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Exact Solution of Average Run Length on Extended EWMA Control Chart for the First-Order Autoregressive Process

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Abstract

Statistical process control methods are widely used in several fields for monitoring and detecting process problems. One of them is the control chart which is effective for monitoring process. Generally, the average run length (ARL) is measured the performance of control chart. The aim of the article is to derive explicit formulas of ARL using a Fredholm integral equation of the second kind on an extended exponentially weighted moving average (EEWMA) control chart for first-order autoregressive (AR(1)) process with exponential white noise. The accuracy of the solution obtained with the EEWMA control chart was compared to the numerical integral equation (NIE) method. The analytical results agree with NIE approximations with an absolute percentage difference less than 1.191×10^{-4} and the computational times of NIE approximately 2-4 seconds whereas the computational time of the explicit formulas is less than one second. In addition, a performance comparison of the ARL using explicit formulas on the EEWMA and EWMA control charts show that they performed better on the EEWMA control chart for all shift sizes and cases. Besides, an exponential smoothing parameter of 0.05 is recommended. Moreover, the ARL performances on the EEWMA and EWMA charts were compared using real data on the concentration of the 24-hour average of particulate matter or PM10 (10 $\mu g/m^3$) in the air pollutant in the air which is an important indicator of air pollution and a major environmental problem. The results indicate that the EEWMA chart performed better than the EWMA chart for all situations.

Keywords: EWMA control chart, change point detection, numerical integral equation, explicit formulas, autoregressive.

Introduction

The statistical process control (SPC) is widely used in the manufacture industry for monitoring, controlling, and improving process. A control chart is an effective tool in statistical process control. It can be applied to other fields such as environment, finance, economics, medicine, health and others; see Lucas and Saccucci (1990) and Srivastava and Wu (1997). The Shewhart control chart was the first to be reported and is widely used for monitoring processes and detecting shifts in the process mean. It is useful for detecting large changes in the process mean, but its performance is degraded when the changes are small (Shewhart 1931). Since then, many other control charts that are useful for detecting small changes in the process mean, such as the Cumulated Sum (CUSUM) (Page 1954) and exponentially weighted moving average (EWMA) control charts (Robert 1959), have been suggested by Yashchin (1993), Zhang (1998) and Prajapati (2015). Patel and Divecha (2011) proposed the modified EWMA control chart that is effective at detecting small and abrupt changes in the process mean for observations that are independent and normally distributed or autocorrelated. Later, Khan et al. (2017) redesigned the modified EWMA control statistic. Recently, Naveed et al. (2018) proposed the extended EWMA (EEWMA) control chart that performed better than other control charts for detecting small shifts in the mean of a monitored process.

The comparative performance method for control charts is the ARL. The ARL_0 is usually measured when the process is in-control and should be large and the ARL_1 is correctly signaled to be out of control and should be as small as possible. The exact solution of the ARL of control charts have been described in previous literature such as explicit formulas, Monte Carlo simulations (MC), Markov chain approach (MCA), martingale approach (MA) and numerical integral equation approach (NIE).

Previous literature about approximation of the ARL to represent an efficient control chart using many methods. Crowder (1987) presented the evaluation of ARL for EWMA control chart using NIE method to an integral for Gaussian observation. Lucas and Saccucci (1990) proposed the evaluation of ARL for EWMA control chart using a finite state Markov chain approximation. Fu et al. (2002) determined the ARL on Shewhart, CUSUM and EWMA control charts based on the Markov chain approach. Sukparungsee and Novikov (2008) used the martingale approach to derive close-form formula for the ARL for EWMA control chart for a variety of light-tailed distributions.

In addition, Areepong (2009) proposed analytical derivation to find explicit formulas for ARL of the EWMA chart when observations are exponential distributed. Suriyakat et al. (2012) presented an explicit formula for the ARL of EWMA control chart for autoregressive AR(1) process observation with exponential white noise. Petcharat et al. (2014) analyzed explicit formulas for ARL of CUSUM control chart for MA(1) process and compared them with the NIE method.

Subsequently, Sukparungsee and Areepong (2017) presented an explicit formula for the ARL of EWMA control chart for $\mathrm{AR}(p)$ process with exponential white noise. Peerajit et al. (2018) evaluated Numerical integral equation method for ARL of CUSUM chart for long-memory process with non-seasonal and seasonal ARFIMA models. Later, Peerajit et al. (2019) compared the efficiency of explicit solutions to the NIE method of ARL on CUSUM control chart for a long memory process with a seasonally adjusted autoregressive fractionally integrated moving-average (ARFIMA) process model.

Recently, Supharakonsakun et al. (2020) proposed explicit formulas of modified EWMA and compared the efficiency of the EWMA and modified EWMA control charts. The performance comparison shows that the modified EWMA control chart is outperforms the EWMA control chart for almost all of exponential smoothing parameters and shift sizes.

As previously mentioned, indicate that the ARL is useful for efficiency comparison of the control charts and that explicit formulas take much less computational time to evaluate the ARL than the other methods. The EWMA control chart is more performance than the others but less performance than modified EWMA control chart, the modified EWMA control chart is more effective than the EWMA control chart for small detecting small shifts in process mean. However, derivation of the explicit formulas for the average run length (ARL) on the EEWMA control chart and comparing them with other control charts has not previously been reported.

Therefore, the objective of this study is to derive explicit formulas of the ARL on the EEWMA control chart for a first-order autoregressive (AR(1)) process with exponential white noise. The explicit formulas for ARL_0 and ARL_1 were compared with the numerical integral equation (NIE) method as the benchmark. Besides, the performance of the explicit formulas for deriving the ARL on the EEWMA control chart was compared with those on the EWMA chart.

