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The Effectiveness of CUSUM Control Chart for Trend Stationary Seasonal Autocorrelated Data

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Abstract

In this paper, the average run length (ARL) for the cumulative sum (CUSUM) control chart based on $SAR(P)_L$ with trend process is developed using Fredholm integral equation approach. The numerical integral equation is used to approximate the ARL. Banach's Fixed Point theorem is used to guarantee the existence and uniqueness of the solution. Finally, the accuracy of the proposed formulas is established by comparing them to the numerical integration method. The results from explicit formula and numerical integration show that the absolute percentage relative error is less than 0.3% In terms of computational time, the explicit formula can perform better than the numerical integration. In addition, real-world application of the derived explicit formulas was illustrated using the silver price data (USD/oz) to evaluate the ARL of $SAR(P)_L$ process on a CUSUM control chart.

Keywords: Average run length, cumulative sum control chart, integral equation approach, autoregressive process.

1. Introduction

The control charts are essential tools in statistical process control (SPC) processes, first introduced by Walter Shewhart in 1924. They are widely used and powerful tools for detecting and monitoring changes in processes. The Cumulative Sum (CUSUM) chart is considered study. The CUSUM control chart was first proposed by Page in 1954, which successfully identifies a slight change in the process. It is usually used to detect changes in observations of normal and independently distributed processes.

Various researchers, including Johnson and Bagshaw (1974), Harris and Ross (1991), and Karaoglan and Bayhan (2011) have examined the usefulness of control charts when the observations of the process are serially-correlated, which can affect the performance of the control chart. In general, control charts are used to detect changes in the process. If the mean of data in a normal process is within the desired value, this is referred to as "the process is in control state." On the other hand, when the mean of the data exceeds or goes below the desired value, this is referred to as "the process is in an out-of-control situation." The control chart's primary function is to detect process changes as fast as feasible. Simultaneously, the control chart must be monitoring the rate of false alarms. However, the CUSUM control chart is one of these introduced methods.

In a situation, the method is an in-control state, the criteria that may be accustomed to evaluating the performance of the control charts are median run length (MRL), standard deviations of run Length (SDRL), and average run length (ARL). The famous criteria accustomed measure the performance of the control chart is that the ARL. The minimum value of every method sets the suitable main method for locating the ARL. There are several methods for estimating the ARL of the CUSUM control chart, including the Monte Carlo simulations (MC) method. This method is the traditional method used to verify the validity and compare it with the other methods. However, it takes much time to process. Markov chain method (see Brook and Evan 1972, Lucus and Saccucci 1990), Martingale method (see Sukparangsri and Navikov 2008) and integral equation method (see Srivastava and Wu 1997, Areepong 2009, Mittitelu et al. 2010, Petcharat et al. 2013, Petcharat et al. 2015, Peerajit et al. 2018).

Actually, if the observations are serially correlated, it can affect the performance of the control chart such as a first order autoregressive (AR(1)) and a first order moving average (MA(1)) processes (see Johnson and Bagshaw 1974). For this reason, some authors evaluate the ARL when the process has a serial correlation, such as Lu and Reynolds (1999) used the integral equation method to calculate the ARL when the observations are AR(1) and ARMA(1,1) processes. Busaba et al. (2012) and Busaba et al. (2013) derived the analytical formula for the ARL of CUSUM control chart for AR(1) with trend process and proposed the explicit formula of ARL for AR(1) process. Then, Petcharat et al. (2013) proved the analytical expression for the ARL of EWMA control chart for the moving average of order q (MA(q)) process. Later, Busababodin (2014) proposed the explicit formula for the ARL of the CUSUM control chart when the observations are a seasonal AR(p) process. Additionally, Phanyaem et al. (2014) presented the exact solution for the ARL of the CUSUM control chart for an autoregressive and moving average (ARMA(1,1)) process. Sukparungsee et al. (2015) derived the analytical formula for the ARL of CUSUM control chart for AR(p) with trend process with exponential distribution white noise. Also, Petcharat et al. (2015) derived an analytical expression for the ARL of CUSUM control chart when the observations are modeled as a MA(q) process. After that, Petcharat (2015) presented the exact solution of ARL for the EWMA chart based on $SAR(P)_L$ process and compared with the CUSUM control chart. It had been found that the EWMA chart was more sensitive to detecting small shifts within the process than the CUSUM control chart. Moreover, Phanyaem (2017) proved the analytical formula for the ARL of CUSUM control chart for $SARMA(1,1)_L$ process and used the numerical integral equation method to approximate the ARL. In 2018, Phanthuna et al. applied the numerical integral equation of the ARL on a modified weighted moving average control chart for the AR(1) process, the method was found to be effective in estimating the ARL. Recently, Sunthornwat and Areepong (2020) proposed an exact formula of ARL on CUSUM control chart for both seasonal and non-seasonal moving averages processes by using integral equations and numerical integral equations as well as applying to the detection of the character of the stock price of Thailand.

