $g_b(x,\alpha,\theta)$ from (1) and $G_b\left(x,\alpha,\theta\right)$ from (2) in Equation (5)

$$f_\tau\left(x,\alpha,\theta,\lambda\right) = \frac{\theta^2}{(\alpha\theta+2)}(\alpha+\theta x^2)e^{-\theta x}\left(1+\lambda-2\lambda\left(1-\left[1+\frac{\theta x(\theta x+2)}{(\alpha\theta+2)}\right]e^{-\theta x}\right)\right)$$

$$f_\tau\left(x,\alpha,\theta,\lambda\right) = \frac{\theta^2}{\left(\alpha\theta+2\right)^2}\left(\alpha+\theta x^2\right)e^{-\theta x}\left(\left(1-\lambda\right)\left(\alpha\theta+2\right)+2\lambda\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right) \tag{6}$$

which is the p.d.f. of transmuted quasi Akash distribution, where $g_b(x,\alpha,\theta)$ and $G_b(x,\alpha,\theta)$ are p.d.f. and c.d.f. of base model, respectively. The transmuted quasi Akash distribution is positively skewed as is clear from the graph of its p.d.f. given below:

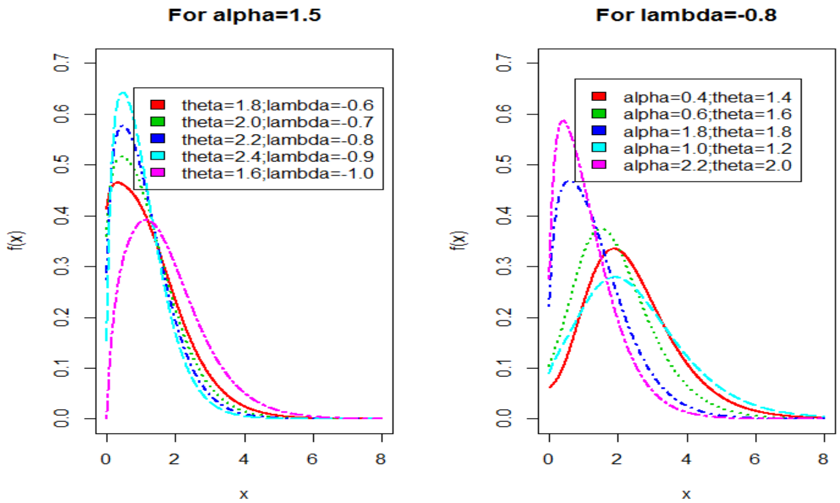

Figure 1 Graphs of probability density

3. Motivation and Need of Proposed Model

Since in the basic model there are two parameters and we added transmuted parameter to basic model to observe how well the transmuted model fits to the survival times data, to check the flexibility of the transmuted model, what role transmuted parameter plays, how much variation in data will be covered by three parameter model. From the graph of p.d.f. along with hazard rate graphs we observed that proposed model is more flexible than base model as skewness is lesser in proposed model than base model, also approximation to normal distribution as sample size increases is more in proposed model than base as is clear from p.d.f. graphs, proposed model possesses increasing, decreasing as well as constant hazard rate which is common phenomenon in most real life situations. Also with the addition of transmuted parameter flexibility of model increases. From the simulation part and application part of paper, we observed significance of transmuted parameter along with goodness of fit of proposed model than basic model to survival times data.

4. Reliability Analysis

We explored survival function, hazard rate and reverse hazard rate of the proposed transmuted quasi Akash distribution in this segment of paper.

4.1. Reliability function R(x)

The reliability function or survival function $R_\tau(x,\alpha,\theta,\lambda)$ is the probability that a system survives beyond a specified time t.

Mathematically,

$$R_\tau(x,\alpha,\theta,\lambda)=P(X>t).$$

It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of transmuted quasi Akash distribution is calculated as

$$R_\tau\left(x,\alpha,\theta,\lambda\right)=1-\left(\frac{1}{(\alpha\theta+2)^2}\left(\begin{array}{l}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\end{array}\right)\right).$$

4.2. Hazard function

The hazard function of TQAD is given as

$$H.R=h_\tau\left(x,\alpha,\theta,\lambda\right)=\frac{f_\tau\left(x,\alpha,\theta,\lambda\right)}{R_\tau\left(x,\alpha,\theta,\lambda\right)}$$

$$=\frac{\theta^2(\alpha+\theta x^2)e^{-\theta x}\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)}{(\alpha\theta+2)^2-\left(\begin{array}{l}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\end{array}\right)}.$$

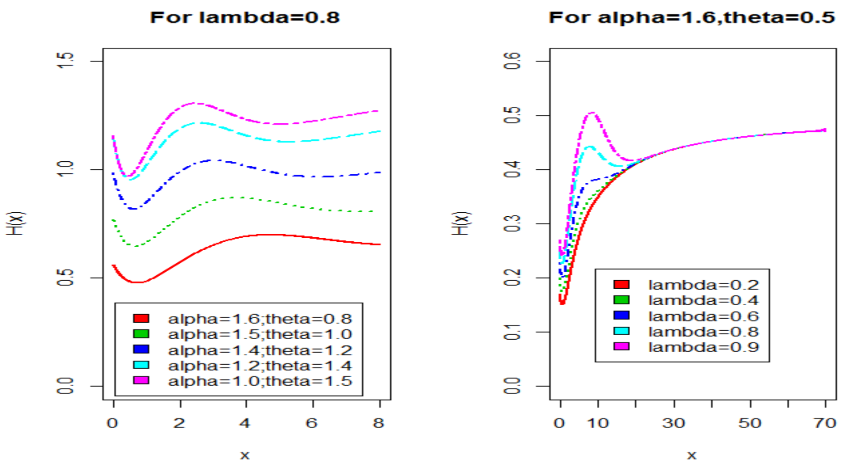

Figure 2 Graphs of hazard function

From graphs of hazard rate of proposed model, it can be observed that hazard rate is increasing, decreasing as well as showing constant behavior for different parameter values and different sample sizes. Such hazard rates are observed in many real life situations.

