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Size Biased Lindley-Quasi Xgamma Distribution Applicable to Survival Times

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Abstract

We have formulated a new non-decreasing hazard rate model as a size biased version of Lindley-quasi Xgamma distribution known as size biased Lindley-quasi Xgamma distribution. The mathematical and statistical properties of the newly introduced model have been obtained. Method of maximum likelihood estimation is employed for estimating unknown parameters of proposed model. Simulation study for checking the performance of maximum likelihood estimates is performed. Proposed model and its related models are fitted to two lifetime data sets and goodness of fit of proposed model over its related models to two lifetime data sets is tested. For testing the goodness of fit of our proposed model we have used Kolmogorov D statistic and loss of information measures AIC, BIC, AICC, and HQIC.

Keywords: Hazard rate, simulation study, Renyi entropy, maximum likelihood estimation, goodness of fit.

1. Introduction

Probability models have found greater applicability in analyzing data and improving decision making. Variety of probability models have been fitted by researchers over decades to different types of real life problems. There are many cases where stochastic process produces observations with unequal probability of being recorded, instead the observations are recorded according to some weight function. When the observations are recorded with probability proportional to some measure of unit size then the resulting distribution is known as size biased distribution. In the area of size biased distributions lot of work has been done by researchers over decades. Warren (1975) applied the size biased distributions in connection with sampling wood cells. Ayesha (2017) introduced size biased Lindley distribution and discussed its various properties. Ducey and Gove (2015) introduced size biased distributions in the generalized beta distribution family. Hassan et al. (2019) introduced a new generalization of Ishita distribution and obtained vital mathematical properties of the distribution along with applications of the proposed model. Das and Roy (2011) studied the length biased weighted generalized Rayleigh distribution with properties and applications. Mir and Ahmad (2009) introduced size biased distribution with applications. Beghriche and Zeghdoudi (2019) introduced size biased gamma Lindley distribution. Rather et.al (2018) studied size biased Ailamujia distribution and obtained its properties and applications. Patil and Rao (1978) introduced weighted distributions and size biased sampling with applications to wild life populations and human families and obtained its properties. Hassan et al. (2018) formulated three parameter Quasi Lindley distribution by using weighting technique and obtained various properties of that model. Hassan et al. (2020) introduced Poisson Pranav distribution and obtained its various mathematical properties along with obtaining applications of the proposed model. R core team (2019) developed R software version 3.5.3 which we have used for analyzing data in this paper. Hassan et al. (2020) proposed an introduced Lindley-quasi Xgamma distribution (LQXD) with probability density function (pdf), cumulative distribution function (cdf) and mean $E(X)$ given below in (1)-(3), respectively

$$f\left(x,\alpha,\theta\right)=\frac{\theta e^{-\theta x}}{\left(\theta+\alpha\right)^{2}}\left(\left(\alpha+\theta\right)\left(\alpha+\frac{x^{2}\theta^{2}}{2}\right)+\theta\left(\theta-1\right)\left(1+\alpha x\right)\right),\ x>0,\theta>0,\alpha\geq 0, \tag{1}$$

$$F(x)=\frac{1}{2(\theta+\alpha)^{2}}\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^{2}\theta^{2}+2\theta x+2\right)e^{-\theta x}\right\}\\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right], \tag{2}$$

$$E(X)=\frac{\left\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\right\}}{\theta(\theta+\alpha)^{2}}. \tag{3}$$

The important statistical properties along with application in real life were studied for the model given in (1).

In this paper, we have obtained size biased version of Lindley-quasi Xgamma (LQXD) distribution with pdf given in (1). The motivation and objective behind working on this paper is to introduce the size biased version of Lindley-quasi Xgamma distribution for modeling of unequally recorded observations. Proposed model is developed for increasing flexibility in respect of skewness, kurtosis, etc., and for better fitting of complex data than base model and related models. The proposed model is described in Section 2 of paper. In Section 3, need of proposed model is discussed. Reliability analysis of proposed size biased Lindley-quasi Xgamma distribution distribution is introduced in Section 4. Statistical properties of proposed model are obtained in Section 5. Expressions for order statistics are obtained in Section 6. In Section 7, Bonferroni and Lorenz curves and indices are discussed. Section 8 presents the Renyi entropy. Section 9 deals with estimation of unknown parameters of proposed model. Section 10 deals with quantiles of proposed model. Simulation analysis is provided in Section 11. Real life applications are presented in Section 12. In Section 13, conclusions are presented. Appendix is given at the end of paper.

2. Size Biased Lindley-Quasi Xgamma Distribution

Suppose X is a non-negative random variable with pdf $f\left(x\right)$. Then, the pdf of the size biased random variable X_{sb} is given by

$$f_{sb}\left(x\right)=\frac{xf\left(x\right)}{E\left(X\right)},\ x>0. \tag{4}$$

Using (4), (1) and (3), the probability density function $f_{sb}\left(x\right)$ of size biased Lindley-quasi Xgamma distribution (SBLQXD) with scale parameter θ and shape parameter α is given in (5) obtained as

$$f_{sb}\left(x\right)=\frac{xf\left(x\right)}{E\left(X\right)},$$

$$f_{sb}(x)=\frac{\theta^2\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)},\ x>0,\theta>0,\alpha\geq 0, \tag{5}$$

where $f(x)$ and $E(X)$ are the pdf and the mean of Lindley-quasi Xgamma distribution given in (1) and (3), respectively.

The plots of pdf for different values of parameters are given in Figure 1 below indicating that proposed model is positively skewed for higher values of θ i.e., for $\theta>1.5$ our model is leptokurtic for $\theta<1.5$ our model is platykurtic.

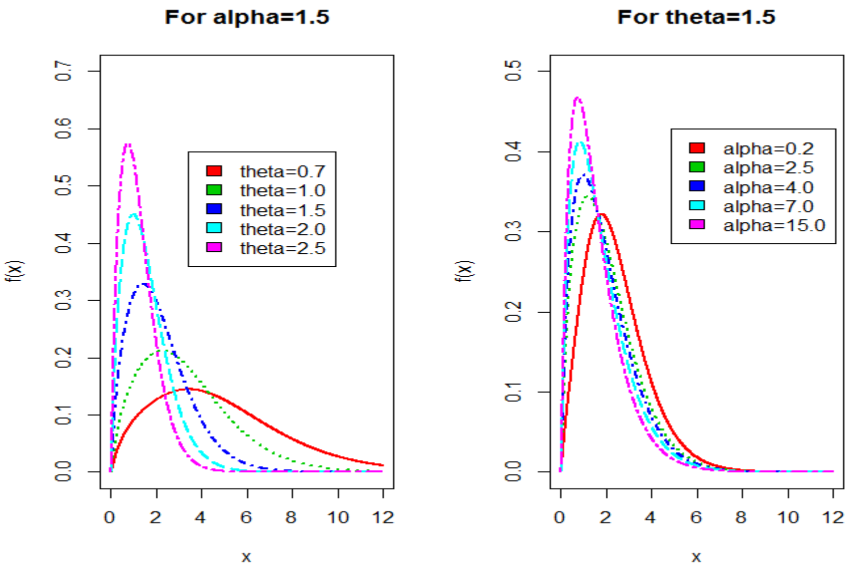

Figure 1 Graph of density function

The corresponding cdf of size biased Lindley-quasi Xgamma distribution is given in (6) and obtained as

$$F_{sb}(x)=1-\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}. \tag{6}$$

The cdf plots of SBLQXD are given in Figure 2 for different values of parameters.