Consequently, this article aims to prove the explicit formulas of ARL of CUSUM control chart for a seasonal autoregressive with trend, $SAR(P)_L$, process with exponential white noise and compare to the numerical integration. This article could be organized in the following manner. Section 2 presents model of seasonal autoregressive with trend, ($SAR(P)_L$) with trend process based on CUSUM control chart. Section 3 provides the explicit formula for ARL of CUSUM control chart when the processes are $SAR(P)_L$ with trend. Section 4 presents the numerical integration of ARL for CUSUM control chart, whereas Section 5 explains the simulation study. In Section 6, real-world data are used to evaluate the ARL by the explicit formula and numerical integral equations. Finally, conclusions are included in Section 7.

2. The Cumulative Sum Chart for SAR(P)L with Trend

In this section, we present the characteristics of the CUSUM control chart for SAR(P)L with trend process. The CUSUM control chart is a powerful tool in detecting a mean shift. Let X_t be the sequence of a seasonal autoregressive, SAR(P)L, with trend process.

The recursive equation of SAR(P)L with trend process with exponential white noise is defined as:

$$X_t = \tau + \gamma t + \phi_1 X_{t-L} + \phi_2 X_{t-2L} + \ldots + \phi_P X_{t-PL} + \varepsilon_t, \tag{1}$$

where ε_t is assumed to be a white noise process with exponential distribution, τ is a constant and γ is the trend slope in time of t, P is an order of moving average parameter, L is seasonal lag, an autoregressive coefficient $0 \le \phi_i \le 1$, and an initial value of SAR(P)L with trend process; $X_{t-L}, X_{t-2L}, \ldots X_{t-PL}$ is equal to 1.

The recursive equation of CUSUM statistics based on SAR(P)L with trend process is defined by

$$C_t = \max(C_{t-1} + X_t - k,\ 0);\ \ t = 1, 2, \ldots, \tag{2}$$

where X_t is a sequence of SAR(P)L with trend process, and k is a reference value of CUSUM control chart. The stopping time of CUSUM control chart is defined as follows

$$\tau_h = \inf\{t > 0; C_t > h\},\ \ h > u, \tag{3}$$

where h is a constant parameter known as the upper control limit.

Let $\mathbb{E}_\infty(.)$ denoted the expectation under density function $f(x,\alpha)$ that the change-point occurs at point θ, where $\theta < \infty$. Thus by definition, the ARL for SAR(P)L with trend process with an initial value $C_0 = u$ is as follow

$$ARL = H(u) = \mathbb{E}_\infty(\tau_h) < \infty. \tag{4}$$

3. Explicit Formulas for ARL of CUSUM control chart for SAR(P)L with Trend Process

The explicit formulas of average run length of CUSUM control chart for a seasonal autoregressive, SAR(P)L with trend process is presented. We derive analytical explicit formulas of ARL by using the Fredholm integral equation (NIE) of the second kind. Firstly, we define the function $H(u)$ is the ARL of CUSUM control chart for SAR(P)L with trend process.