4.3. Reverse hazard rate

The reverse hazard rate of the transmuted quasi Akash distribution is given as

$$R.H.R.=\frac{f_{\tau}\left(x,\alpha,\theta,\lambda\right)}{F_{\tau}\left(x,\alpha,\theta,\lambda\right)}=\frac{\theta^{2}\left(\alpha+\theta x^{2}\right)e^{-\theta x}\left(\left(1-\lambda\right)\left(\alpha\theta+2\right)+2\lambda\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)}{\begin{pmatrix}\left(\left(\alpha\theta+2\right)-\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)\\ \left(\left(\alpha\theta+2\right)+\lambda\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)\end{pmatrix}}.$$

5. Statistical Properties

Here we have explored the different structural and statistical properties of the proposed transmuted quasi Akash model. These include moments, coefficient of variation, index of dispersion, kurtosis, skewness, moment generating function, characteristic function and mean deviation.

5.1. Moments

Suppose X is a random variable following transmuted quasi Akash distribution with parameters $\left(\alpha,\theta,\lambda\right)$. Then, the r^{th} moment about origin for a given probability distribution is given by

$$\mu_r'=E(X^r)=\int_0^{\infty}x^r f_{\tau}\left(x,\alpha,\theta,\lambda\right)dx$$

$$=\int_0^{\infty}x^r\frac{\theta^2}{\left(\alpha\theta+2\right)^2}(\alpha+\theta x^2)e^{-\theta x}\left(\left(1-\lambda\right)\left(\alpha\theta+2\right)+2\lambda\left(\left(\alpha\theta+2\right)+\theta x\left(\theta x+2\right)\right)e^{-\theta x}\right)dx$$

$$\mu_r'=\frac{r!}{\theta^r\left(\alpha\theta+2\right)^2}\begin{pmatrix}\left(\alpha\theta+2\right)\left((1-\lambda)\left(\alpha\theta+(r+1)(r+2)\right)+\frac{\lambda}{2^{r+2}}\left(4\alpha\theta+(r+2)(r+1)\right)\right)+\\ \frac{\lambda(r+1)}{2^{r+2}}\left(\frac{(r+2)\left(4\alpha\theta+(r+3)(r+4)\right)}{4}+4\alpha\theta+(r+3)(r+2)\right)\end{pmatrix}. \quad (7)$$

Put $r=1$ in (7) we get

$$\mu_1'=\frac{1}{4\theta(\alpha\theta+2)^2}\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right),$$

which is mean of the transmuted quasi Akash distribution. Put $r=2,3,4$ in equation (7) we get

$$\mu_2'=\frac{1}{4\theta^2(\alpha\theta+2)^2}\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)$$

$$\mu_3'=\frac{3}{16\theta^3(\alpha\theta+2)^2}\left(4(\alpha\theta+2)(8\alpha\theta+160-7\alpha\theta\lambda-155\lambda)+\lambda(36\alpha\theta+330)\right)$$

$$\mu_4'=\frac{3}{8\theta^4(\alpha\theta+2)^2}\left((\alpha\theta+2)(64\alpha\theta+1920-60\alpha\theta\lambda-1890\lambda)+5\lambda(10\alpha\theta+126)\right).$$

The moments about mean are obtained as below

$$\mu_2=\left(\frac{1}{\theta^2\left(4(\alpha\theta+2)^2\right)^2}\begin{pmatrix}4(\alpha\theta+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda\left(4\alpha\theta+25\right)\right)-\\ \left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^2\end{pmatrix}\right)$$

which is the variance of transmuted quasi Akash distribution.

$$\mu_3=\left(\frac{1}{32\theta^3(\alpha\theta+2)^6}\begin{pmatrix}6(\alpha\theta+2)^4\left(4(\alpha\theta+2)(8\alpha\theta+160-7\alpha\theta\lambda-155\lambda)+\lambda(36\alpha\theta+330)\right)\\-6(\alpha\theta+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)\\\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)+\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^3\end{pmatrix}\right)$$

$$\mu_4=\frac{3}{256\theta^4(\alpha\theta+2)^8}\begin{bmatrix}32(\alpha\theta+2)^6\{(\theta\alpha+2)(64\alpha\theta+1920-60\alpha\theta\lambda-1890\lambda)+5\lambda(10\alpha\theta+126)\}\\-16(\alpha\theta+2)^4\{4(\theta\alpha+2)(8\alpha\theta+160-7\alpha\theta\lambda-155\lambda)+\lambda(36\alpha\theta+330)\}\\\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}\\+8(\alpha\theta+2)^2\{2(\theta\alpha+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\}\\\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}^2\\-\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}^4\end{bmatrix}.$$

5.2. Coefficient of variation, skewness, kurtosis and index of dispersion

The coefficient of variation (C.V.), coefficient of skewness $\sqrt{\beta_1}$, coefficient of kurtosis β_2, and index of dispersion γ of the TQAD are determined as

$$C.V.=\frac{(\mu_2)^{0.5}}{\mu_1'}=\frac{\begin{pmatrix}4(\alpha\theta+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)-\\\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^2\end{pmatrix}^{0.5}}{\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{2\begin{bmatrix}6(\alpha\theta+2)^4\{4(\theta\alpha+2)(8\alpha\theta+160-7\alpha\theta\lambda-155\lambda)+\lambda(36\alpha\theta+330)\}\\-6(\alpha\theta+2)^2\{2(\theta\alpha+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\}\\\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}\\+\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}^3\end{bmatrix}}{\begin{Bmatrix}4(\theta\alpha+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)\\-\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^2\end{Bmatrix}^{3/2}}.$$

As can be observed from graph of p.d.f., skewness increases for decrease in value of parameters

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3\begin{bmatrix}32(\alpha\theta+2)^6\{(\theta\alpha+2)(64\alpha\theta+1920-60\alpha\theta\lambda-1890\lambda)+5\lambda(10\alpha\theta+126)\}\\-16(\alpha\theta+2)^4\{4(\theta\alpha+2)(8\alpha\theta+160-7\alpha\theta\lambda-155\lambda)+\lambda(36\alpha\theta+330)\}\\\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}\\+8(\alpha\theta+2)^2\{2(\theta\alpha+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\}\\\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}^2\\-\{(\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\}^4\end{bmatrix}}{\begin{Bmatrix}4(\theta\alpha+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)\\-\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^2\end{Bmatrix}^{2}}.$$

Also from graph of p.d.f., it can been observed that proposed model is platykurtic as well as leptokurtic for different values of parameter,

$$\gamma=\frac{\mu_2}{\mu_1'}=\frac{\begin{pmatrix}4(\alpha\theta+2)^2\left(2(\alpha\theta+2)(4\alpha\theta+48-3\alpha\theta\lambda-45\lambda)+3\lambda(4\alpha\theta+25)\right)-\\ \left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)^2\end{pmatrix}}{4(\alpha\theta+2)^2\left((\alpha\theta+2)(4\alpha\theta+24-2\alpha\theta\lambda-21\lambda)+\lambda(7\alpha\theta+27)\right)}.$$

If $\gamma>1,$ the model is over dispersed and if $\gamma<1,$ the model is under dispersed.