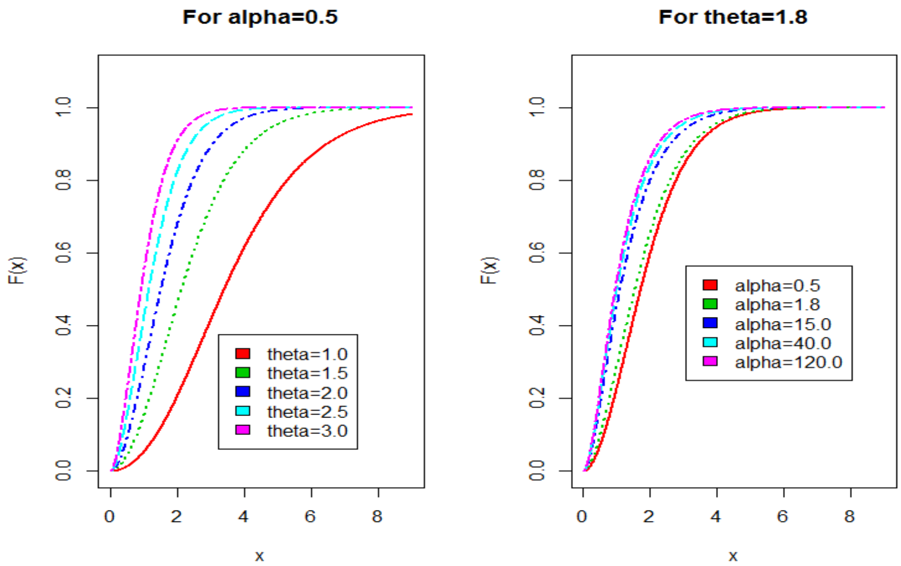

Figure 2 Graph of distribution function

3. Motivation and Need of Proposed Model

As can be seen from graph of hazard rate in Figure 4, the proposed model has non decreasing hazard rate which is common phenomenon in many real life situations. Also from graphs of probability density function it is observed that proposed model is platykurtic as well as leptokurtic for different parameter values. Simulation, hazard rate and applicability for survival times have shown flexibility and need of proposed model in real life. It's the flexibility and applicability of proposed model which motivated us to work on this model.

4. Reliability Measures

This division of paper presents survival function, hazard rate, reverse hazard rate of the proposed size biased Lindley-quasi Xgamma distribution for random variable X, where X represents the lifetime of a system.

4.1. Reliability function R(x)

The reliability function or survival function $R(x)$ gives the numerical value of odds of surviving of a system beyond a specified time (t) .

Mathematically,

$$R_{sb}(x) = P(X > t) = 1 - F_{sb}(x).$$

The reliability function or the survival function of size biased Lindley-quasi Xgamma distribution is obtained as

$$R_{sb}\left(x\right)=\left[1+\frac{\theta x\left(\left(\alpha+\theta\right)\left(2\alpha+6+3x\theta+x^{2}\theta^{2}\right)+2\left(\theta-1\right)\left(\theta+2\alpha+x\theta\alpha\right)\right)}{2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}.$$

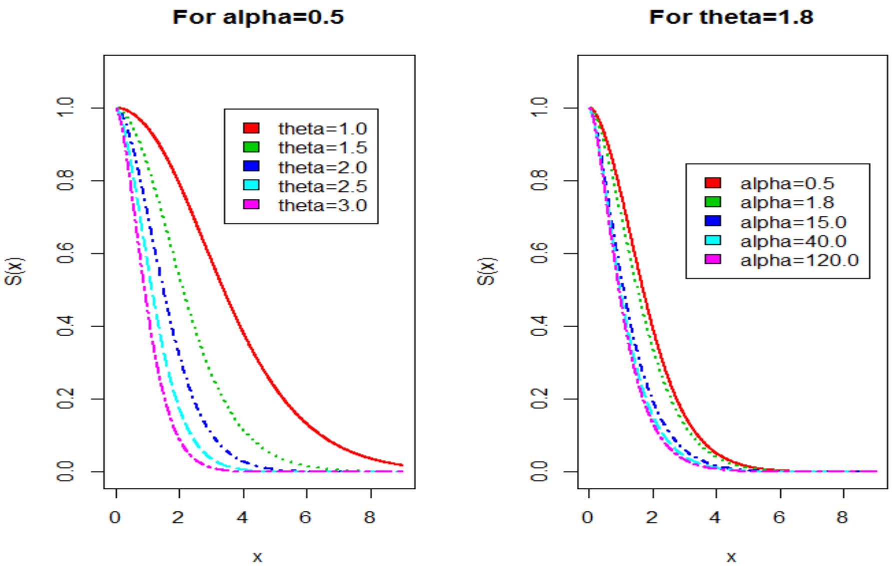

Figure 3 Graph of survival function

The above graph represents the survival function of SBLQXD for different parameter values.

4.2. Hazard function

The hazard function which is defined as chance that a system which is surviving up to time "t" will fail in the small time interval after "t" is obtained for SBLQXD as

$$H.R = h_{sb}(x) = \frac{f_{sb}(x)}{R_{sb}(x)}$$

$$= \frac{\left[2\theta^2\left((\alpha+\theta)\left(\alpha x + \frac{x^3\theta^2}{2}\right) + \theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}\right]}{\left[\left[2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)+\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)\right]e^{-\theta x}\right]}.$$

The graphs of hazard rate of SBLQXD for different values of parameter are given below. From the graphs of hazard rate, it is revealed that our proposed model possesses non-decreasing hazard rate and it can be also seen that hazard rate becomes constant as value of x increases. There are many situations in real life where hazard rate is non-decreasing, like lifetime of human beings, animals etc.

4.3. Reverse hazard rate

The reverse hazard rate of the size biased Lindley-quasi Xgamma distribution is given as

$$R.H.R = h_{rsb}(x) = \frac{f_{sb}(x)}{F_{sb}(x)}$$

$$=\frac{\left[2\theta^{2}\left(\left(\alpha+\theta\right)\left(\alpha x+\frac{x^{3}\theta^{2}}{2}\right)+\theta\left(\theta-1\right)\left(x+\alpha x^{2}\right)\right)e^{-\theta x}\right]}{\left[2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)-\left[2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)+\theta x\begin{pmatrix}\left(\alpha+\theta\right)\left(2\alpha+6+3x\theta+x^{2}\theta^{2}\right)+\\2\left(\theta-1\right)\left(\theta+2\alpha+x\theta\alpha\right)\end{pmatrix}\right]e^{-\theta x}\right]}.$$

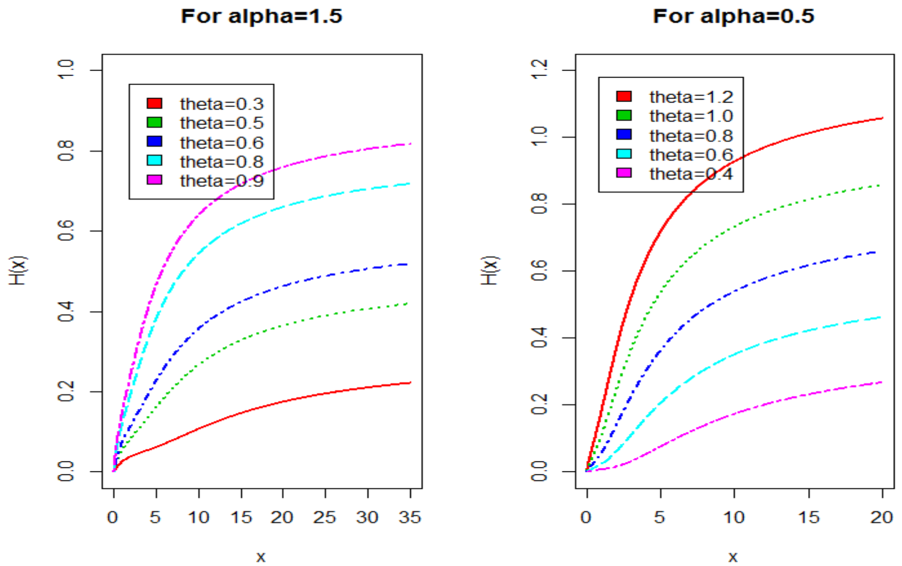

Figure 3 Graph of hazard function

4.4. Mean residual life

For a continuous random variable X following size biased Lindley-quasi Xgamma distribution mean residual life $m(x)$ is