2. Materials and Methods

2.1. Exponentially weighted moving average (EWMA) control chart

The EWMA control chart was initially proposed by Robert (1959) (see also Crowder (1978), Lucas and Saccucci (1990), Suriyakat et al. (2012)). It is usually used to monitor and detect small changes in process mean. The EWMA control chart can be expressed by the recursive equation below

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t, t = 1, 2, ... \tag{1}$$

where X_t is a process with mean, λ is an exponential smoothing parameter with $0 < \lambda < 1$ and Z_0 is the initial value of EWMA statistics, $Z_0 = u.$ The upper control limit (UCL) and Lower control limit (LCL) of EWMA control chart are given by

$$UCL = \mu_0 + Q\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{2}$$

$$LCL = \mu_0 - Q\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{3}$$

where μ_0 is the target mean, σ is the process standard deviation and Q is suitable control limit width. The stopping time of the EWMA control chart is given by

$$\tau_h = \inf\{t \geq 0 : Z_t > h\},\ h > u, \tag{4}$$

where τ_h is the stopping time, h is UCL.

2.2. Extended exponentially weighted moving average (EEWMA) control chart

The extended EWMA control chart was proposed by Naveed et al. (2018). It is developed form the EWMA control chart. This is effective to monitored and detected small changes in process mean. The EWMA control chart can be expressed by the recursive equation below

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1},\ t = 1, 2, \ldots, \tag{5}$$

where λ_1 and λ_2 are exponential smoothing parameters with $0 < \lambda_1 \leq 1$ and $0 \leq \lambda_2 < \lambda_1$ and the initial value is a constant, $E_0 = u.$ The upper control limit (UCL) and Lower control limit (LCL) of the extended EWMA control chart are given by

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \tag{6}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \tag{7}$$

where μ_0 is the target mean, σ is the process standard deviation and L is suitable control limit width. The stopping time of the EEWMA control chart is given by

$$\tau_b = \inf\{t \geq 0 : E_t > b\},\ b > u, \tag{8}$$

where τ_b is the stopping time, b is UCL.

3. Explicit formulas of ARL on the EEWMA control chart of AR(1)

Let $L(u)$ denote the ARL for the first-order autoregressive process, to define function $L(u)$ as

$$ARL = L(u) = E_\infty(\tau_b) \geq T, \tag{9}$$

where E_∞ is the expectation. The first-order autoregressive process denoted by AR(1) can be described by

$$X_t = \eta + \phi_1 X_{t-1} + \varepsilon_t, \tag{10}$$

where η is a constant and autoregressive coefficient $(-1 \leq \phi_1 \leq 1)$, ε_t is the error term of time t and assumed to be a white noise process with exponential distribution, $\varepsilon_t \sim Exp(\alpha)$. The probability density function of ε_t is given by $f(x) = \dfrac{1}{\alpha} e^{-\frac{x}{\alpha}}$.

Let $L(u)$ denote the ARL for AR(1) process, the EEWMA statistics E_t can be written as

$$E_t = (1-\lambda_1+\lambda_2)Z_0 + (\lambda_1\phi_1 - \lambda_2)X_0 + \lambda_1\eta + \lambda_1\varepsilon_t,$$

where $0 < \lambda_1 \leq 1$, $0 \leq \lambda_2 < \lambda_1$ and the initial value $E_0 = u$, $X_0 = v$. Thus, the EEWMA statistics E_1 can be written as

$$E_t = (1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v + \lambda_1\eta + \lambda_1\varepsilon_t$$

If $\varepsilon_t \geq 0$, $LCL = 0$ and $UCL = b$, respectively. Then

$$0 \leq (1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v + \lambda_1\eta + \lambda_1\varepsilon_1 \leq b.$$

Let $L(u)$ denote the ARL on the EEWMA control chart. The function $L(u)$ can be derived by Fredholm integral equation of the second kind, $L(u)$ is defined as follows

$$L(u) = 1 + \int L(Z_1) f(\varepsilon_1) d\varepsilon_1. \tag{11}$$

Consequently, the function $L(u)$ is obtained as follows

$$L(u) = 1 + \int L(1-\lambda_1+\lambda_2)u + \lambda_1\eta + (\lambda_1\phi_1 - \lambda_2)v + \lambda_1 y_1) f(y_1) dy_1.$$

Changing the integration variable, the function $L(u)$ is given by

$$L(u) = 1 + \frac{1}{\lambda_1}\int_0^b L(y) f\left(\frac{y-(1-\lambda_1+\lambda_2)u - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta\right) dy. \tag{12}$$

The $L(u)$ is Fredholm integral equation of the second kind. If $\varepsilon_t \sim Exp(\alpha)$, then

$$L(u) = 1 + \frac{1}{\lambda_1\alpha}\int_0^b L(y) e^{-\frac{y}{\lambda_1\alpha}} e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} dy = 1 + \frac{G(u)}{\lambda_1\alpha} d \tag{13}$$

Suppose that $G(u)$ by given by

$$G(u) = e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \text{ and } d = \int_0^b L(y) e^{-\frac{y}{\lambda_1\alpha}} dy.$$

Therefore, $d = 1 + \int_0^b \left[1 + \dfrac{G(y)}{\lambda_1\alpha} d\right] \cdot e^{-\frac{y}{\lambda_1\alpha}} dy$, solving the constant d,

$$d = \int_0^b e^{-\frac{y}{\lambda_1\alpha}}dy + \int_0^b \frac{G(y)}{\lambda_1\alpha} d \cdot e^{-\frac{y}{\lambda_1\alpha}}dy = -\lambda_1\alpha\left(e^{-\frac{b}{\lambda_1\alpha}} - 1\right) - \frac{d}{\lambda_1 - \lambda_2} \cdot e^{\frac{(\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha}+\frac{\eta}{\alpha}} \cdot \left(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha}} - 1\right)$$

$$= \frac{-\lambda_1\alpha\left(e^{-\frac{b}{\lambda_1\alpha}} - 1\right)}{1 + \frac{1}{\lambda_1 - \lambda_2} \cdot e^{\frac{(\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha}+\frac{\eta}{\alpha}} \cdot \left(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha}} - 1\right)}.$$