Table1 Mean, variance and index of dispersion for different parameter values

Properties	$\theta=0.5,\alpha=1.5,$ $\lambda=0.4$	$\theta=1.0,\alpha=0.1,$ $\lambda=0.8$	$\theta=1.7,\alpha=0.3,$ $\lambda=0.8$	$\theta=1.7,\alpha=0.5,$ $\lambda=-0.8$
Mean	4.125620	2.141950	1.064414	1.873626
Variance	10.76769	1.763070	0.6414804	1.193107
Index of Dispersion	2.609957	0.8231145	0.6026603	0.636790

It can be seen from Table 1 that proposed model is overdespersed as well as under dispersed for different parameter values. Hence proposed model finds greater applicability in real life.

5.3. Mean deviation about mean and median of transmuted quasi Akash distribution (TQAD)

We have derived the expressions for mean deviation about mean and median of TQAD in this section.

Theorem 1 *If X follows TQAD(α,θ,λ), , then the mean deviation about mean $\delta_1(X)$ and mean deviation about median $\delta_2(X)$ are given as*

$$\delta_1(X)=\begin{pmatrix}\left(\dfrac{2}{(\alpha\theta+2)^2}\begin{pmatrix}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta\mu(\theta\mu+2)\right)e^{-\theta\mu}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta\mu(\theta\mu+2)\right)e^{-\theta\mu}\right)\end{pmatrix}\right)\\ -2\left(\dfrac{\theta^2}{\alpha\theta+2}\left(V(1-\lambda)+2\lambda\left(U+\dfrac{\theta}{\theta\alpha+2}(A+B)\right)\right)\right)\end{pmatrix}$$

and

$$\delta_2(X)=\mu-2\left(\frac{\theta^2}{\alpha\theta+2}\left(V'(1-\lambda)+2\lambda\left(U'+\frac{\theta}{\theta\alpha+2}(A'+B')\right)\right)\right),$$

respectively.

Proof: Mean deviation about mean and mean deviation about median are defined as

$$\delta_1(X)=\int_0^\infty|x-\mu|f_\tau(x,\alpha,\theta,\lambda)dx$$

and

$$\delta_2(X)=\int_0^\infty|x-M|f_\tau(x,\alpha,\theta,\lambda)dx,$$

respectively, where μ and M are mean and median respectively of random variable $X \sim TQAD$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be obtained by using the simplified relationships.

$$\delta_1(X) = \int_0^{\mu} (\mu - x) f_\tau(x,\alpha,\theta,\lambda)\,dx + \int_{\mu}^{\infty} (x-\mu) f_\tau(x,\alpha,\theta,\lambda)\,dx$$

$$\delta_1(X) = 2\mu F_\tau(\mu) - 2\int_0^{\mu} x f_\tau(x,\alpha,\theta,\lambda)\,dx \tag{8}$$

and

$$\delta_2(X) = \mu - 2\int_0^{M} x f_\tau(x,\alpha,\theta,\lambda)\,dx, \tag{9}$$

where $f_\tau(x,\alpha,\theta,\lambda) = \dfrac{\theta^2}{(\alpha\theta+2)^2}(\alpha+\theta x^2)e^{-\theta x}\left((1-\lambda)(\alpha\theta+2) + 2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)$.

Now,

$$\int_0^{\mu} x f_\tau(x,\alpha,\theta,\lambda)\,dx = \left(\frac{\theta^2}{\alpha\theta+2}\left(V(1-\lambda) + 2\lambda\left(U + \frac{\theta}{\theta\alpha+2}(A+B)\right)\right)\right), \tag{10}$$

where

$$V = \frac{1}{\theta^3}\left(\alpha\theta + 6 - e^{-\theta\mu}\left(\alpha\theta + 6 + \alpha\mu\theta^2 + 6\theta\mu + 3\mu^2\theta^2 + \mu^3\theta^3\right)\right),$$

$$U = \frac{1}{16\theta^3}\left(4\alpha\theta + 6 - e^{-2\theta\mu}\left(4\alpha\theta + 6 + 8\alpha\mu\theta^2 + 12\theta\mu + 12\mu^2\theta^2 + 8\mu^3\theta^3\right)\right),$$

$$A = \frac{1}{64\theta^4}\left(24\alpha\theta + 120 - e^{-2\theta\mu}\left(\begin{array}{c} 24\alpha\theta + 120 + 48\mu^2\theta^3\alpha + 32\mu^3\theta^4\alpha + 32\mu^5\theta^5 + \\ 48\alpha\mu\theta^2 + 240\theta\mu + 240\mu^2\theta^2 + 160\mu^3\theta^3 + 80\mu^4\theta^4 \end{array}\right)\right),$$

$$B = \frac{1}{16\theta^4}\left(8\alpha\theta + 24 - e^{-2\theta\mu}\left(\begin{array}{c} 8\alpha\theta + 24 + 16\mu^2\theta^3\alpha + 16\alpha\mu\theta^2 + \\ 48\theta\mu + 48\mu^2\theta^2 + 32\mu^3\theta^3 + 16\mu^4\theta^4 \end{array}\right)\right),$$

and

$$\int_0^{M} x f_\tau(x,\alpha,\theta,\lambda)\,dx = \left(\frac{\theta^2}{\alpha\theta+2}\left(V'(1-\lambda) + 2\lambda\left(U' + \frac{\theta}{\theta\alpha+2}(A'+B')\right)\right)\right), \tag{11}$$