$$m(x)=\frac{\left\{\begin{array}{l}2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)\left[\frac{2\left(\left(\alpha+\theta\right)\left(\alpha+6\right)+\left(\theta-1\right)\left(\theta+3\alpha\right)\right)}{\theta\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)}\right]-\\ \frac{1}{\theta}\left\{\begin{array}{l}2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)(1-e^{-\theta x})+(\alpha+\theta)\\ \left((2\alpha+6)\left(1-(x\theta+1)e^{-\theta x}\right)+6-e^{-\theta x}(3\theta^{2}x^{2}+6\theta x+6)+6-e^{-\theta x}(x^{3}\theta^{3}+3x^{2}\theta^{2}+6\theta x+6)\right)\\ +2(\theta-1)\left((\theta+2\alpha)(1-(x\theta+1)e^{-\theta x})+\alpha(2-e^{-\theta x}(x^{2}\theta^{2}+2\theta x+2))\right)\end{array}\right\}\end{array}\right\}}{\left[2\left(\theta^{2}+\alpha^{2}+3\alpha\theta+2\theta+\alpha\right)+\theta x\left(\left(\alpha+\theta\right)\left(2\alpha+6+3x\theta+x^{2}\theta^{2}\right)+2\left(\theta-1\right)\left(\theta+2\alpha+x\theta\alpha\right)\right)\right]e^{-\theta x}}. \quad (7)$$

The proof of (7) is given in Appendix II.

5. Statistical Properties

Moments, characteristic function, mean deviation, skewness, coefficient of variation characterize probability models. Here we have obtained these statistical properties for proposed size biased Lindley-quasi Xgamma distribution.

5.1. Moments

Assuming X to be a random variable having size biased Lindley-quasi Xgamma distribution with parameters θ and α. Then the r^{th} moment about origin for a given probability distribution is given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f\left(x\right)dx;\ \ r = 1,\ 2,\ 3\ldots$$

$$= \int_0^\infty x^r \frac{\theta^2\left(\left(\alpha+\theta\right)\left(\alpha x + \frac{x^3\theta^2}{2}\right) + \theta\left(\theta-1\right)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}dx$$

$$\mu_r' = \left[\frac{(r+1)!\left(\left(\alpha+\theta\right)\left(\alpha+\frac{(r+3)(r+2)}{2}\right)+(\theta-1)\left(\theta+\alpha(r+2)\right)\right)}{\theta^r\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]. \tag{8}$$

Put $r=1$ in (8), we get

$$\mu_1' = \left[\frac{2\left(\left(\alpha+\theta\right)\left(\alpha+6\right)+\left(\theta-1\right)\left(\theta+3\alpha\right)\right)}{\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right],$$

which is mean of the SBLQXD.

Put $r=2$ in (8), we get

$$\mu_2' = \left[\frac{6\left(\left(\alpha+\theta\right)\left(\alpha+10\right)+\left(\theta-1\right)\left(\theta+4\alpha\right)\right)}{\theta^2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right].$$

Put $r=3$ in (8), we get

$$\mu_3' = \left[\frac{24\left(\left(\alpha+\theta\right)\left(\alpha+15\right)+\left(\theta-1\right)\left(\theta+5\alpha\right)\right)}{\theta^3\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right].$$

Put $r=4$ in (8), we get

$$\mu_4' = \left[\frac{120\left(\left(\alpha+\theta\right)\left(\alpha+21\right)+\left(\theta-1\right)\left(\theta+6\alpha\right)\right)}{\theta^4\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right].$$

The moments about mean are given as

$$\mu_2 = \left[\frac{\left\{\left(6\left(\left(\alpha+\theta\right)\left(\alpha+10\right)+\left(\theta-1\right)\left(\theta+4\alpha\right)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)-4\left(\left(\alpha+\theta\right)\left(\alpha+6\right)+\left(\theta-1\right)\left(\theta+3\alpha\right)\right)^2\right\}}{\theta^2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^2}\right],$$

which is the variance of SBLQXD.

$$\mu_3 = \left[\frac{\begin{array}{l}\left(24\left((\alpha+\theta)(\alpha+15)+(\theta-1)(\theta+5\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^2\right)- \\ 36\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left(\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\right)\left(\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)\right) \\ +16\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^3\end{array}}{\theta^3\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^3}\right]$$

$$\mu_4 = \left[\frac{\begin{array}{l}\left(120\left((\alpha+\theta)(\alpha+21)+(\theta-1)(\theta+6\alpha)\right)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^3- \\ 192\left(\left((\alpha+\theta)(\alpha+15)+(\theta-1)(\theta+5\alpha)\right)\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^2\right)+ \\ 144\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left(\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\right)\left(\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right) \\ -48\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^4\end{array}}{\theta^4\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^4}\right]$$

5.2. Coefficient of variation, skewness, kurtosis and Index of Dispersion of Size Biased Lindley-Quasi Xgamma Distribution (SBLQXD).

The coefficient of variation (CV), coefficient of skewness $(\sqrt{\beta_1})$, coefficient of kurtosis (β_2) and index of dispersion (γ) of the SBLQXD are determined as

$$CV = \frac{(\mu_2)^{\frac{1}{2}}}{\mu_1'}$$

$$= \frac{\left[\left(6\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)-4\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right]^{\frac{1}{2}}}{2\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)},$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}}$$

$$= \left\{\frac{\left[\begin{array}{l}\left(24\left((\alpha+\theta)(\alpha+15)+(\theta-1)(\theta+5\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^2\right)- \\ 36\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left(\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\right)\left(\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)\right) \\ +16\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^3\end{array}\right]}{\left[\left(6\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)-4\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right]^{3/2}}\right\},$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}$$

$$=\left\{\frac{\left[\begin{array}{l}\left(120\left((\alpha+\theta)(\alpha+21)+(\theta-1)(\theta+6\alpha)\right)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^3-\\ 192\left(\left((\alpha+\theta)(\alpha+15)+(\theta-1)(\theta+5\alpha)\right)\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^2\right)+\\ 144\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left(\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\right)\left(\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right)\\ -48\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^4\end{array}\right]}{\left[\left(6\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)-4\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right]^2}\right\},$$

$$\gamma=\frac{\mu_2}{\mu_1'}=\frac{\left[\left(6\left((\alpha+\theta)(\alpha+10)+(\theta-1)(\theta+4\alpha)\right)\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)-4\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)^2\right]}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)}.$$

5.3. Mean deviation about mean and median of size biased Lindley-quasi Xgamma distribution (SBLQXD)

We have derived the expressions for mean deviation about mean and median of SBLQXD in this section.