We assume that the lower control limit is zero and upper control limit is h. Let $\mathbb{P}_c$ denote the probability measure and $\mathbb{E}_c$ denote the expectation corresponding to initial value $C_0 = u$. The ARL of CUSUM control chart based on SAR(P)L with trend process after it is reset at $u \in [0,h]$ as follows:

$$H(u) = 1 + \mathbb{E}_c\left[I\{0 < C_1 < h\} H(C_1)\right] + \mathbb{P}_c\{C_1 = 0\} H(0). \tag{5}$$

Let $C_t = C_{t-1} + X_t - k;\ t = 1, 2, \ldots$ where $X_t = \tau + \gamma t + \phi_1 X_{t-L} + \phi_2 X_{t-2L} + \ldots + \phi_P X_{t-PL} + \varepsilon_t$ and $C_0 = u$. First, to calculate $\mathbb{E}_c\left[I\{0 < C_1 < h\} H(C_1)\right]$ for $t = 1$.

$$\mathbb{E}_c\left[I\{0 < C_1 < h\} H(C_1)\right] = \int_{k-u-\tau-\gamma t-\phi_1 X_{1-L}-\ldots-\phi_P X_{1-PL}}^{h+k-u-\tau-\gamma t-\phi_1 X_{1-L}-\ldots-\phi_P X_{1-PL}} H(u + \tau + \gamma t + \phi_1 X_{1-L} + \ldots + \phi_P X_{1-PL} + y - k)\alpha e^{-\alpha y} dy$$

$$= \int_0^h H(y)\alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1 X_{1-L}+\phi_2 X_{1-2L}+\ldots+\phi_P X_{1-PL}-y)} dy$$

$$= \alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1 X_{1-L}+\phi_2 X_{1-2L}+\ldots+\phi_P X_{1-PL})} \int_0^h H(y) e^{-\alpha y} dy$$

$$\begin{aligned} \mathbb{P}_c\{C_1 = 0\}H(0) &= \mathbb{P}_c\{u + \tau + \gamma t + \phi_1 X_{1-L} + \phi_2 X_{1-2L} + \ldots + \phi_P X_{1-PL} - k = 0\}H(0) \\ &= \left[1 - \mathbb{P}_c\{\xi_1 > k - u - \tau - \gamma t - \phi_1 X_{1-L} - \phi_2 X_{1-2L} - \ldots - \phi_P X_{1-PL}\}\right]H(0) \\ &= [1 - e^{-\alpha(k-u-\tau-\gamma t-\phi_1 X_{1-L}-\phi_2 X_{1-2L}-\ldots-\phi_P X_{1-PL})}]H(0). \end{aligned}$$

In this case, (5) can be written as

$$H(u) = 1 + \alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1 X_{t-L}+\phi_2 X_{t-2L}+\ldots+\phi_P X_{t-PL})} \int_0^h H(y) e^{-\alpha y} dy + \left(1 - e^{-\alpha\left(k-u-\tau-\gamma t-\phi_1 X_{t-L}-\phi_2 X_{t-2L}-\ldots-\phi_P X_{t-PL}\right)}\right) H(0). \quad (6)$$

Petcharat et al. (2015) used integral equation method to analyze ARL for MA(q) process. In this section, we show that the ARL of CUSUM control chart is the unique solution to the integral equation by using following theorem.

Theorem 1 (Banach fixed point theorem) *Let* (M,d) *be non-empty complete metric space with a contraction mapping* $T: M \to M$. *Therefore,* T *admits a unique fixed-point* $m^* \in M$ (i.e., $T(m^*) = m^*$). *Also,* m^* *can be found as follows: start with an arbitrary element* $m_0 \in M$ *and define a sequence* $\{m_n\}$ *by* $m_n = T(m_{n-1})$, *then* $m_n \to m^*$.

According the right hand side of (6) is continuous, such that the solution of (6) is also continuous function. On the metric space of all continuous functions $(C(I), \|\ \|_\infty)$ where I denotes the compact interval and the norm $\|H\|_\infty = Sup_{u\in I}|H(u)|$ and the operator T is named on contraction, if it exists a number of $0 \le K < 1$ such that

$$\|T(H_1) - T(H_2)\| \le K \|H_1 - H_2\| \text{ for all } H_1, H_2 \in I.$$

Now, let $C(I_1)$ be the class of all continuous functions defined on a compact interval $I_1 = [0, h]$ and define the operator T by

$$T(H(u)) = 1 + \alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1 X_{t-L}+\phi_2 X_{t-2L}+\ldots+\phi_P X_{t-PL})} \int_0^h H(y) e^{-\alpha y} dy + \left(1 - e^{-\alpha\left(k-u-\tau-\gamma t-\phi_1 X_{t-L}-\phi_2 X_{t-2L}-\ldots-\phi_P X_{t-PL}\right)}\right) H(0). \quad (7)$$

Therefore, the integral equation can be written as $T\left(H(u)\right) = H(u)$. According to the Banach's fixed point theorem, if the operator T is a contraction, then fixed point equations $T\left(H(u)\right) = H(u)$. have a unique solution.