where

$$V' = \frac{1}{\theta^3}\left(\alpha\theta + 6 - e^{-\theta M}\left(\alpha\theta + 6 + \alpha M\theta^2 + 6\theta M + 3M^2\theta^2 + M^3\theta^3\right)\right),$$

$$U' = \frac{1}{16\theta^3}\left(4\alpha\theta + 6 - e^{-2\theta M}\left(4\alpha\theta + 6 + 8\alpha M\theta^2 + 12\theta M + 12M^2\theta^2 + 8M^3\theta^3\right)\right),$$

$$A' = \frac{1}{64\theta^4}\left(24\alpha\theta + 120 - e^{-2\theta M}\left(\begin{array}{c} 24\alpha\theta + 120 + 48M^2\theta^3\alpha + 32M^3\theta^4\alpha + 32M^5\theta^5 \\ +48\alpha M\theta^2 + 240\theta M + 240M^2\theta^2 + 160M^3\theta^3 + 80M^4\theta^4 \end{array}\right)\right),$$

$$B' = \frac{1}{16\theta^4}\left(8\alpha\theta + 24 - e^{-2\theta M}\left(\begin{array}{c} 8\alpha\theta + 24 + 16M^2\theta^3\alpha + 16\alpha M\theta^2 \\ +48\theta M + 48M\theta^2 + 32M^3\theta^3 + 16M^4\theta^4 \end{array}\right)\right).$$

Using expressions (8), (9), (10) and (11) and expression for c.d.f. (6) we obtain mean deviation about mean $\delta_1(X)$ and mean deviation about median $\delta_2(X)$,

$$\delta_1(X)=\left(\begin{array}{l}\left(\dfrac{2}{(\alpha\theta+2)^2}\left(\begin{array}{l}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta\mu(\theta\mu+2)\right)e^{-\theta\mu}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta\mu(\theta\mu+2)\right)e^{-\theta\mu}\right)\end{array}\right)\right)\\ -2\left(\dfrac{\theta^2}{\alpha\theta+2}\left(V(1-\lambda)+2\lambda\left(U+\dfrac{\theta}{\theta\alpha+2}(A+B)\right)\right)\right)\end{array}\right)$$

and

$$\delta_2(X)=\mu-2\left(\frac{\theta^2}{\alpha\theta+2}\left(V'(1-\lambda)+2\lambda\left(U'+\frac{\theta}{\theta\alpha+2}(A'+B')\right)\right)\right).$$

5.4. Moment generating function and characteristic function of transmuted quasi Akash distribution (TQAD)

We will derive moment generating function and characteristic function of TQAD in this section.

Theorem 2 *If X has the TQAD, then the moment generating function* $M_{X(t)}$ *and characteristic generating function* $\phi_{X(t)}$ *are*

$$M_{X(t)}=\left(\frac{\theta^2}{(\theta\alpha+2)^2}\left(\begin{array}{c}\dfrac{(1-\lambda)(\alpha\theta+2)}{(\theta-t)^3}\left(\alpha(\theta-t)^2+2\theta\right)+\dfrac{2\lambda}{(2\theta-t)^5}\\ \left(\begin{array}{l}(\alpha\theta+2)(2\theta-t)^2\left(\alpha(2\theta-t)^2+2\theta\right)+2\theta^2(\alpha\\ (2\theta-t)^2+12\theta)+2\theta(2\theta-t)(\alpha(2\theta-t)^2+6\theta\end{array}\right)\end{array}\right)\right)$$

and

$$\phi_{X(t)}=\left(\frac{\theta^2}{(\theta\alpha+2)^2}\left(\begin{array}{c}\dfrac{(1-\lambda)(\alpha\theta+2)}{(\theta-it)^3}\left(\alpha(\theta-it)^2+2\theta\right)+\dfrac{2\lambda}{(2\theta-it)^5}\\ \left(\begin{array}{l}(\alpha\theta+2)(2\theta-it)^2\left(\alpha(2\theta-it)^2+2\theta\right)+2\theta^2(\alpha\\ (2\theta-it)^2+12\theta)+2\theta(2\theta-it)(\alpha(2\theta-it)^2+6\theta\end{array}\right)\end{array}\right)\right),$$

respectively. And hence show that quasi Akash distribution is a particular case of transmuted quasi Akash distribution.

Proof: We begin with the well-known definition of the moment generating function given by

$$M_X(t)=E(e^{tx})=\int_0^\infty e^{tx}f_\tau(x,\alpha,\theta,\lambda)\,dx$$

$$=\int_0^\infty e^{tx}\frac{\theta^2}{(\alpha\theta+2)^2}(\alpha+\theta x^2)e^{-\theta x}\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)dx,$$

$$M_{X(t)} = \left(\frac{\theta^2}{(\theta\alpha+2)^2} \left(\begin{array}{c} \frac{(1-\lambda)(\alpha\theta+2)}{(\theta-t)^3}\left(\alpha(\theta-t)^2+2\theta\right)+\frac{2\lambda}{(2\theta-t)^5} \\ \left(\begin{array}{c} (\alpha\theta+2)(2\theta-t)^2\left(\alpha(2\theta-t)^2+2\theta\right)+2\theta^2(\alpha \\ (2\theta-t)^2+12\theta)+2\theta(2\theta-t)(\alpha(2\theta-t)^2+6\theta \end{array} \right) \end{array} \right) \right), \qquad (12)$$

which is the m.g.f. of transmuted quasi Akash distribution. For $\lambda = 0$ in (12). We get

$$M_{X(t)} = \left(\frac{\theta^2}{(\theta\alpha+2)} \left(\frac{\alpha}{(\theta-t)} + \frac{2\theta}{(\theta-t)^3} \right) \right)$$

which is m.g.f. of quasi Akash distribution with parameters α and θ. Also we know that $\phi_{X(t)} = M_{X(it)}$. Therefore,

$$\phi_{X(t)} = \left(\frac{\theta^2}{(\theta\alpha+2)^2} \left(\begin{array}{c} \frac{(1-\lambda)(\alpha\theta+2)}{(\theta-it)^3}\left(\alpha(\theta-it)^2+2\theta\right)+\frac{2\lambda}{(2\theta-it)^5} \\ \left(\begin{array}{c} (\alpha\theta+2)(2\theta-it)^2\left(\alpha(2\theta-it)^2+2\theta\right)+2\theta^2(\alpha \\ (2\theta-it)^2+12\theta)+2\theta(2\theta-it)(\alpha(2\theta-it)^2+6\theta \end{array} \right) \end{array} \right) \right),$$

which is the characteristic function of transmuted quasi Akash distribution.