Theorem 1 *If X has the $SBLQXD(\theta,\alpha)$, then the mean deviation about mean $(\delta_1(X))$ and mean deviation about median $(\delta_2(X))$ are given as*

$$\delta_1(X)=\left[\begin{array}{l}2\mu\left\{1-\left[1+\frac{\theta\mu\left((\alpha+\theta)\left(2\alpha+6+3\mu\theta+\mu^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+\mu\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta\mu}\right\}\\ -2\left\{\mu-\left\{\mu+\frac{\theta\mu\left(\begin{array}{l}(\alpha+\theta)\left(4\alpha+2\alpha\mu\theta+24+12\mu\theta+4\mu^2\theta^2+\mu^3\theta^3\right)+\\ 2(\theta-1)\left(2\theta+\mu\theta^2+6\alpha+3\mu\alpha\theta+\mu^2\theta^2\alpha\right)\end{array}\right)}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta\mu}\right\}\end{array}\right],$$

and

$$\delta_2(X)=\left[\mu-2\left\{\mu-\left\{\mu+\frac{\theta M\left(\begin{array}{l}(\alpha+\theta)\left(4\alpha+2\alpha M\theta+24+12M\theta+4M^2\theta^2+M^3\theta^3\right)+\\ 2(\theta-1)\left(2\theta+M\theta^2+6\alpha+3M\alpha\theta+M^2\theta^2\alpha\right)\end{array}\right)}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta M}\right\}\right],$$

respectively.

Proof: Mean deviation about mean and mean deviation about median are defined as

$$\delta_1(X)=\int_0^\infty|x-\mu|f_{sb}(x)dx \text{ and } \delta_2(X)=\int_0^\infty|x-M|f_{sb}(x)dx,$$

respectively, where μ and M are mean and median respectively of random variable $X\sim SBLQXD$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be obtained by using the simplified relationships.

$$\delta_1(X)=\int_0^{\mu}(\mu-x)f_{sb}(x)dx+\int_{\mu}^{\infty}(x-\mu)f_{sb}(x)dx=2\mu F_{sb}(\mu)-2\int_0^{\mu}xf_{sb}(x)dx. \tag{9}$$

And

$$\delta_2(X)=\int_0^{M}(M-x)f_{sb}(x)dx+\int_{M}^{\infty}(x-M)f_{sb}(x)dx=\mu-2\int_0^{M}xf_{sb}(x)dx, \tag{10}$$

where $f_{sb}\left(x\right)=\dfrac{\theta^2\left(\left(\alpha+\theta\right)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta\left(\theta-1\right)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}$.

Now,

$$\int_0^{\mu}xf(x)dx=\left[\mu-\left\{\mu+\frac{\theta\mu\begin{pmatrix}\left(\alpha+\theta\right)\left(4\alpha+2\alpha\mu\theta+24+12\mu\theta+4\mu^2\theta^2+\mu^3\theta^3\right)+\\2\left(\theta-1\right)\left(2\theta+\mu\theta^2+6\alpha+3\mu\alpha\theta+\mu^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta\mu}\right]. \tag{11}$$

And

$$\int_0^{M}xf(x)dx=\left[\mu-\left\{\mu+\frac{\theta M\begin{pmatrix}\left(\alpha+\theta\right)\left(4\alpha+2\alpha M\theta+24+12M\theta+4M^2\theta^2+M^3\theta^3\right)+\\2\left(\theta-1\right)\left(2\theta+M\theta^2+6\alpha+3M\alpha\theta+M^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta M}\right]. \tag{12}$$

Using expressions (9), (10), (11) and (12) and expression for cdf in (6), we obtain mean deviation about mean $(\delta_1(X))$ and mean deviation about median $(\delta_2(X))$,

$$\delta_1(X)=\begin{bmatrix}2\mu\left\{1-\left[1+\dfrac{\theta\mu\left(\left(\alpha+\theta\right)\left(2\alpha+6+3\mu\theta+\mu^2\theta^2\right)+2\left(\theta-1\right)\left(\theta+2\alpha+\mu\theta\alpha\right)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta\mu}\right\}\\-2\left\{\mu-\left\{\mu+\dfrac{\theta\mu\begin{pmatrix}\left(\alpha+\theta\right)\left(4\alpha+2\alpha\mu\theta+24+12\mu\theta+4\mu^2\theta^2+\mu^3\theta^3\right)+\\2\left(\theta-1\right)\left(2\theta+\mu\theta^2+6\alpha+3\mu\alpha\theta+\mu^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta\mu}\right\}\end{bmatrix},\text{ and}$$

$$\delta_2(X)=\left[\mu-2\left\{\mu-\left\{\mu+\frac{\theta M\begin{pmatrix}\left(\alpha+\theta\right)\left(4\alpha+2\alpha M\theta+24+12M\theta+4M^2\theta^2+M^3\theta^3\right)+\\2\left(\theta-1\right)\left(2\theta+M\theta^2+6\alpha+3M\alpha\theta+M^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta M}\right\}\right].$$

5.4. Probability weighted moments

The probability weighted moment $M_{h,r,\beta}$ of size biased Lindley-quasi Xgamma distribution is

$$M_{h,r,\beta}=\left\{\begin{array}{l}\dfrac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\sum_{i=0}^{\infty}\binom{r}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{p=0}^{l}\binom{l}{p}\sum_{v=0}^{j-k}\binom{j-k}{v}\\ \dfrac{\begin{pmatrix}\theta^{2j+k-l-p-v}(\alpha+\theta)^k 2^{p+j-k}(\alpha+3)^p 3^{l-p}\\(\theta-1)^{j-k}(\theta+2\alpha)^v\alpha^{j-k-v}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^j}\sum_{s=0}^{\infty}\binom{\beta}{s}\sum_{t=0}^{s}\binom{s}{t}\sum_{m=0}^{t}\binom{t}{m}\sum_{c=0}^{m}\binom{m}{c}\sum_{q=0}^{s-t}\binom{s-t}{q}\\ \dfrac{\begin{pmatrix}\theta^{2s+t-m-c-q}(\alpha+\theta)^t 2^{c+s-t}(\alpha+3)^c 3^{m-c}\\(\theta-1)^{s-t}(\theta+2\alpha)^q\alpha^{s-t-q}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^s}\\ \left(\begin{array}{l}(\alpha+\theta)\left(\dfrac{\alpha\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+1\end{pmatrix}!}{\left(\theta(1+i+s)\right)^{2j-k-l-v-p+2s+t-m-c-q+h+2}}+\dfrac{\theta^2\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+3\end{pmatrix}!}{2\left(\theta(1+i+s)\right)^{2j-k-l-v-p+2s+t-m-c-q+h+4}}\right)\\ +\theta(\theta-1)\left(\dfrac{\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+1\end{pmatrix}!}{\left(\theta(1+i+s)\right)^{2j-k-l-v-p+2s+t-m-c-q+h+2}}+\dfrac{\alpha\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+2\end{pmatrix}!}{\left(\theta(1+i+s)\right)^{2j-k-l-v-p+2s+t-m-c-q+h+3}}\right)\end{array}\right)\end{array}\right\}.$$

The proof of $M_{h,r,\beta}$ is given in Appendix I.

5.5. Moment generating function, characteristic function and probability generating function of size biased Lindley-quasi Xgamma distribution (SBLQXD)

We will derive moment generating function, characteristic function and probability generating function of SBLQXD in this segment of paper.

Theorem 2 *If $X \sim SBLQXD(\theta,\alpha)$ then the moment generating function $M_X(t)$ and characteristic function $\varphi_X(t)$ are*

$$M_X(t)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-t)^2}+\frac{3\theta^2}{(\theta-t)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-t)^2}+\frac{2\alpha}{(\theta-t)^3}\right)\right)\right],$$

$$\varphi_X(t)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-it)^2}+\frac{3\theta^2}{(\theta-it)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-it)^2}+\frac{2\alpha}{(\theta-it)^3}\right)\right)\right],$$

and

$$P(s)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-\log s)^2}+\frac{3\theta^2}{(\theta-\log s)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-\log s)^2}+\frac{2\alpha}{(\theta-\log s)^3}\right)\right)\right],$$

respectively.