Finally, substituting constant d in (13), then

$$L(u) = 1 - \frac{(\lambda_1 - \lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha}} \cdot (e^{-\frac{b}{\lambda_1\alpha}} - 1)}{(\lambda_1 - \lambda_2)e^{-\left\{\frac{(\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha}+\frac{\eta}{\alpha}\right\}} + \left(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha}} - 1\right)}. \quad (14)$$

As mentioned above, the value of the parameter α is equal to α_0 when the process is "in-control". Therefore, substituting $\alpha = \alpha_0$ in (14) give the explicit formulas of the ARL_0 for the first-order autoregressive process on the EEWMA control chart can be defined as:

$$ARL_0 = 1 - \frac{(\lambda_1 - \lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_0}} \cdot \left(e^{-\frac{b}{\lambda_1\alpha_0}} - 1\right)}{(\lambda_1 - \lambda_2)e^{-\left\{\frac{(\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha_0}+\frac{\eta}{\alpha_0}\right\}} + \left(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha_0}} - 1\right)}. \quad (15)$$

The explicit formulas of ARL_1 can be obtained in a similar manner. When the process is "out-of-control", the value of parameter α, substituting $\alpha = \alpha_1$ in (14) give the explicit formulas of ARL_1 for the first-order autoregressive process on the EEWMA control chart can be defined as:

$$ARL_1 = 1 - \frac{(\lambda_1 - \lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1}} \cdot \left(e^{-\frac{b}{\lambda_1\alpha_1}} - 1\right)}{(\lambda_1 - \lambda_2)e^{-\left\{\frac{(\lambda_1\phi_1-\lambda_2)v}{\lambda_1\alpha_1}+\frac{\eta}{\alpha_1}\right\}} + \left(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - 1\right)}. \quad (16)$$

The performance of control chart is measured by the ARL. The ARL_0 is defined as the expected of false alarm time (τ) before an in-control process is taken to signal to be out of control. A sufficient large in-control ARL_0 is desired. When the process is out-of-control, the performance of control chart is usually used as ARL_1. It is the expected number of observations taken from out-of-control process until the control chart signals that the process is out-of-control. Ideally, ARL_1 should be small.

As mentioned above, the explicit formulas of ARL_1 on the EEWMA control chart should be less than the explicit formulas of ARL_1 on the EWMA control chart (Suriyakat et al. 2012).

4. Existence and Uniqueness of ARL

The solution of ARL shows that there uniquely exists the integral equation for explicit formulas by the Banach's fixed-point theorem. In this research, let T be an operation in the class of all continuous functions defined by

$$T(L(u)) = 1 + \frac{1}{\lambda_1}\int_0^b L(y) f\left(\frac{y - (1-\lambda_1+\lambda_2)u - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta\right)dy. \quad (17)$$

According to Banach's fixed-point theorem, if an operator T is a contraction, and then the fixed-point equation $T(L(u)) = L(u)$ has a unique solution (Supharakonsakun et al. 2020). To show that (17) exists and has a unique solution, theorem can be used Banach fixed-point Theorem. The Banach fixed-point theorem is also called the contraction mapping theorem, appeared in explicit form in Banach' thesis in 1922 (Banach 1922). It is in general use to establish the existence of a solution to an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical (Jleli and Samet 2014). The details are as follows below.

Theorem 1 (Banach fixed-point) *Let* (X,d) *be a complete metric space and* $T: X \to X$ *be a contraction mapping with contraction constant* $0 \le r < 1$ *such that* $\|T(L_1) - T(L_2)\| \le r\|L_1 - L_2\|$, $\forall L_1, L_2 \in X$. *Then there exists a unique* $L(\cdot) \in X$ *such that* $T(L(u)) = L(u)$, *i.e., a unique fixed-point in* X (Sofonea et al. 2005).

Proof: To show that T defined in (17) is a contraction mapping for $L_1, L_2 \in G[0,b]$, such that $\|T(L_1) - T(L_2)\| \le r\|L_1 - L_2\|$, $\forall L_1, L_2 \in G[0,b]$ with $0 \le r < 1$ under the norm $\|L\|_\infty = \sup_{u\in[0,b]} |L(u)|$.
From (13) and (17),

$$\begin{aligned}
\|T(L_1) - T(L_2)\|_\infty &= \sup_{u\in[0,b]} \left| \frac{1}{\lambda_1\alpha} e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \int_0^b (L_1(y) - L_2(y)) e^{-\frac{y}{\lambda_1\alpha}} dy \right| \\
&\le \sup_{u\in[0,b]} \left| \|L_1 - L_2\| \frac{1}{\lambda_1\alpha} e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \cdot (-\lambda_1\alpha)(e^{-\frac{b}{\lambda_1\alpha}} - 1) \right| \\
&= \|L_1 - L_2\|_\infty \sup_{u\in[0,b]} \left| e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \right| \left| 1 - e^{-\frac{b}{\lambda_1\alpha}} \right| \\
&\le r\|L_1 - L_2\|_\infty
\end{aligned}$$

where $r = \sup_{u\in[0,b]} \left| e^{\frac{(1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \right| \left| 1 - e^{-\frac{b}{\lambda_1\alpha}} \right|$, $0 \le r < 1$. Therefore, the existence and the uniqueness of the solution are guaranteed by Banach's fixed-point theorem.