6. Order Statistics of Transmuted Quasi Akash Distribution

Assuming $X_{(1)}, X_{(2)}, ..., X_{(n)}$ to be the ordered statistics of the random sample $x_1, x_2, ..., x_n$ obtained from the transmuted quasi Akash distribution with cumulative distribution function $F_\tau(x,\alpha,\theta,\lambda)$ and probability density function $f_\tau(x,\alpha,\theta,\lambda)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{(r)}(x,\alpha,\theta,\lambda) = \frac{n!}{(r-1)!(n-r)!} f_\tau(x,\alpha,\theta,\lambda)\left(F_\tau(x,\alpha,\theta,\lambda)\right)^{r-1}\left(1-F_\tau(x,\alpha,\theta,\lambda)\right)^{n-r}, \; r = 1,2,\ldots,n.$$

Using (4) and (6), the probability density function of r^{th} order statistics of transmuted quasi Akash distribution is given by

$$f_{(r)}(x,\alpha,\theta,\lambda) = \left[\begin{array}{c} \frac{n!}{(r-1)!(n-r)!} \frac{\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2\right)e^{-\theta x} \left[\begin{array}{c} (1-\lambda)(\theta\alpha+2)+ \\ 2\lambda\{(\theta\alpha+2)+\theta x(\theta x+2)\}e^{-\theta x} \end{array} \right] \\ \left[\frac{1}{(\theta\alpha+2)^2} \left\{ \begin{array}{c} \left[(\theta\alpha+2)-\{(\theta\alpha+2)+\theta x(\theta x+2)\}e^{-\theta x}\right] \\ \left[(\theta\alpha+2)+\lambda\{(\theta\alpha+2)+\theta x(\theta x+2)\}e^{-\theta x}\right] \end{array} \right\} \right]^{r-1} \\ \left[1-\frac{1}{(\theta\alpha+2)^2} \left\{ \begin{array}{c} \left[(\theta\alpha+2)-\{(\theta\alpha+2)+\theta x(\theta x+2)\}e^{-\theta x}\right] \\ \left[(\theta\alpha+2)+\lambda\{(\theta\alpha+2)+\theta x(\theta x+2)\}e^{-\theta x}\right] \end{array} \right\} \right]^{n-r}. \end{array} \right]$$

Then, the p.d.f. of first order statistic $X_{(1)}$ of transmuted quasi Akash distribution is given by

$$f_{(1)}(x,\alpha,\theta,\lambda)=\left\{\begin{array}{l}\dfrac{n\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2\right)e^{-\theta x}\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left(1-\left(\dfrac{1}{(\alpha\theta+2)^2}\left(\begin{array}{l}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\end{array}\right)\right)\right)^{n-1}\end{array}\right\}$$

and the pdf of n^{th} order statistic $X_{(n)}$ of transmuted quasi Akash model is given as

$$f_{(n)}(x,\alpha,\theta,\lambda)=\left\{\begin{array}{l}\dfrac{n\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2\right)e^{-\theta x}\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left(\dfrac{1}{(\alpha\theta+2)^2}\left(\begin{array}{l}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\end{array}\right)\right)^{n-1}\end{array}\right\}.$$

7. Bonferroni and Lorenz Curves and Indices of TQAD

The Bonferroni curve $B(p)$, Lorenz curve $L(p)$, Bonferroni index B and Gini index G have find applicability in fields of economics, demography, reliability, life testing and medical sciences. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q xf_\tau(x,\alpha,\theta,\lambda)\,dx, \tag{13}$$

$$L(p)=\frac{1}{\mu}\int_0^q xf_\tau(x,\alpha,\theta,\lambda)\,dx, \tag{14}$$

where $\mu=E(X)$ is the mean of TQAD and $q=F^{-1}(p)$. The Bonferroni and Gini indices are defined as

$$B=1-\int_0^1 B(p)dp, \tag{15}$$

$$G=1-2\int_0^1 L(p)dp. \tag{16}$$

Using the p.d.f. (6) of TQAD, we get

$$\int_0^q xf_\tau(x,\alpha,\theta,\lambda)\,dx=\left(\frac{\theta^2}{\alpha\theta+2}\left(V''(1-\lambda)+2\lambda\left(U''+\frac{\theta}{\theta\alpha+2}(A''+B'')\right)\right)\right), \tag{17}$$

where

$$V''=\frac{1}{\theta^3}\left(\alpha\theta+6-e^{-\theta q}\left(\alpha\theta+6+\alpha q\theta^2+6\theta q+3q^2\theta^2+q^3\theta^3\right)\right),$$

$$U''=\frac{1}{16\theta^3}\left(4\alpha\theta+6-e^{-2\theta q}\left(4\alpha\theta+6+8\alpha q\theta^2+12\theta q+12q^2\theta^2+8q^3\theta^3\right)\right),$$

$$A''=\frac{1}{64\theta^4}\left(24\alpha\theta+120-e^{-2\theta q}\left(\begin{array}{l}24\alpha\theta+120+48q^2\theta^3\alpha+32q^3\theta^4\alpha+32q^5\theta^5+\\48\alpha q\theta^2+240\theta q+240q^2\theta^2+160q^3\theta^3+80q^4\theta^4\end{array}\right)\right),$$

$$B'' = \frac{1}{16\theta^4}\left(8\alpha\theta + 24 - e^{-2\theta q}\begin{pmatrix} 8\alpha\theta + 24 + 16q^2\theta^3\alpha + 16\alpha q\theta^2 + \\ 48\theta q + 48q^2\theta^2 + 32q^3\theta^3 + 16q^4\theta^4 \end{pmatrix}\right).$$