Proof: We begin with the well-known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tX}\right)=\int_0^\infty e^{tx} f_{sb}(x)\,dx$$

$$=\int_0^\infty e^{tx}\frac{\theta^2\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\,dx$$

$$M_X(t)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-t)^2}+\frac{3\theta^2}{(\theta-t)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-t)^2}+\frac{2\alpha}{(\theta-t)^3}\right)\right)\right],$$

which is the moment generating function of size biased Lindley-quasi Xgamma distribution. Also we know that $\varphi_X(t)=M_X(it)$. Therefore,

$$\varphi_X(t)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-it)^2}+\frac{3\theta^2}{(\theta-it)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-it)^2}+\frac{2\alpha}{(\theta-it)^3}\right)\right)\right],$$

which is the characteristic function of SBLQXD distribution. Also we know the relationship between moment generating function is $e^t=s$. So probability generating function of SBLQXD is

$$P(s)=\left[\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\left((\alpha+\theta)\left(\frac{\alpha}{(\theta-\log s)^2}+\frac{3\theta^2}{(\theta-\log s)^4}\right)+\theta(\theta-1)\left(\frac{1}{(\theta-\log s)^2}+\frac{2\alpha}{(\theta-\log s)^3}\right)\right)\right].$$

6. Order Statistics of Size Biased Lindley-Quasi Xgamma Distribution

Consider $X_{(1)},X_{(2)},....,X_{(n)}$ to be the ordered statistics of the random sample $x_1,x_2,...,x_n$ obtained from the size biased Lindley-quasi Xgamma distribution with cumulative distribution function $F_{sb}(x)$ and probability density function $f_{sb}(x)$, then the probability density function of v^{th} order statistics $X_{(v)}$ is given by

$$f_{vsb}(x)=\frac{n!}{(v-1)!(n-v)!}f_{sb}(x)\left[F_{sb}(x)\right]^{v-1}\left[1-F_{sb}(x)\right]^{n-v},\ v=1,\ 2,\ 3\ldots n.$$

Using the equations (5) and (6), the probability density function of v^{th} order statistics of size biased Lindley-quasi Xgamma distribution is given by

$$f_{(v)sb}(x)=\left[\begin{array}{l}\frac{n!}{(v-1)!(n-v)!}\left\{\frac{\theta^2\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}\\ \left[1-\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^{v-1}\\ \left[\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^{n-v}\end{array}\right].$$

Then, the pdf of first order statistic $X_{(1)}$ of size biased Lindley-quasi Xgamma distribution is given by

$$f_{(1)sb}(x)=\left[\begin{array}{l} n\left\{\dfrac{\theta^2\left((\alpha+\theta)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\} \\ \left[\left[1+\dfrac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^{n-1}. \end{array}\right],$$

and the pdf of n^{th} order statistic $X_{(n)}$ of size biased Lindley-quasi Xgamma distribution is given as

$$f_{(n)sb}(x)=\left[\begin{array}{l} n\left\{\dfrac{\theta^2\left((\alpha+\theta)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\} \\ \left[1-\left[1+\dfrac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^{n-1} \end{array}\right].$$

7. Bonferroni and Lorenz Curves and Indices of SBLQXD

The Bonferroni curve $(B(p))$, Lorenz curve $(L(p))$, Bonferroni index (B) and Gini index (G) have find applicability in fields of economics, demography, reliability, life testing and medical sciences. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q xf_{sb}(x)dx, \tag{13}$$

$$L(p)=\frac{1}{\mu}\int_0^q xf_{sb}(x)dx, \tag{14}$$

where $\mu=E(X)$ is the mean of SBLQXD and $q=F^{-1}(p)$. The Bonferroni and Gini indices are defined as

$$B=1-\int_0^1 B(p)dp, \tag{15}$$

$$G=1-2\int_0^1 L(p)dp. \tag{16}$$

Using the pdf in (5) of SBLQXD, we get

$$\int_0^q xf(x)dx=\left[\mu-\left\{\mu+\frac{\theta q\left(\begin{array}{l}(\alpha+\theta)\left(4\alpha+2\alpha q\theta+24+12q\theta+4q^2\theta^2+q^3\theta^3\right)+\\ 2(\theta-1)\left(2\theta+q\theta^2+6\alpha+3q\alpha\theta+q^2\theta^2\alpha\right)\end{array}\right)}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta q}\right]. \tag{17}$$

Using Equation (17) in (13) and (14), we get

$$B(p)=\frac{1}{p\mu}\left[\mu-\left\{\mu+\frac{\theta q\begin{pmatrix}(\alpha+\theta)\left(4\alpha+2\alpha q\theta+24+12q\theta+4q^2\theta^2+q^3\theta^3\right)+\\2(\theta-1)\left(2\theta+q\theta^2+6\alpha+3q\alpha\theta+q^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta q}\right], \quad (18)$$

and

$$L(p)=\frac{1}{\mu}\left[\mu-\left\{\mu+\frac{\theta q\begin{pmatrix}(\alpha+\theta)\left(4\alpha+2\alpha q\theta+24+12q\theta+4q^2\theta^2+q^3\theta^3\right)+\\2(\theta-1)\left(2\theta+q\theta^2+6\alpha+3q\alpha\theta+q^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta q}\right]. \quad (19)$$

Using Equations (18) and (19) in (15) and (16), we get

$$B=1-\frac{1}{\mu}\left[\mu-\left\{\mu+\frac{\theta q\begin{pmatrix}(\alpha+\theta)\left(4\alpha+2\alpha q\theta+24+12q\theta+4q^2\theta^2+q^3\theta^3\right)+\\2(\theta-1)\left(2\theta+q\theta^2+6\alpha+3q\alpha\theta+q^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta q}\right],$$

$$L=1-\frac{2}{\mu}\left[\mu-\left\{\mu+\frac{\theta q\begin{pmatrix}(\alpha+\theta)\left(4\alpha+2\alpha q\theta+24+12q\theta+4q^2\theta^2+q^3\theta^3\right)+\\2(\theta-1)\left(2\theta+q\theta^2+6\alpha+3q\alpha\theta+q^2\theta^2\alpha\right)\end{pmatrix}}{2\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right\}e^{-\theta q}\right].$$

8. Renyi Entropy

Entropy measures the variation of uncertainty of random variable. Renyi entropy $T_R(\gamma)$ of random variable X following size biased Lindley-quasi Xgamma distribution is obtained as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left(\int_0^\infty f^\gamma(x)dx\right),$$

where $\gamma>0$ and $\gamma\neq 1$,

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left(\int_0^\infty\frac{\theta^{2\gamma}\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)^\gamma e^{-\gamma\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^\gamma}dx\right). \quad (20)$$

Using the binomial expansion,

$$\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)^\gamma$$

$$\sum_{t=0}^{\gamma}\binom{r}{t}(\alpha+\theta)^{t}\alpha^{t}\sum_{l=0}^{\infty}\binom{t}{l}\frac{\theta^{2l}}{(2\alpha)^{l}}\theta^{r-t}(\theta-1)^{r-t}\sum_{v=0}^{\infty}\binom{r-t}{v}\alpha^{v}x^{\gamma+2l+v}. \tag{21}$$