Theorem 2 *On metric space* $(C(I),\|\ \|_\infty)$ *with the norms* $\|H\|_\infty = \sup_{u\in I}|H(x)|$ *the operation* T *is contraction.*

Proof: To show T is contraction, for any $u\in I$ and $H_1,H_2\in C(I)$ we have the inequality $\|T(H_1)-T(H_2)\|\le K\|H_1-H_2\|$ where $K<1$. According to (7), we get

$$\begin{aligned}\|T(H_1)-T(H_2)\| &= \sup_{u\in[0,h]}|H_1(0)-H_2(0)|(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})\\ &\quad +\alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}\int_0^h\big(H_1(y)-H_2(y)\big)e^{-\alpha y}dy\\ &\le \sup_{u\in[0,h]}\Big|\|H_1(0)-H_2(0)\|(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})\\ &\quad +\|H_1-H_2\|\alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}\int_0^h e^{-\alpha y}dy\Big|\\ &= \|H_1-H_2\|\sup_{u\in[0,h]}(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})-\alpha h})\\ &\le K\|H_1-H_2\|,\end{aligned}$$

where $K=(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})-\alpha h})<1$. By using the triangle inequality for norms and the fact that $|H_1(0)-H_2(0)|\le\sup_{u\in[0,h]}|H_1(y)-H_2(y)|=\|H_1-H_2\|_\infty$.

As a result, Theorems 1 and 2 ensure the solution's uniqueness (Banach fixed point). Following that, we proved the explicit formula for the ARL of the CUSUM control chart for SAR(P)$_L$ with trend process using the Fredholm integral equation.

Let $G=\int_0^h H(y)e^{-\alpha y}dy,$ we obtain that

$$H(u) = 1+\alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}G+(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})H(0). \tag{8}$$

Now, we let $u=0$, then we have

$$\begin{aligned}H(0) &= 1+\alpha e^{\alpha(-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}G+(1-e^{-\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})H(0)\\ &= \frac{1+(e^{\alpha(-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})})}{(e^{-\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})}\alpha k\\ &= e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})}+\alpha G.\end{aligned} \tag{9}$$

Then, substituting (9) into (8), we obtain that

$$\begin{aligned}H(u) &= 1+\alpha e^{\alpha(u-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}G\\ &\quad +(1-e^{-\alpha(k-u-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})})e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})}+\alpha G\\ &= 1+\alpha Ge^{\alpha(u-k+\tau+\gamma t+\phi_1X_{t-L}+\phi_2X_{t-2L}+\ldots+\phi_PX_{t-PL})}+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})}+\alpha G\\ &= 1+\alpha G+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-\ldots-\phi_PX_{t-PL})}-e^{\alpha u}.\end{aligned} \tag{10}$$

To find a constant G as following form

$$G=\int_0^h H(y)e^{-\alpha y}dy$$

$$=\int_0^h (1+\alpha k+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha y})\,e^{-\alpha y}dy$$

$$=(1+\alpha k+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})})\int_0^h e^{-\alpha y}dy-\int_0^h e^{\alpha y-\alpha y}dy$$

$$=\frac{e^{\alpha h}}{\alpha}(1-e^{-\alpha h})(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})})-he^{\alpha h}.$$

Thus, a constant d can be found as follows

$$G=\frac{e^{\alpha h}}{\alpha}(1-e^{-\alpha h})(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})})-he^{\alpha h}.$$

Substituting a constant G into (10) as follows

$$\begin{aligned}H(u)&=1+\alpha\left(\frac{e^{\alpha h}}{\alpha}(1-e^{-\alpha h})(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})})-he^{\alpha h}\right)\\&\quad+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha u}\\&=1+(e^{\alpha h}(1-e^{-\alpha h})(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})})-\alpha he^{\alpha h})\\&\quad+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha u}\\&=1+(e^{\alpha h}-e^{(-\alpha h+\alpha h)})(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}\\&\quad-\alpha he^{\alpha h}+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha u})\\&=1+(e^{\alpha h}-1)(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}\\&\quad-\alpha he^{\alpha h}+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha u})\\&=e^{\alpha h}+e^{\alpha h+\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}\\&\quad-\alpha he^{\alpha h}+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-e^{\alpha u}\\&=e^{\alpha h}(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-\alpha h)-e^{\alpha u}.\end{aligned}$$