Using (17) in (13) and (14), we get

$$B(p) = \frac{1}{p\mu}\left(\frac{\theta^2}{\alpha\theta+2}\left(V''(1-\lambda) + 2\lambda\left(U'' + \frac{\theta}{\theta\alpha+2}(A''+B'')\right)\right)\right), \tag{18}$$

and

$$L(p) = \frac{1}{\mu}\left(\frac{\theta^2}{\alpha\theta+2}\left(V''(1-\lambda) + 2\lambda\left(U'' + \frac{\theta}{\theta\alpha+2}(A''+B'')\right)\right)\right). \tag{19}$$

Using (18) and (19) in (15) and (16), we get

$$B = 1 - \frac{1}{\mu}\left(\frac{\theta^2}{\alpha\theta+2}\left(V''(1-\lambda) + 2\lambda\left(U'' + \frac{\theta}{\theta\alpha+2}(A''+B'')\right)\right)\right),$$

$$L = 1 - \frac{2}{\mu}\left(\frac{\theta^2}{\alpha\theta+2}\left(V''(1-\lambda) + 2\lambda\left(U'' + \frac{\theta}{\theta\alpha+2}(A''+B'')\right)\right)\right).$$

8. Estimation of Parameters of Transmuted Quasi Akash Distribution

For estimating the parameters of transmuted quasi Akash distribution method of maximum likelihood estimation is used. Assuming $x_1, x_2, ..., x_n$ to be the random sample of size n drawn from transmuted quasi Akash distribution having density function given by (6), then the likelihood function of transmuted quasi Akash distribution is given as

$$L(x \mid \alpha,\theta,\lambda) = \prod_{i=1}^{n}\left(\frac{\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2_i\right)e^{-\theta x_i}\left((1-\lambda)(\alpha\theta+2) + 2\lambda\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)\right).$$

The log likelihood function becomes

$$\log L = \left(2n\log\theta - 2n\log(\alpha\theta+2) - \theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log\left(\left(\alpha+\theta x^2_i\right)\begin{pmatrix}(1-\lambda)(\alpha\theta+2) \\ +2\lambda\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\end{pmatrix}\right)\right). \tag{20}$$

Differentiating the log-likelihood function (20) partially with respect to α, θ and λ equating the result to zero, we obtain the following normal equations,

$$\frac{\partial \log L}{\partial \theta} = \left[\begin{array}{l} \dfrac{2n}{\theta} - \dfrac{2n\alpha}{(\alpha\theta+2)} \\ -\displaystyle\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\left[\dfrac{\begin{array}{c} x_i^2\left\{(1-\lambda)(\theta\alpha+2) + 2\lambda((\theta\alpha+2)+\theta x_i(\theta x_i+2))e^{-\theta x_i}\right\} \\ +(\alpha+\theta x_i^2)\left\{\begin{array}{l}\alpha(1-\lambda) + 2\lambda(e^{-\theta x_i}(\alpha+2\theta x_i^2+2x_i) \\ -x_i e^{-\theta x_i}((\alpha\theta+2)+\theta x_i(\theta x_i+2)))\end{array}\right\}\end{array}}{\left\{(\alpha+\theta x_i^2)\left((1-\lambda)(\theta\alpha+2) + 2\lambda((\alpha\theta+2)+\theta x_i(\theta x_i+2))e^{-\theta x}\right)\right\}}\right]\end{array}\right] = 0 \tag{21}$$

$$\frac{\partial \log L}{\partial \alpha}=\left(-\frac{2n\theta}{(\alpha\theta+2)}+\sum_{i=1}^{n}\frac{\left\{\begin{array}{c}\left(\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)\right)+\\ \left(\alpha+\theta x_i^2\right)\left((1-\lambda)\theta+2\lambda\theta e^{-\theta x_i}\right)\end{array}\right\}}{\left(\left(\alpha+\theta x_i^2\right)\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)\right)}\right)=0 \tag{22}$$

$$\frac{\partial \log L}{\partial \lambda}=\left(\sum_{i=1}^{n}\frac{\left\{\left(\alpha+\theta x_i^2\right)\left(-(\alpha\theta+2)+2\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)\right\}}{\left(\left(\alpha+\theta x_i^2\right)\left((1-\lambda)(\alpha\theta+2)+2\lambda\left((\alpha\theta+2)+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)\right)}\right)=0. \tag{23}$$

MLEs of α,θ and λ cannot be obtained by solving above complex equations (21), (22), (23) as these equations are not in closed form. So, we solve above equations by using iteration method through R software.

9. Simulation Study

We analyzed the performance of estimates obtained through maximum likelihood method by carrying out simulation study for the proposed model TQAD. We generated sample of size (30, 80 and 180) through inverse c.d.f. technique and repeated the process 200 times.

In the inverse c.d.f. method, the random numbers from a particular distribution are generated by solving the equation obtained on equating the CDF of a distribution to a number u. The number u is itself being generated from $U(0,1)$. Thus following the same procedure for the generation of random numbers from the TQAD, we proceeded as

$$F_\tau(x,\alpha,\theta,\lambda)=u,$$

$$\left(\frac{1}{(\alpha\theta+2)^2}\left(\begin{array}{c}\left((\alpha\theta+2)-\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\\ \left((\alpha\theta+2)+\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\right)\end{array}\right)\right)=u.$$

The above complex equation is solved by using Mathematica software. We computed variance ad mean square error of maximum likelihood estimates and observed that decreasing trend is observed in variance as well as mean square error as the sample size increases. So maximum likelihood estimates perform well for estimation of parameters of proposed model TQAD.

Table 2 Simulation study of TQAD

Parameter	Sample size (n)	$\theta=3.7,\alpha=4.8,\lambda=0.2$		$\theta=1.7,\alpha=2.8,\lambda=-0.2$	
		Variance	MSE	Variance	MSE
α	30	0.04966788	0.04994605	0.04459	0.125899
θ		0.2709754	23.66466	0.481149	28.12672
λ		0.1087913	0.139824	0.165115	0.166973
α	80	0.01220147	0.01280251	0.0256704	0.1200356
θ		0.09777657	23.55018	0.1456518	27.75126
λ		0.04689785	0.07644949	0.032095	0.0322734
α	180	0.002039544	0.00471921	0.002194598	0.119048
θ		0.07250198	23.28892	0.08312316	27.74681
λ		0.01812964	0.05424853	0.01551866	0.0156404

10. Model Comparison Based on Simulated Data

For comparing proposed transmuted quasi Akash distribution with the base model on the basis of simulated data we use inverse c.d.f. technique discussed in Section 9 to generate one data set by taking a set of random parameter combination with sample sizes (n =250, 300, 400, 500). It is evident from Table 3 that transmuted parameter plays a highly significant role as the sample size increases. Likelihood ratio test reveals that transmuted parameter exhibits a highly significant role as sample size increases. Likelihood ratio (LR) statistic for testing $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$ is $\omega = 2\left(L(\hat{\Theta}) - L(\hat{\Theta}_0)\right)$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically $n \to \infty$ distributed as χ_k^2, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if LR statistic value is greater than 6.635 at 99% confidence level.