5. Numerical Integral Equation Method of ARL for the EEWMA Control Chart of AR(1)

From a Fredholm integral equation of the second kind in (12) is given as

$$L(u) = 1 + \frac{1}{\lambda_1} \int_0^b L(y) f\left(\frac{y - (1-\lambda_1+\lambda_2)u - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta \right) dy.$$

It cannot be solved analytically for $L(u)$ and it is necessary to use numerical methods to solve them. A quadrature rule is used to approximate the integral by a finite sum of areas of rectangles with base b/m and heights chosen as the values of $f(a_j)$ at the midpoints of intervals of length beginning at zero. Specifically, once the choice of a quadrature rule is made, the interval $[0,b]$ is divided into $0 \le a_1 \le a_2 \le \ldots \le a_m \le b$ with a set of constant weights $w_j = b/m \ge 0$.

The approximation for an integral is evaluated by the quadrature rule as follows

$$\int_0^b L(y)f(y)dy \approx \sum_{j=1}^{m} w_j f(a_j). \tag{16}$$

Let $L(a_i)$ be a numerical approximation to the integral equation which can be found as the solution of linear equations as follows

$$\tilde{L}(a_i) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)a_i - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta\right),\ i = 1,2,...,m. \tag{17}$$

The system of m linear equation is showed as

$$L_{m\times 1} = 1_{m\times 1} + R_{m\times m}L_{m\times 1} \text{ or } (I_m - R_{m\times m})L_{m\times 1} = 1_{m\times 1} \text{ or } L_{m\times 1} = (I_m - R_{m\times m})^{-1}1_{m\times 1}.$$

If the inverse $(I_m - R_{m\times m})^{-1}$ exists, then a unique solution of equation is

$$L_{m\times 1} = (I_m - R_{m\times m})^{-1}1_{m\times 1}.$$

where $L_{m\times 1} = \begin{bmatrix} \tilde{L}(a_1) \\ \tilde{L}(a_2) \\ \vdots \\ \tilde{L}(a_m) \end{bmatrix}$, $I_m = diag(1,1,...,1)$ and $1_{m\times 1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Let $R_{m\times m}$ be a matrix, the definition of the m to m^{th} element of the matrix R is given by

$$\left[R_{ij}\right] \approx \frac{1}{\lambda_1} w_j f\left(\frac{a_j - (1-\lambda_1+\lambda_2)a_i - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta\right).$$

Finally, substituting, a_i by u in $\tilde{L}(a_i)$.

Therefore, the approximation of ARL is evaluated by the numerical integral equation (NIE) method for the function $L(u)$ is as follows

$$\tilde{L}(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)u - (\lambda_1\phi_1 - \lambda_2)v}{\lambda_1} - \eta\right), \tag{18}$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m},\ j = 1,2,...,m.$

6. Numerical Results

In this section, the ARL was approximated by NIE method using the Gauss-Legendre quadrature rule on the EEWMA control chart with 500 nodes (see Areepong and Sukparungsee 2015, Petcharat et al. 2015, Phanyaem 2017). The absolute percentage difference to measure the accuracy of ARL is defined as:

$$Diff(\%) = \frac{\left|L(u) - \tilde{L}(u)\right|}{L(u)} \times 100, \tag{19}$$

where $L(u)$ is explicit formulas and $\tilde{L}(u)$ is approximation of ARL using NIE method. The numerical results are computed by Mathematica. The explicit formulas of ARL (15), (16) and NIE method (18) on the EEWMA control chart for the first-order autoregressive process when given $\text{ARL}_0 = 370$, $\lambda_1 =$ 0.05, 0.10, $\lambda_2 = 0.04$ and $\phi_1 = 0.1, -0.1, 0.2, -0.2$. The 'in-control' process had parameter value as $\alpha = \alpha_0 = 1$ with shift size $(\delta = 0)$. On the other hand, the 'out-of-control' process was presented with parameter values as $\alpha_1 = \alpha_0(1+\delta)$ with δ = 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.30, 0.50, and 1.00, respectively were determined. The results in Tables 1 and 2 show that the ARL values and in parentheses ( ) the computation time for calculation. The ARL values derived from explicit formulas give results close to those from NIE. The analytical results agree with NIE approximations with an absolute percentage difference of less than 1.191×10^{-4} and computational times of approximately 2-4 seconds. The computational time of the explicit formulas is less than one second.

7. Performance comparison of the ARL on the EWMA and EEWMA control charts for AR(1)

In the section, the numerical comparative results of ARL on the EWMA and the EEWMA control charts are investigated. Tables 3 and 4 present the ARL of explicit formulas on the EWMA and EEWMA control charts for first-order autoregressive process when $\text{ARL}_0 = 370, 500,$ $\eta = 1,$ $\lambda_1 = 0.05, 0.10,$ $\lambda_2 = 0.04, \phi_1 = 0.1, -0.1, 0.2, -0.2$ which are obtained by the upper control limit (Table 5). For $\text{ARL}_0 = 370,$ the results are shown in Table 3 and the details are as follows.

Under $\lambda_1 = 0.05,$ shows that the EEWMA control chart reduced the ARL_1 more than the EWMA control chart when the shift sizes (=0.01 to 0.03) whereas the large shift size (=0.05 and 1.00), the performance of the EEWMA control chart is close to the EWMA control chart both $\phi_1 = 0.1$ and $\phi_1 = -0.1.$ Under $\lambda_1 = 0.10,$ shows that the EEWMA control chart reduced the ARL_1 more than the EWMA control chart whereas only the large shift size (=1.00), the performance of the EEWMA control chart is close to the EWMA control chart. For $\text{ARL}_0 = 500,$ the results are shown in Table 4 and the details are related the results in Table 3.

As mentioned above, the results indicated that the EEWMA control chart reduced the ARL_1 more than the EWMA control chart when detecting shift sizes process. Therefore, the performance of the EEWMA control chart is more efficient than the performance of EWMA control chart for all shift sizes and exponential smoothing parameters whereas the large shift size (=1.00), the performance of the EEWMA control chart is close to the EWMA control chart.