Using (21) in (20), we get

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\left(\sum_{t=0}^{\gamma}\binom{r}{t}(\alpha+\theta)^{t}\sum_{l=0}^{\infty}\binom{t}{l}\frac{1}{(2)^{l}}(\theta-1)^{r-t}\sum_{v=0}^{\infty}\binom{r-t}{v}\alpha^{v+t-l}\theta^{3\gamma+2l-t}\int_{0}^{\infty}x^{\gamma+2l+v}e^{-\gamma\theta x}dx\right)}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^{\gamma}}\right\},$$

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\left(\sum_{t=0}^{\gamma}\binom{r}{t}(\alpha+\theta)^{t}\sum_{l=0}^{\infty}\binom{t}{l}\frac{1}{(2)^{l}}(\theta-1)^{r-t}\sum_{v=0}^{\infty}\binom{r-t}{v}\alpha^{v+t-l}\theta^{2\gamma-t-v}(\gamma+2l+v)!\right)}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)^{\gamma}(\gamma)^{\gamma+2l+v}}\right\}.$$

9. Estimation of Parameters of SBLQXD

We used method of maximum likelihood estimation for estimating the unknown parameters of proposed model. Considering $X_{(1)}, X_{(2)}, ..., X_{(n)}$ to be the random sample of size n drawn from size biased Lindley-quasi Xgamma distribution having probability density function given by (5), then the likelihood function of SBLQXD is given as

$$L(x\mid\theta,\alpha)=\prod_{i=1}^{n}\left[\frac{\theta^2\left((\alpha+\theta)\left(\alpha x+\frac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right].$$

Taking log on both sides of likelihood function we get log likelihood function as:

$$\log L=\left\{\begin{array}{l}2n\log\theta-n\log\left(\alpha^2+\theta^2+3\alpha\theta+2\theta+\alpha\right)-\\ \theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left((\alpha+\theta)(\alpha x+\frac{x^3\theta^2}{2})+\theta(\theta-1)(x+\alpha x^2)\right)\end{array}\right\}. \tag{22}$$

Differentiating the log-likelihood function with respect to θ and α. This is done by partially differentiate (22) with respect to θ and α and equating the result to zero, we obtain the following normal equations,

$$\frac{\partial\log L}{\partial\theta}=\left[\frac{2n}{\theta}-\frac{n(2\theta+3\alpha+2)}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}+\sum_{i=1}^{n}\left(\frac{\alpha\theta x^3+\alpha x+\frac{3}{2}x^3\theta^2+\left(x+\alpha x^2\right)(2\theta-1)}{\left((\alpha+\theta)(\alpha x+\frac{x^3\theta^2}{2})+\theta(\theta-1)(x+\alpha x^2)\right)}\right)-\sum_{i=1}^{n}x_i\right]=0, \tag{23}$$

$$\frac{\partial\log L}{\partial\alpha}=\left[\sum_{i=1}^{n}\left[\frac{2\alpha x+\frac{x^3\theta^2}{2}+\theta x+2\theta(\theta-1)\alpha x}{\left((\alpha+\theta)(\alpha x+\frac{x^3\theta^2}{2})+\theta(\theta-1)(x+\alpha x^2)\right)}\right]-\frac{n(2\alpha+3\theta+1)}{\left(\theta^2+\alpha^2+3\alpha\theta++2\theta+\alpha\right)}\right]=0. \tag{24}$$

MLEs of θ and α cannot be obtained by solving above complex equations (23) (24) as these equations are not in closed form. So we solve above equations by using iteration method through R software.

10. Quantile and Random Number Generation from WQLD

Inverse CDF method is one of the methods used for the generation of random numbers from a particular distribution. In this method the random numbers from a particular distribution are generated by solving the equation obtained on equating the CDF of a distribution to a number u. The number u is itself being generated from $U(0,1)$. Thus following the same procedure for the generation of random numbers from the WQLD we will proceed.

$$F_{sb}(x) = u$$

$$1-\left[1+\frac{\theta x\left(\left(\alpha+\theta\right)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2\left(\theta-1\right)\left(\theta+2\alpha+x\theta\alpha\right)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}=u. \quad (25)$$

Equation (25) is a complex equation. It can't be solved manually. It is solved through Mathematica software to find the value of x and hence find the quantiles of the proposed model.

11. Simulation Study

In this part of paper, we have carried out the simulation study for checking the performance of maximum likelihood (ML) estimates by taking different sample sizes (n=20, 40, 60, 100). We have used the inverse CDF technique for data simulation for SBLQXD using R software. The process was repeated 1,000 times for calculation of bias, variance and mean square error (MSE) as are given values in Table 1. For two parameter combinations of SBLQXD, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite good and consistent in case of size biased Lindley-quasi Xgamma distribution.

Table 1 Simulation study of ML estimators for SBLQXD

Parameter	n	$\alpha=0.5,\theta=1.5$			$\alpha=0.2,\theta=0.5$		
		Bias	Variance	MSE	Bias	Variance	MSE
α	20	0.7666484	1.6234460	2.2111960	0.5808761	1.4819190	1.8193360
θ		0.2959874	0.4141496	0.5017582	0.2961204	0.4066763	0.4943636
α	40	0.5278281	1.0201080	1.2987100	0.3489703	0.8939187	1.0156990
θ		0.2400262	0.1896502	0.2472628	0.2210686	0.1823114	0.2311827
α	60	0.3961951	0.7833881	0.9403586	0.2673077	0.7405056	0.8119590
θ		0.2214441	0.1241444	0.1731819	0.2209415	0.1249370	0.1737520
α	100	0.2073943	0.4892112	0.5322236	0.0741192	0.4796740	0.4851676
θ		0.2209416	0.0678405	0.1166556	0.2115969	0.0773928	0.1221659

12. Applications of Size Biased Lindley-Quasi Xgamma Distribution

We fitted size biased Lindley-quasi Xgamma distribution and its related distributions to two lifetime data sets to check the superiority of our model over its related models.

Data set 1: The data set given in Table 2 represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli observed and reported by Bjerkedal (1960). It is known that guinea pigs have high susceptibility of human tuberculosis and that is why they were used in this particular study. The regimen number is the common logarithm of the number of bacillary units per 0.5 ml. (log(4.0) 6.6). Corresponding to regimen 6.6, there were 72 observations as listed in Table 2. This data set was recently used by Shukla (2019).

Table 2 Survival times (in days) of 72 guinea pigs injected with different doses of tubercle bacilli

12	15	22	24	24	32	32	33	34
38	38	43	44	48	52	53	54	54
55	56	57	58	58	59	60	60	60
60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84
85	87	91	95	96	98	99	109	110
121	127	129	131	143	146	146	175	175
211	233	258	258	263	297	341	341	376

Data Set 2: The non-censored data given in Table 3 represents the survival times (in months) of 46 patients of melanoma (non-censored data) has been taken from Kayid et.al. (2010).

Table 3 Survival times (in months) of 46 patients of melanoma (non-censored data)

3.25	3.50	4.75	4.75	5.00	5.25	5.75	5.75
6.25	6.50	6.50	6.75	6.75	7.78	8.00	8.50
8.50	9.25	9.50	9.50	10.00	11.50	12.5	13.25
13.5	14.25	14.50	14.75	15.00	16.25	16.25	16.50
17.5	21.75	22.50	24.50	25.50	25.75	27.50	29.50
31.00	32.50	34.00	34.50	35.25	58.50		

These data sets are used here only for illustrative purposes. The required numerical evaluations are carried out using R software version 3.5.3. We have fitted size biased Lindley-quasi Xgamma distribution, Lindley-quasi Xgamma distribution, another two-parameter Sujatha distribution and exponential distribution to these two real life data sets. The summary statistic of these two data sets is given in Table 4. The MLEs of the parameters with standard errors in parentheses, model functions are displayed in Table 5 for these two data sets. The corresponding log-likelihood values, AIC, AICC, HQIC, BIC, Kolmogorov statistic, p-value and Shannon's entropy are given in Tables 6 and 7 for data sets 1 and 2, respectively.