Table 3 Model comparison based on simulated data

$\theta = 3.7, \alpha = 4.8, \lambda = 0.2$				Parameter Estimates		LR Statistic
Criterion	TQAD	QAD	Sample size (n)	TQAD	QAD	
−logL	30.63562	34.27193	250	$\hat{\alpha} = 0.082425$	$\hat{\alpha} = 0.077269$	7.2720
AIC	67.27124	72.54386		$\hat{\theta} = 3.622912$	$\hat{\theta} = 5.424733$	
BIC	77.83562	79.58679		$\hat{\lambda} = 1.000000$		
−logL	40.96714	46.18117	300	$\hat{\alpha} = 0.094282$	$\hat{\alpha} = 0.093549$	10.4280
AIC	87.9342	96.36233		$\hat{\theta} = 3.52124$	$\hat{\theta} = 5.253082$	
BIC	99.04563	103.7699		$\hat{\lambda} = 1.000000$		
−logL	63.41193	70.9082	400	$\hat{\alpha} = 0.078228$	$\hat{\alpha} = 0.0740132$	14.9920
AIC	132.8239	145.8165		$\hat{\theta} = 3.473820$	$\hat{\theta} = 5.179104$	
BIC	144.7983	153.7994		$\hat{\lambda} = 1.000000$		
−logL	74.3975	84.0509	500	$\hat{\alpha} = 0.081669$	$\hat{\alpha} = 0.0790841$	19.3068
AIC	154.7951	172.1019		$\hat{\theta} = 3.4924028$	$\hat{\theta} = 5.2009173$	
BIC	167.4389	180.5311		$\hat{\lambda} = 1.000000$		

As can be seen from Table 3, LR statistic value is greater than 6.635 at 1% level of significance for all the sample sizes (250, 300, 400 and 500). So transmuted parameter plays a statistically significant role. Also, AIC and BIC values are lesser for transmuted quasi Akash distribution than quasi Akash distribution for all the samples of size (250, 300, 400, 500). So transmuted quasi Akash distribution is better than quasi Akash distribution.

11. Applications of Transmuted Quasi Akash Distribution

We had analyzed two real data set to show that the transmuted quasi Akash distribution which can be a better model than the quasi Akash distribution and exponential distribution. We also tested the significance of transmuted parameter.

Data set 1: This data set corresponds to survival time in weeks of 40 rats which were given dosages of Cytoxan at a concentration of 60 mg/kg supplied by Professor S.C. Choi and was applied by McLachlan, Lawoko and Ganesalingam(1982). This data set is accessible on book Mixture Models: Inference and Applications to Clusterings by McLachlan and Basford (1988, p.121).

Table 4 Survival time in weeks of 40 rats

13.50	13.25	8.00	4.75	11.50	5.50	17.75	8.50
10.50	12.75	12.00	12.75	9.25	5.50	5.25	11.50
11.25	12.50	11.75	13.50	8.50	15.00	8.50	17.50
8.75	16.00	17.00	15.75	15.00	11.50	13.50	6.75
6.75	6.75	5.00	18.00	8.50	13.00	17.75	11.00

Data set 2: The data set given in Table 2 represents the relief times (in minutes) of twenty patients receiving an analgesic Gross and Clark (1975). This data set was used by Shanker, Hagos and Sujatha (2015) in the paper on modelling of lifetime data using one parameter Akash, Lindley and exponential distributions.

Table 5 Relief times of 20 patients receiving an analgesic

1.1	1.4	1.3	1.7	1.9
1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4
3.0	1.7	2.3	1.6	2.0

These data are used here only for illustrative purposes. The required numerical evaluations are carried out using R software version 3.5.3. We have fitted quasi Akash distribution, exponential distribution and transmuted quasi Akash models to these data sets. The summary statistics of the data set 1 and 2 is displayed in Table 6. MLEs of the parameters and model functions are displayed in Table 7 and 8 for data sets 1 and 2, respectively. The corresponding log-likelihood values, LR statistic, AIC, AICC, BIC, HQIC, Kolmogorov statistic and p-value are displayed in Tables 9 and 10 for data sets 1 and 2, respectively.

Table 6 Summary statistics of data set 1 and 2

Data set	n	min	first quartile	median	mean	third quartile	max
1	40	4.75	8.50	11.50	11.29	13.50	18.00
2	20	1.10	1.48	1.70	1.90	2.05	4.10

Table 7 ML Estimates (standard error in parenthesis), model function of related models and proposed model for data set 1

Distribution	Parameter Estimates	Model function
Quasi Akash (QAD)	$\hat{\theta}=0.26561860$ (0.01822084) $\hat{\alpha}=0.001$	$\frac{\theta^2}{(\alpha\theta+2)}\left(\alpha+\theta x^2\right)e^{-\theta x}$
Transmuted Quasi Akash (TQAD)	$\hat{\theta}=0.34522241$ (0.03905684) $\hat{\alpha}=0.00100000$ $\hat{l}=-1.000$	$\frac{\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2\right)e^{-\theta x}\left(\begin{array}{l}(1-\lambda)(\alpha\theta+2)+\\ 2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\end{array}\right)$
Exponential Distribution (ED)	$\hat{\theta}=11.293750$ (1.785699)	$\frac{1}{\theta}e^{-x/\theta}$

Table 8 ML Estimates (standard error in parenthesis), model function of related models and proposed model for data set 2