8. Application

In the section, real data was applied to determine the ARL of explicit formulas on the EEWMA and the EWMA control charts for the concentration of 24-hour average of Particulate Matter or PM10 in terms of 10 microgram per cubic meter ($10\ \mu g/m^3$) pollutant in the air which is one of the major causes of air pollution in Thailand. The concentration of 24-hour average of PM10 pollutant observed form February 1st, 2020 to February 29th, 2020. This data is a stationary time series. By looking at the autocorrelation function (ACF) and partial autocorrelation function (PACF). The data was analyzed and fitted with first-order autoregressive process with the significant of mean and standard deviation equals 6.528571 and 3.381196, respectively, and then $\text{ARL}_0 = 370, 500,$ $\lambda_1 = 0.05, 0.10,$ $\lambda_2 = 0.04.$ Tables 6 and 7 show that the results for the ARL on EEWMA and EWMA control charts for real data are in similar agreement to the simulation results in Tables 3 and 4.

ARL_1 on the EEWMA control chart was reduced more sensitively than on the EWMA control chart for very small shift sizes whereas on the EWMA control chart, ARL_1 reduction was as sensitive as on the EEWMA control chart for large shift sizes and all exponential smoothing parameter values. These results indicate that the performance of the EEWMA control chart was more efficient than the EWMA control chart for all cases when monitoring and detecting changes in the process mean, as illustrated in Figure 1.

The concentration of 24-hour average of PM10 is an important indicator of air pollution, which is a major environmental problem and a concentration higher than the international standard can greatly affect public health. The concentration of 24-hour average of PM10 pollutant analyzed. There are 29 observations of daily form February 1st, 2020 to February 29th, 2020. The upper and lower control limits were established by (2) and (3) for the EWMA control chart and then (6) and (7) for the EEWMA control chart. The detection of the process with real data for exponential smoothing parameter $\lambda = 0.05$ is shown in Figure 2. The results show that the EEWMA control chart detected the out-of-control process at the 9th observation whereas the EWMA chart detected it at the 24th observation (these observations were plotted above the upper control limit).

9. Conclusions

In this study, the results showed the derivation of explicit formulas of the ARL on the EEWMA control chart for a first-order autoregressive (AR(1)) process with exponential white noise. The accuracy of the solution obtained by deriving explicit formulas for the ARL on an EEWMA chart for an AR(1) process was compared to the NIE method as the benchmark. The analytical results agree with the NIE approximations with an absolute percentage difference of less than 1.191×10^{-4}. A performance comparison of the ARL using explicit formulas on the EEWMA and EWMA control charts show that they performed better on the EEWMA control chart for all shift sizes and cases. Besides, an exponential smoothing parameter of 0.05 is recommended. Moreover, the ARL performances on the EEWMA and EWMA charts were compared using real data on the concentration of the 24-hour average of PM10 in the air. The results indicate that the EEWMA chart performed better than the EWMA chart for all scenarios.

Table 1 Comparison of ARL of the EEWMA control chart for AR(1) using explicit formulas against NIE when $\eta = 1, \lambda_1 = 0.05, \lambda_2 = 0.04,$ and $ARL_0 = 370$

b	ϕ_1	Shift sizes	Explicit	NIE	*Diff* (%)
1.55816×10^{-11}	0.1	0.00	370.0023413617	370.0022779640 (2.734)	1.1713×10^{-5}
		0.01	278.4400407776	278.4400524166 (3.312)	4.180×10^{-6}
		0.03	160.4188440586	160.4188575594 (2.688)	8.416×10^{-6}
		0.05	94.52256676494	94.52257068707 (2.813)	4.149×10^{-6}
		0.07	56.94963892038	56.94964924043 (2.719)	1.812×10^{-5}
		0.09	35.09708752734	35.09708130464 (2.703)	1.773×10^{-5}
		0.10	27.79511028281	27.79510949752 (2.954)	2.825×10^{-6}
		0.30	1.464428844114	1.464428863607 (2.549)	1.331×10^{-6}
		0.50	1.023252801222	1.023252797228 (2.891)	3.903×10^{-7}
		1.00	1.000169552980	1.000169552951 (2.657)	2.912×10^{-9}
2.10874×10^{-12}	−0.1	0.00	370.0024023440	370.0022343929 (3.031)	4.539×10^{-5}
		0.01	273.0004290896	273.0001907503 (2.578)	8.730×10^{-5}
		0.03	151.3974185330	151.3976034750 (2.844)	1.222×10^{-4}
		0.05	86.02652619443	86.02663298535 (3.890)	1.241×10^{-4}
		0.07	50.08780630459	50.08781627181 (3.109)	1.990×10^{-5}
		0.09	29.90672934606	29.90671695217 (2.970)	4.144×10^{-5}
		0.10	23.34052438882	23.34049657985 (2.765)	1.191×10^{-4}
		0.30	1.292735646345	1.292735590186 (3.062)	4.344×10^{-6}
		0.50	1.011938372222	1.011938382502 (2.985)	1.016×10^{-6}
		1.00	1.000143835187	1.000062375037 (2.874)	8.145×10^{-5}
4.2355×10^{-11}	0.2	0.00	370.0006850042	370.0007084819 (3.531)	6.345×10^{-6}
		0.01	281.1994398816	281.1994378784 (2.516)	7.124×10^{-7}
		0.03	165.1297229815	165.1297123448 (2.688)	6.441×10^{-6}
		0.05	99.08334709714	99.08335012564 (2.672)	3.057×10^{-6}
		0.07	60.73203935142	60.73203689086 (2.938)	4.052×10^{-6}
		0.09	38.03177900107	38.03177694188 (2.986)	5.414×10^{-6}
		0.10	30.34506028706	30.34506138600 (2.829)	3.621×10^{-6}
		0.30	1.584977552377	1.584977513985 (2.860)	2.485×10^{-6}
		0.50	1.032451757441	1.032451755257 (2.985)	2.115×10^{-7}
		1.00	1.000279544352	1.000279544370 (2.609)	1.784×10^{-9}
7.7577×10^{-13}	−0.2	0.00	370.0052053728	370.0059955846 (3.172)	2.136×10^{-4}
		0.01	270.3238650623	270.3231506360 (3.891)	2.643×10^{-4}
		0.03	147.0818611628	147.0817559030 (3.875)	7.157×10^{-5}
		0.05	82.07375038550	82.07345992182 (4.046)	3.539×10^{-4}
		0.07	46.97980745368	46.97972168631 (3.094)	1.826×10^{-4}
		0.09	27.61601576727	27.61607467871 (4.640)	2.133×10^{-4}
		0.10	21.39937463998	21.39933075146 (3.172)	2.051×10^{-4}
		0.30	1.232411285532	1.232411783310 (3.984)	4.039×10^{-5}
		0.50	1.008554324283	1.008554311999 (3.156)	1.218×10^{-6}
		1.00	1.000037832677	1.000037832758 (3.453)	8.103×10^{-9}