Table 4. Summary statistic of data sets 1 and 2

Data Set	No. of observations	Min.	First quartile	median	mean	Third quartile	Max.
1	72	12.00	54.75	70.00	99.82	112.75	376.00
2	46	3.25	6.75	12.88	15.66	22.81	58.50

Table 5 ML Estimates, standard error of estimates in parenthesis, model function of related models and proposed model for data sets 1 and 2

Data Set	Distribution	ML estimates with standard errors	Model function
1	Size Biased Lindley-Quasi Xgamma Distribution (SBLQXD)	$\hat{\theta}=0.02198$ (0.00263) $\hat{\alpha}=19.5497$ (22.2551)	$\dfrac{\theta^2\left((\alpha+\theta)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}$
	Another Two-Parameter Sujatha Distribution (ATPSD)	$\hat{\theta}=2.991697e-02$ (2.037893e−03 $\hat{\alpha}=1.028127e+01$ (1.648809e+03)	$\dfrac{\theta^3}{(\theta^2+\alpha\theta+2\alpha)}\left(1+\alpha x+\alpha x^2\right)e^{-\theta x}$
	Exponential Distribution (ED)	$\hat{\theta}=99.8193$ (11.7638)	$\dfrac{1}{\theta}e^{-\frac{x}{\theta}}$
	Lindley-Quasi Xgamma Distribution (LQXD)	$\hat{\theta}=0.035612$ (0.002954) $\hat{\alpha}=0.5699$ (0.0634)	$\dfrac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\dfrac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}$
2	Size Biased Lindley-Quasi Xgamma Distribution (SBLQXD)	$\hat{\theta}=0.13412$ (0.0372) $\hat{\alpha}=41.4168$ (231.3287)	$\dfrac{\theta^2\left((\alpha+\theta)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}$
	Another Two-Parameter Sujatha Distribution (ATPSD)	$\hat{\theta}=0.1858$ (0.01589) $\hat{\alpha}=6.3286$ (30.2251)	$\dfrac{\theta^3}{(\theta^2+\alpha\theta+2\alpha)}\left(1+\alpha x+\alpha x^2\right)e^{-\theta x}$
	Exponential Distribution (ED)	$\hat{\theta}=15.6582$ (2.3086)	$\dfrac{1}{\theta}e^{-\frac{x}{\theta}}$
	Lindley-Quasi Xgamma Distribution (LQXD)	$\hat{\theta}=0.06482$ (0.009840) $\hat{\alpha}=40.19040$ (54.4436)	$\dfrac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\dfrac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}$

Table 6 Model comparison, Kolmogorov statistic, p-value of proposed model and its related models for data set 1

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	K-S distance (D)	p-value	Shannon Entropy $(H(X))$
SBLQXD	393.792	791.584	796.137	791.757	793.396	0.1246	0.21360	5.469
LQXD	419.525	843.051	847.225	843.225	844.864	0.2292	0.00103	5.826
ATPSD	397.240	798.480	803.034	798.654	800.293	0.2144	0.00266	5.517
ED	403.442	808.884	811.160	808.941	809.790	0.2115	0.00317	5.603

Table 7 Model comparison, Kolmogorov statistic, p-value of proposed model and its related models for data set 2

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	K-S distance (D)	p-value	Shannon Entropy $(H(X))$
SBLQXD	165.831	335.663	339.320	335.942	337.033	0.08597	0.88580	3.605
LQXD	172.8183	349.6367	353.2939	349.915	351.006	0.22031	0.02300	3.756
ATPSD	166.759	337.518	341.176	337.797	338.888	0.29129	0.00081	3.625
ED	172.545	347.091	348.920	347.188	347.776	0.2181	0.02500	3.750

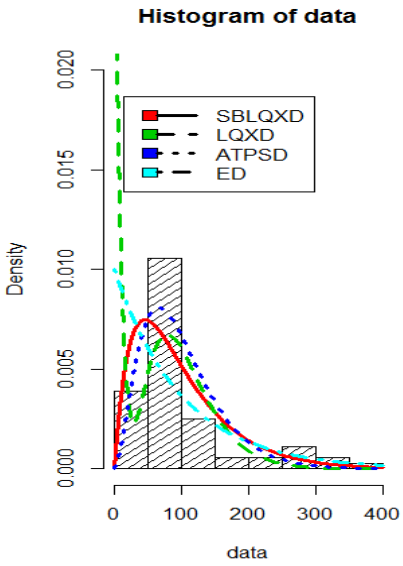

Figure 5 Curve fitting of data set 1

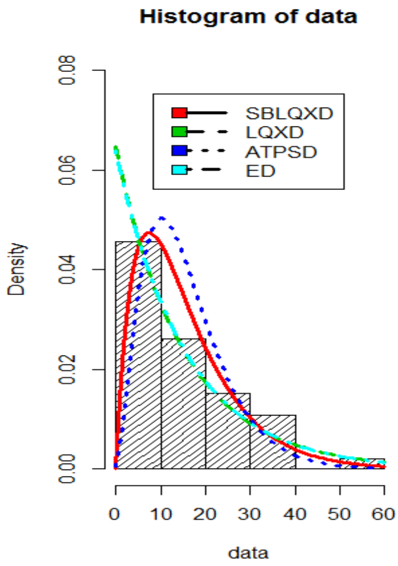

Figure 6 Curve fitting of data set 2

For testing the goodness of fit of our proposed model size biased Lindley-quasi Xgamma distribution and its related models, another two-parameter Sujatha distribution, Lindley-Quasi Xgamma distribution and exponential distribution to the two data sets, we computed Kolmogorov statistic and p-value. The better model possesses lesser Kolmogorov statistic value and higher p-value It can be seen from Tables 6 and 7 that size biased Lindley-quasi Xgamma distribution possesses lesser Kolmogorov statistic value and higher p-value for both the data sets as compared to Lindley-quasi Xgamma distribution, another two-parameter Sujatha distribution and exponential distribution.

Also for comparing models, we computed the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC which represent the loss of information resulting from fitting probability models to data. The better distribution corresponds to lesser AIC, AICC, BIC and HQIC values. Also we computed the Shannon's entropy $(H(X))$ which represents the average uncertainty. The better model possesses lesser Shannon's entropy value,

$$AIC = 2k - 2\log L, \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad BIC = k\log n - 2\log L,$$

$$HQIC = 2k\log(\log n) + 2\log L, \quad H(X) = -\frac{\log L}{n},$$

where k is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of the log-likelihood function under the considered model. From Tables 6 and 7, it has been observed that the size biased Lindley-quasi Xgamma distribution possesses the lesser AIC, AICC, BIC, HQIC and $H(X)$ values as compared to Lindley-quasi Xgamma distribution, another two-parameter Sujatha distribution and exponential distribution for data sets 1 and 2 respectively. Hence we can conclude that the Size Biased Lindley-Quasi Xgamma distribution leads to a better fit than Lindley-quasi Xgamma distribution, another two-parameter Sujatha distribution and exponential distribution for data sets 1 and 2, respectively.