Under the $SAR(P)_L$ with trend process, we have the ARL of CUSUM control chart as follows

$$H(u)=e^{\alpha h}(1+e^{\alpha(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-\alpha h)-e^{\alpha u}. \tag{11}$$

Suppose that the process in-control process has the mean $\alpha=\alpha_0$ which is known. The explicit formula of ARL_0 of CUSUM control chart for $SAR(P)_L$ with trend process as follows

$$ARL_0=e^{\alpha_0 h}(1+e^{\alpha_0(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-\alpha_0 h)-e^{\alpha_0 u}. \tag{12}$$

On the other hand, the process is out-of-control with exponential parameter $\alpha=\alpha_1$ where $\alpha_1=\alpha_0(1+\delta)$. The explicit formula for ARL_1 of CUSUM control chart for $SAR(P)_L$ with trend process is as follows

$$ARL_1=e^{\alpha_1 h}(1+e^{\alpha_1(k-\tau-\gamma t-\phi_1X_{t-L}-\phi_2X_{t-2L}-...-\phi_PX_{t-PL})}-\alpha_1 h)-e^{\alpha_1 u}, \tag{13}$$

where α is a parameter of exponential white noise, τ is a constant, γ is the trend slope in time of t, h is upper control limit, $X_{t-L},X_{t-2L},...,X_{t-PL}$ are the initial values, and ϕ_i is an autoregressive coefficient, $0\le\phi_i\le1$.

4. Numerical Integration of ARL of CUSUM Control Chart for $SAR(P)_L$ with Trend

In this section, we present the scheme to evaluate numerically the solutions of the integral equation by using composite midpoint rule.

Since $y \sim Exp(\alpha)$, then $F(u) = 1-e^{-\alpha u}$ and $f(u) = \dfrac{dF(u)}{du} = \alpha e^{-\alpha u}$. Consequently, the integral equation in (6) can be rewritten as follows

$$\begin{aligned}\tilde{H}(u) = {} & 1 + H(0)F(k-u-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & +\int_0^h H(y)f(y+k-u-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})dy. \end{aligned} \tag{14}$$

The numerical approximation to integral equation is denoted by $\tilde{H}(a_i)$, which can be found as the solution of linear equations as follows

$$\begin{aligned}\tilde{H}(a_i) = {} & 1 + \tilde{H}(a_1)F(a-a_i-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & +\sum_{j=1}^{m} w_j \tilde{H}(a_j) f(a_j+a-a_i-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}). \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned}\tilde{H}(a_1) = {} & 1 + \tilde{H}(a_1)\big[F(a-a_1-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & + w_1 f(a-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})\big] \\ & +\sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j+a-a_1-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ \tilde{H}(a_2) = {} & 1 + \tilde{H}(a_1)\big[F(a-a_2-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & + w_1 f(a_1+a-a_2-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})\big] \\ & +\sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j+a-a_2-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & \vdots \\ \tilde{H}(a_m) = {} & 1 + \tilde{H}(a_1)\big[F(a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ & + w_1 f(a_1+a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})\big] \\ & +\sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j+a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}), \end{aligned}$$

or in matrix form as

$$\mathbf{H}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{H}_{m\times 1}, \tag{16}$$

where $$\mathbf{H}_{m\times 1} = \begin{pmatrix} \tilde{H}(a_1) \\ \tilde{H}(a_2) \\ \vdots \\ \tilde{H}(a_m) \end{pmatrix}, \quad \mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

$$\mathbf{R} = \begin{pmatrix} F(a-a_1-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})+w_1 f(a-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) & \ldots & w_m f(a_m+a-a_1-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ F(a-a_1-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})+w_1 f(a_1+a-a_2-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) & \ldots & w_m f(a_m+a-a_2-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \\ \vdots & & \vdots \\ F(a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL})+w_1 f(a_1+a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) & \ldots & w_m f(a_m+a-a_m-\tau-\gamma t-\phi_1 X_{t-L}-\ldots-\phi_P X_{t-PL}) \end{pmatrix},$$