The values in parentheses are the computation time in numerical integration methods (seconds).

Table 2 Comparison of ARL of the EEWMA control chart for AR(1) using explicit formulas against NIE when $\eta = 1, \lambda_1 = 0.10, \lambda_2 = 0.04,$ and $ARL_0 = 370$

b	ϕ_1	Shift sizes	Explicit	NIE	*Diff* (%)
5.4268×10^{-5}	0.1	0.00	370.0066203326	370.0066203326 (2.203)	7.989×10^{-13}
		0.01	320.4876367969	320.4876367969 (2.249)	1.499×10^{-11}
		0.03	242.4908534499	242.4908534499 (2.297)	1.814×10^{-11}
		0.05	185.4661106391	185.4661106391 (2.203)	9.164×10^{-12}
		0.07	143.3098777147	143.3098777147 (2.171)	5.573×10^{-12}
		0.09	111.8181680205	111.8181680205 (2.297)	2.682×10^{-12}
		0.10	99.11812076701	99.11812076701 (2.594)	1.520×10^{-12}
		0.30	13.61926715886	13.61926715886 (2.219)	1.248×10^{-11}
		0.50	3.756376503662	3.756376503662 (2.360)	4.256×10^{-12}
		1.00	1.220851083454	1.220851083454 (2.360)	3.274×10^{-12}
7.3426×10^{-6}	-0.1	0.00	370.0014297063	370.0014297063 (2.233)	1.730×10^{-11}
		0.01	314.0403047463	314.0403047462 (2.219)	3.057×10^{-11}
		0.03	228.4826158938	228.4826158937 (2.204)	3.020×10^{-11}
		0.05	168.3416197999	168.3416197999 (2.437)	2.318×10^{-11}
		0.07	125.5206391243	125.5206391243 (2.375)	5.020×10^{-11}
		0.09	94.66215092741	94.66215092743 (2.187)	2.494×10^{-11}
		0.10	82.54496894955	82.54496894959 (2.250)	4.714×10^{-11}
		0.30	8.926815304909	8.926815304904 (2.437)	5.779×10^{-11}
		0.50	2.411454610704	2.411454610704 (2.281)	1.328×10^{-11}
		1.00	1.081133419333	1.081133419333 (2.438)	4.642×10^{-12}
1.47581×10^{-4}	0.2	0.00	370.0024187541	370.0024187540 (2.953)	3.865×10^{-11}
		0.01	323.7574272230	323.7574272229 (2.656)	4.077×10^{-11}
		0.03	249.8214192146	249.8214192145 (3.780)	3.883×10^{-11}
		0.05	194.6899697138	194.6899697138 (2.969)	3.236×10^{-11}
		0.07	153.1576527480	153.1576527479 (2.719)	3.199×10^{-11}
		0.09	121.5662768719	121.5662768718 (2.782)	2.879×10^{-11}
		0.10	108.6543731987	108.6543731987 (2.749)	2.669×10^{-11}
		0.30	16.93403104606	16.93403104606 (2.875)	1.653×10^{-11}
		0.50	4.855333862877	4.855333862876 (2.687)	1.277×10^{-11}
		1.00	1.364673834134	1.364673834134 (2.735)	1.464×10^{-12}
2.70114×10^{-6}	-0.2	0.00	370.0027468348	370.0027468345 (2.375)	6.621×10^{-11}
		0.01	310.8704655454	310.8704655449 (2.251)	1.451×10^{-10}
		0.03	221.7962911027	221.7962911025 (2.360)	9.198×10^{-11}
		0.05	160.3995932946	160.3995932945 (2.188)	4.612×10^{-11}
		0.07	117.4948858085	117.4948858083 (2.453)	1.141×10^{-10}
		0.09	87.12448406709	87.12448406718 (2.188)	1.087×10^{-10}
		0.10	75.35645166739	75.35645166728 (2.234)	1.392×10^{-10}
		0.30	7.285835957278	7.285835957267 (2.297)	1.504×10^{-10}
		0.50	2.010580050905	2.010580050903 (2.078)	9.253×10^{-11}
		1.00	1.049195252943	1.049195252943 (2.343)	6.666×10^{-12}

The values in parentheses are the computation time in numerical integration methods (seconds).