13. Conclusions

We formulated a non-decreasing hazard rate model known as Size Biased Lindley-Quasi Xgamma distribution as a size biased version of Lindley-quasi Xgamma distribution. We obtained the important statistical properties like moments, reliability, moment generating function, order statistics, Renyi entropy, Bonferroni and Gini indices of formulated model. For obtaining the estimates of unknown parameters maximum likelihood estimation method is used. For testing the suitability of ML estimates simulation study has been carried which showed that ML estimation method performs well for proposed model. For testing the goodness of fit our model and for investigating the application of our proposed model in real life we fitted our proposed model and its related models to two real life data sets and computed log-likelihood values, AIC, AICC, HQIC, BIC, Kolmogorov statistic, p-value and Shannon's entropy. We observed that our model possesses lesser values of AIC, BIC, AICC, HQIC, D, $H(X)$ and possesses higher p value. Hence our model finds greater applicability in modeling survival times.

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Appendix

Appendix I Probability Weighted Moments

Proof: The probability weighted moment $M_{h,r,\beta}$ of size biased Lindley-quasi Xgamma distribution is computed as follows:

$$M_{h,r,\beta} = E\left[X^h F^r (1-F)^\beta\right] = \int_0^1 \left[x(F)\right]^h F^r (1-F)^\beta dF = \int_0^\infty x^h \left[F(x)\right]^r \left[1-F(x)\right]^\beta f(x)dx, \quad (26)$$

where $f(x)$ and $F(x)$ are the pdf and cdf of SBLQXD.

$$M_{h,r,\beta} = \int_0^\infty \left\{ \begin{array}{l} x^h \left[1-\left[1+\dfrac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^r \\ \left[\left[1+\dfrac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^\beta \\ \dfrac{\theta^2\left((\alpha+\theta)\left(\alpha x+\dfrac{x^3\theta^2}{2}\right)+\theta(\theta-1)\left(x+\alpha x^2\right)\right)e^{-\theta x}}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)} \quad dx \end{array} \right\}.$$

Using the binomial expansion,

$$\left[1-\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^r$$

$$= \sum_{i=0}^{\infty}\binom{r}{i}(-1)^i\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]^i e^{-i\theta x}. \quad (27)$$

Further using binomial expansion,

$$\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]^i$$

$$= \sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{p=0}^{l}\binom{l}{p}\sum_{v=0}^{j-k}\binom{j-k}{v}\frac{\begin{pmatrix}\theta^{2j+k-l-p-v}(\alpha+\theta)^k 2^{p+j-k}(\alpha+3)^p 3^{l-p}\\(\theta-1)^{j-k}(\theta+2\alpha)^v\alpha^{j-k-v}x^{2j+k-l-p-v}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^j}. \quad (28)$$

Using (28) in (27)

$$\left[1-\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^r$$

$$= \sum_{i=0}^{\infty}\binom{r}{i}(-1)^i e^{-i\theta x}\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{p=0}^{l}\binom{l}{p}\sum_{v=0}^{j-k}\binom{j-k}{v}\frac{\begin{pmatrix}\theta^{2j+k-l-p-v}(\alpha+\theta)^k 2^{p+j-k}(\alpha+3)^p 3^{l-p}\\(\theta-1)^{j-k}(\theta+2\alpha)^v\alpha^{j-k-v}x^{2j+k-l-p-v}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^j}.$$

(29)

Also, $\left[\left[1+\frac{\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)}{2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]e^{-\theta x}\right]^{\beta}$

$$=\sum_{s=0}^{\infty}\binom{\beta}{s}\sum_{t=0}^{s}\binom{s}{t}\sum_{m=0}^{t}\binom{t}{m}\sum_{c=0}^{m}\binom{m}{c}\sum_{q=0}^{s-t}\binom{s-t}{q}\frac{e^{-s\theta x}\begin{pmatrix}\theta^{2s+t-m-c-q}(\alpha+\theta)^t 2^{c+s-t}(\alpha+3)^c 3^{m-c}\\(\theta-1)^{s-t}(\theta+2\alpha)^q\alpha^{s-t-q}x^{2s+t-m-c-q}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^s}. \tag{30}$$

Putting the values of (29), (30), (5) in (26), we get

$$M_{h,r,\beta}=\left\{\begin{array}{l}\frac{\theta^2}{\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\sum_{i=0}^{\infty}\binom{r}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{p=0}^{l}\binom{l}{p}\sum_{v=0}^{j-k}\binom{j-k}{v}\\ \frac{\begin{pmatrix}\theta^{2j+k-l-p-v}(\alpha+\theta)^k 2^{p+j-k}(\alpha+3)^p 3^{l-p}\\(\theta-1)^{j-k}(\theta+2\alpha)^v\alpha^{j-k-v}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^j}\sum_{s=0}^{\infty}\binom{\beta}{s}\sum_{t=0}^{s}\binom{s}{t}\sum_{m=0}^{t}\binom{t}{m}\sum_{c=0}^{m}\binom{m}{c}\sum_{q=0}^{s-t}\binom{s-t}{q}\\ \frac{\begin{pmatrix}\theta^{2s+t-m-c-q}(\alpha+\theta)^t 2^{c+s-t}(\alpha+3)^c 3^{m-c}\\(\theta-1)^{s-t}(\theta+2\alpha)^q\alpha^{s-t-q}\end{pmatrix}}{\left(2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\right)^s}\\ \left(\begin{array}{l}(\alpha+\theta)\left(\frac{\alpha\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+1\end{pmatrix}!}{(\theta(1+i+s))^{2j-k-l-v-p+2s+t-m-c-q+h+2}}+\frac{\theta^2\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+3\end{pmatrix}!}{2(\theta(1+i+s))^{2j-k-l-v-p+2s+t-m-c-q+h+4}}\right)\\ +\theta(\theta-1)\left(\frac{\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+1\end{pmatrix}!}{(\theta(1+i+s))^{2j-k-l-v-p+2s+t-m-c-q+h+2}}+\frac{\alpha\begin{pmatrix}2j-k-l-v-p+2s\\+t-m-c-q+h+2\end{pmatrix}!}{(\theta(1+i+s))^{2j-k-l-v-p+2s+t-m-c-q+h+3}}\right)\end{array}\right)\end{array}\right\}.$$

Appendix II Mean Residual life

Proof: The mean residual life $m(x)$ of SBLQXD is obtained as

$$m(x)=E\left[X-x\mid X>x\right]=\frac{1}{1-F(x)}\int_x^{\infty}\left[1-F(t)\right]dt. \tag{31}$$

Using value of $F(x)$ in (31), we get

$$m(x)=\frac{\left\{\begin{array}{l}2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)\left[\frac{2\left((\alpha+\theta)(\alpha+6)+(\theta-1)(\theta+3\alpha)\right)}{\theta\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)}\right]-\\ \frac{1}{\theta}\left\{\begin{array}{l}2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)(1-e^{-\theta x})+(\alpha+\theta)\\ \left((2\alpha+6)\left(1-(x\theta+1)e^{-\theta x}\right)+6-e^{-\theta x}(3\theta^2x^2+6\theta x+6)+6-e^{-\theta x}(x^3\theta^3+3x^2\theta^2+6\theta x+6)\right)\\ +2(\theta-1)\left((\theta+2\alpha)(1-(x\theta+1)e^{-\theta x})+\alpha(2-e^{-\theta x}(x^2\theta^2+2\theta x+2))\right)\end{array}\right\}\end{array}\right\}}{\left[2\left(\theta^2+\alpha^2+3\alpha\theta+2\theta+\alpha\right)+\theta x\left((\alpha+\theta)\left(2\alpha+6+3x\theta+x^2\theta^2\right)+2(\theta-1)(\theta+2\alpha+x\theta\alpha)\right)\right]e^{-\theta x}}. \tag{32}$$

It can be seen from (32) that $m(0)=E(X)=\mu_1'$.