Distribution	Parameter Estimates	Model function
Quasi Akash (QAD)	$\hat{\theta}=1.57$ (0.1600205) $\hat{\alpha}=0.001$	$\frac{\theta^2}{(\alpha\theta+2)}\left(\alpha+\theta x^2\right)e^{-\theta x}$
Transmuted Quasi Akash (TQAD)	$\hat{\theta}=2.050$ (0.7685197) $\hat{\alpha}=0.001$ (0.2940350) $\hat{l}=-1.00$ (1.4457468)	$\frac{\theta^2}{(\alpha\theta+2)^2}\left(\alpha+\theta x^2\right)e^{-\theta x}\left(\begin{array}{l}(1-\lambda)(\alpha\theta+2)+\\ 2\lambda\left((\alpha\theta+2)+\theta x(\theta x+2)\right)e^{-\theta x}\end{array}\right)$
Exponential Distribution (ED)	$\hat{\theta}=1.9000004$ (0.4248527)	$\frac{1}{\theta}e^{-x/\theta}$

Table 9 Model comparison and Likelihood ratio statistic of related models and proposed model for data set 1

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Kolmogorov Statistic (D)	p-value	LR
QAD	118.204	240.409	243.786	240.733	241.630	0.1678	0.21	10.8784
TQAD	112.765	231.530	236.597	232.197	233.362	0.13397	0.4694	
ED	136.97	275.94	277.628	276.045	276.550	0.34334	0.00016	

Table 10 Model comparison and Likelihood ratio statistic of related models and proposed model for data set 2

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Kolmogorov Statistic (D)	p-value	LR
QAD	22.89	49.79	51.78	50.50	50.18	0.25283	0.155	7.3
TQAD	19.24	44.49	47.80	45.99	45.07	0.15429	0.727	
ED	32.837	67.674	68.669	67.896	67.86	0.43951	0.00088	

For testing the significance of transmuted parameter λ of proposed model and for checking superiority of transmuted quasi Akash distribution over quasi Akash distribution and exponential distribution for data sets 1 and 2, we computed likelihood ratio (LR) statistic. For testing $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$ the LR statistic for testing H_0 is $\omega = 2\left(L(\hat{\Theta}) - L(\hat{\Theta}_0)\right) = 10.878$ for data set 1 and $\omega = 2\left(L(\hat{\Theta}) - L(\hat{\Theta}_0)\right) = 7.300$ for data set 2, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are MLEs under H_1 and H_0. LR statistic $\omega \sim \chi_1^2 = 6.6735$ as $n \to \infty$, where degrees of freedom 1 is the difference in dimensionality. From table 9 $\omega = 10.878 > 6.635$ and from Table 10 $\omega = 7.3 > 6.635$ at 1% level of significance, so we reject H_0 and conclude that transmuted parameter λ plays statistically a significant role.

Further in order to compare the two distribution models, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC. The better distribution corresponds to lesser AIC, AICC, BIC and HQIC values.

$$\text{AIC} = 2k - 2\log L, \quad \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

$$\text{BIC} = k\log n - 2\log L, \quad \text{HQIC} = 2k\log\left(\log(n)\right) + 2\ \log L.$$

where k is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of the log-likelihood function under the considered model. From Tables 9 and 10, it has been observed that the transmuted quasi Akash distribution possesses the lesser AIC, AICC BIC and HQIC values as compared to quasi Akash distribution and exponential distribution for data sets 1 and 2, respectively. Hence, we can conclude that the transmuted quasi Akash distribution leads to a better fit than the quasi Akash distribution and exponential distribution for data sets 1 and 2 respectively.

For testing goodness of fit of proposed model, quasi Akash model and exponential model to two data sets we computed Kolmogorov statistic and p-value and from Tables 9 and 10 and we observed that Kolmogorov statistic value is lesser for proposed model as compared to its related models and p-value is higher for proposed model as compared to related models for both the data sets. Hence our model provides better fit than related models to two data sets.

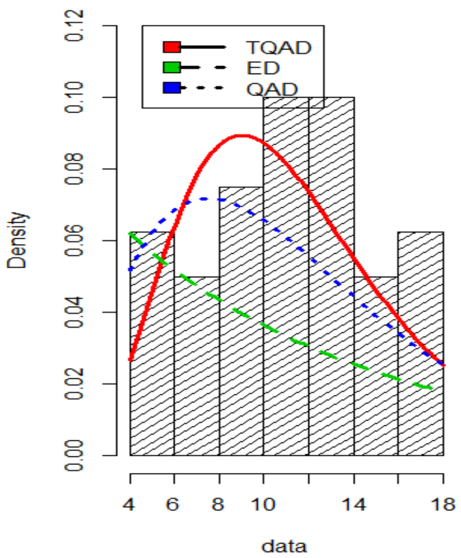

Figure 3 Curve fitting of data set 1

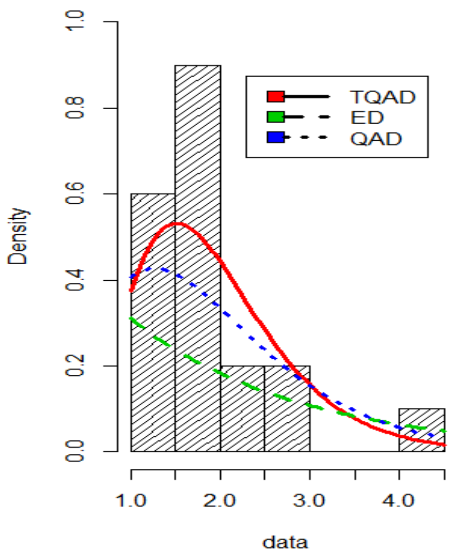

Figure 4 Curve fitting of data set 2

12. Conclusions

We obtained transmuted quasi Akash Distribution by using quadratic rank transmutation map technique. The various important properties of this distribution have been obtained. It has been observed that our proposed model is positively skewed as is clear from graph of probability density function. Various reliability measures have been obtained for the proposed model. From the hazard rate and graph of p.d.f. flexibility of proposed model over its base model is observed. The unknown parameters of proposed model are estimated by using maximum likelihood estimation method. Simulation study has been carried out for testing performance of maximum likelihood estimates and for comparing base model with proposed model. We fitted proposed model and its related models to two lifetime data sets and found that our proposed model gives better results for life time data than its sub model. The significance of transmuted parameter has been tested and it has been concluded that transmuted parameter plays a significant role.

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