Table 3 Comparison of ARL of the EWMA and EEWMA control charts for AR(1) using explicit formulas when $\eta = 1$, $\lambda_1 = 0.05, 0.10$, $\lambda_2 = 0.04$, $\phi_1 = 0.1, -0.1$ and $ARL_0 = 370, 500$

λ_1	ϕ_1	δ	$ARL_0 = 370$		$ARL_0 = 500$	
			EWMA	EEWMA	EWMA	EEWMA
0.05	0.1	0.00	370.002	370.002	500.002	500.002
		0.01	303.697	278.440	410.337	376.182
		0.03	206.994	160.419	279.565	216.582
		0.05	143.202	94.523	193.300	127.471
		0.07	100.499	56.950	135.553	76.661
		0.09	71.514	35.097	96.357	47.109
		0.10	60.634	27.795	81.643	37.235
		0.30	4.539	1.464	5.786	1.628
		0.50	1.437	1.023	1.591	1.031
	−0.1	0.00	370.005	370.002	500.002	500.002
		0.01	297.765	273.000	402.312	368.826
		0.03	195.339	151.397	263.802	204.383
		0.05	130.285	86.027	175.831	115.981
		0.07	88.297	50.088	119.051	67.382
		0.09	60.781	29.907	81.841	40.091
		0.10	50.720	23.341	68.236	31.211
		0.30	3.231	1.293	4.016	1.396
		0.50	1.224	1.012	1.303	1.016
0.10	0.1	0.00	370.001	370.007	500.006	500.001
		0.01	335.120	320.488	452.795	432.814
		0.03	276.448	242.491	373.393	327.090
		0.05	229.671	185.466	310.100	249.885
		0.07	192.095	143.310	259.266	192.870
		0.09	161.695	111.818	218.145	150.317
		0.10	148.689	99.118	200.556	133.167
		0.30	36.374	13.619	48.737	17.941
		0.50	13.150	3.756	17.382	4.696
	−0.1	0.00	370.003	370.001	500.004	500.003
		0.01	328.221	314.040	443.297	424.022
		0.03	260.074	228.483	350.883	308.019
		0.05	207.901	168.342	280.205	226.613
		0.07	167.588	125.521	225.645	168.732
		0.09	136.168	94.662	183.156	127.070
		0.10	123.092	82.545	165.484	110.723
		0.30	22.920	8.927	30.411	11.629
		0.50	7.111	2.411	9.182	2.891

Table 4 Comparison of ARL of the EWMA and EEWMA control charts for AR(1) using explicit formulas when $\eta = 1$, λ_1 = 0.05, 0.10, λ_2 =0.04 , ϕ_1 = 0.2, −0.2 and $ARL_0 = 370, 500$

λ_1	ϕ_1	δ	$ARL_0 = 370$		$ARL_0 = 500$	
			EWMA	EEWMA	EWMA	EEWMA
0.05	0.2	0.00	370.009	370.001	500.002	500.003
		0.01	306.715	281.199	414.411	379.916
		0.03	213.086	165.130	287.798	222.954
		0.05	150.140	99.083	202.679	133.639
		0.07	107.228	60.732	144.650	81.776
		0.09	77.585	38.032	104.564	51.078
		0.10	66.310	30.345	89.317	40.684
		0.30	5.458	1.585	7.028	1.791
		0.50	1.610	1.032	1.825	1.044
	−0.2	0.00	370.001	370.005	500.007	500.001
		0.01	294.837	270.324	398.362	365.202
		0.03	189.758	147.082	256.260	198.544
		0.05	124.272	82.074	167.702	110.634
		0.07	82.768	46.980	111.577	63.178
		0.09	56.043	27.616	75.435	36.992
		0.10	46.399	21.399	62.394	28.586
		0.30	2.771	1.232	3.395	1.314
		0.50	1.161	1.009	1.217	1.012
0.10	0.2	0.00	370.003	370.002	500.001	500.005
		0.01	338.733	323.757	457.813	437.286
		0.03	285.296	249.821	385.693	337.088
		0.05	241.796	194.690	326.958	262.443
		0.07	206.146	153.158	278.801	206.258
		0.09	176.741	121.566	239.065	163.553
		0.10	163.981	108.654	221.817	146.108
		0.30	46.381	16.934	62.643	22.410
		0.50	18.392	4.855	24.661	6.173
	−0.2	0.00	370.004	370.003	500.006	500.004
		0.01	324.851	310.870	438.675	419.696
		0.03	252.325	221.796	340.273	298.921
		0.05	197.905	160.400	266.542	215.830
		0.07	156.657	117.495	210.727	157.855
		0.09	125.094	87.124	168.065	116.871
		0.10	112.135	75.356	150.564	101.003
		0.30	18.319	7.286	24.200	9.426
		0.50	5.357	2.011	6.825	2.353

Table 5 Upper control limit of EWMA and EEWMA control charts using explicit formulas

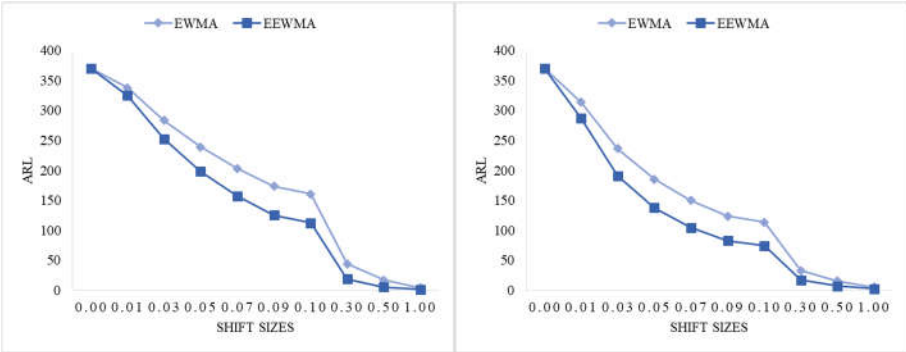
λ_1	ϕ_1	$ARL_0 = 370$ EWMA h	$ARL_0 = 370$ EEWMA b	$ARL_0 = 500$ EWMA h	$ARL_0 = 500$ EEWMA b
0.05	0.1	1.03372×10^{-7}	1.55816×10^{-11}	1.3979×10^{-7}	2.1071×10^{-11}
	−0.1	1.399×10^{-8}	2.10874×10^{-12}	1.89185×10^{-8}	2.85165×10^{-12}
	0.2	2.81×10^{-7}	4.2355×10^{-11}	3.7999×10^{-7}	5.7277×10^{-11}
	−0.2	5.14657×10^{-9}	7.7577×10^{-13}	6.9598×10^{-9}	1.04906×10^{-12}
0.10	0.1	4.4492×10^{-3}	5.4268×10^{-5}	5.9664×10^{-3}	7.2634×10^{-5}
	−0.1	5.9113×10^{-4}	7.3426×10^{-6}	7.8801×10^{-4}	9.827×10^{-6}
	0.2	1.25676×10^{-2}	1.47581×10^{-4}	1.70753×10^{-2}	1.9756×10^{-4}
	−0.2	1.27075×10^{-4}	2.70114×10^{-6}	2.8921×10^{-4}	3.61505×10^{-6}

Table 6 Comparison of *ARL* of the EWMA and EEWMA control charts for AR(1) using explicit formulas when $\lambda_2 = 0.04$, η =1.189693, $\phi_1 = 0.632661$, $ARL_0 = 370$, and $\alpha_0 = 0.532714$

δ	$\lambda_1 = 0.05$ EWMA $h = 0.012769$	$\lambda_1 = 0.05$ EEWMA $b = 0.000315005$	$\lambda_1 = 0.10$ EWMA $h = 0.2068145$	$\lambda_1 = 0.10$ EEWMA $b = 0.0356301$
0.00	370.005	370.006	370.001	370.002
0.01	337.969	324.912	313.865	286.752
0.03	283.413	252.537	236.116	190.843
0.05	239.204	198.265	185.295	137.954
0.07	203.129	157.136	149.839	104.957
0.09	173.498	125.656	123.933	82.717
0.10	160.679	112.727	113.490	74.184
0.30	44.158	18.751	33.114	17.169
0.50	17.207	5.552	15.686	7.388
1.00	4.126	1.476	5.429	2.539

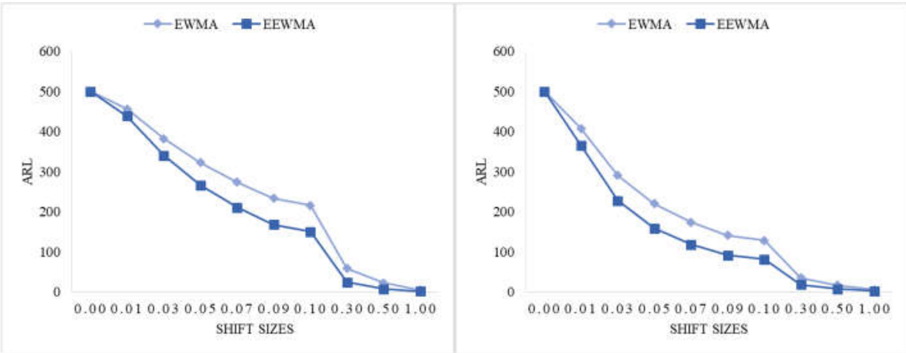
Table 7 Comparison of ARL of the EWMA and EEWMA control charts for AR(1) using explicit formulas when $\lambda_2 = 0.04$, $\eta = 1.189693$, $\phi_1 = 0.632661$, $ARL_0 = 500$, and $\alpha_0 = 0.532714$

δ	$\lambda_1 = 0.05$ EWMA $h = 0.01681$	$\lambda_1 = 0.05$ EEWMA $b = 0.00041388$	$\lambda_1 = 0.10$ EWMA $h = 0.2112752$	$\lambda_1 = 0.10$ EEWMA $b = 0.0367438$
0.00	500.004	500.006	500.001	500.006
0.01	456.506	438.416	407.489	365.084
0.03	382.475	339.830	290.484	227.551
0.05	322.532	266.144	220.223	158.771
0.07	273.656	210.459	173.853	118.149
0.09	233.538	167.945	141.271	91.712
0.10	216.192	150.514	128.437	81.765
0.30	58.882	24.536	35.267	18.083
0.50	22.667	7.013	16.409	7.679
1.00	5.159	1.627	5.584	2.596



Exponential smoothing parameter $\lambda = 0.05$ Exponential smoothing parameter $\lambda = 0.10$

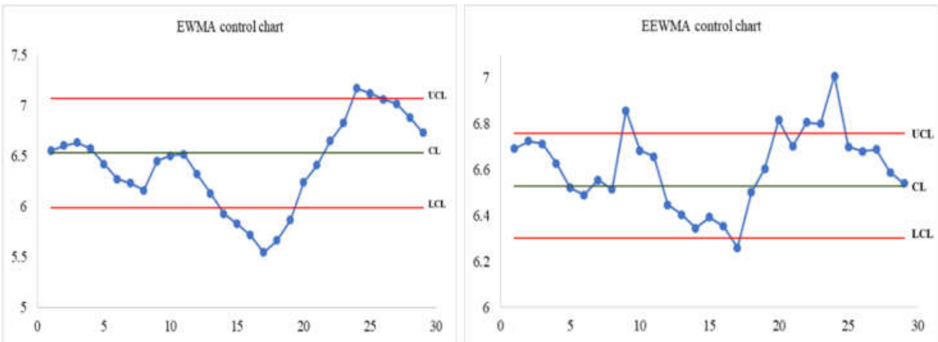
(a) (b)



Exponential smoothing parameter $\lambda = 0.05$ Exponential smoothing parameter $\lambda = 0.10$

(c) (d)

Figure 1 The ARL on the EWMA and EEWMA control charts with real data: (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$ for $ARL_0 = 370$, (c) $\lambda = 0.05$ and (d) $\lambda = 0.10$ for $ARL_0 = 500$



(a) (b)

Figure 2 The detection of the process with real data for exponential smoothing parameter $\lambda = 0.05$: (a) the EWMA control chart and (b) the EEWMA control chart

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