and $\mathbf{I}_m = \text{diag}(1,1,...,1)$. If $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$ there exist

$$\mathbf{H}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}. \tag{17}$$

Here, $\tilde{H}(u)$ denotes the numerical integration solution of $H(u)$, so the integral equation in (6) can be approximated by

$$\begin{aligned}\tilde{H}(u) = 1 &+ \tilde{H}(a_1)F(a-u-\tau-\gamma t-\phi_1 X_{t-L}-...-\phi_P X_{t-PL}) \\ &+ \sum_{j=1}^{m} w_j \tilde{H}(a_j) f(a_j + a-u-\tau-\gamma t-\phi_1 X_{t-L}-...-\phi_P X_{t-PL}),\end{aligned} \tag{18}$$

where $w_j = \dfrac{h}{m}$ and $a_j = \dfrac{h}{m}\left(j-\dfrac{1}{2}\right); j = 1,2,...,m.$

5. Comparison of Average Run Length

In this section, we compare the ARL obtained by using explicit formulas (12) and (13) for CUSUM control chart on $SAR(P)_L$ with trend process with exponential white noise and the ARL estimated using numerical integral equation (18) by utilizing the composite midpoint rule on $m = 500$ subintervals. We set $H(u)$ is ARL from explicit formula solution and $\tilde{H}(u)$ is ARL from numerical integration solution. We also compare computational time between two methods and absolute percentage relative error (APRE), which defined as

$$\text{APRE}(\%) = \frac{\left|H(u)-\tilde{H}(u)\right|}{H(u)} \times 100.$$

The computational time of the two methods are approximated by central processing unit (CPU) time (Operating system: Window 8 OEM, intel(R) core(TM) i5-8265U CPU@1.60GHz 1.80 GHz Ram 8.00 GB (7.89 GB usable)) in minutes.

In Tables 1 and 2, the parameters k and h for CUSUM control chart were selected by setting ARL_0=370 and α_0=1 in the case of $SAR(2)_4$ and $SAR(3)_4$ processes with parameter $(\phi_1,\phi_2) = (0.13, 0.25), (0.30, -0.50)$, $(\phi_1,\phi_2,\phi_3) = (0.15, 0.25, 0.12)$ and $(0.10, 0.15, 0.20)$, respectively.

Tables 1 and 2 show that ARL_0 from explicit solution are closed to numerical integration on $m = 500$ subintervals with APRE less than 0.3%. Nevertheless, the CPU time of explicit formula are much less than the CPU time from numerical method.

Table 1 Comparison of ARL_0 between using explicit formulas and numerical integration for $SAR(2)_4$ with trend process with parameter $\alpha_0 = 1$ for $ARL_0 = 370$

k	h	$SAR(2)_4$ with trend process for $\phi_1 = 0.13$ $\phi_2 = 0.25$		
		Explicit Formulas	Numerical Integration	APRE (%)
4.50	2.018	370.517 (<0.001) [a]	369.784 (9.821)[a]	0.19783
5.00	1.503	370.027 (<0.001) [a]	369.720 (11.813)	0.08297
5.50	0.997	370.304 (<0.001) [a]	369.938 (9.826)	0.09884
k	h	$SAR(2)_4$ with trend process for $\phi_1 = 0.30$ $\phi_2 = -0.50$		
4.50	1.422	370.408 (<0.001) [a]	369.987 (10.418)[a]	0.11366
5.00	0.917	370.504 (<0.001) [a]	369.814 (10.507)	0.18623
5.50	0.415	370.440 (<0.001) [a]	369.570 (10.824)	0.23481

[a] The values in parentheses are CPU times in numerical integration method (minutes).

Table 2 Comparison of ARL_0 between using explicit formulas and numerical integration for $SAR(3)_4$ with trend process with parameter α_0 = 1 for ARL_0 = 370

k	h	$SAR(3)_4$ with trend process for $\phi_1 = 0.15$, $\phi_2 = 0.25$, $\phi_3 = 0.12$		
		Explicit Formulas	Numerical Integration	APRE (%)
4.50	2.164	370.324 (<0.001) [a]	369.537 (11.297)[a]	0.21252
5.00	1.164	370.303 (<0.001)	369.701 (11.599)	0.16257
5.50	1.139	370.605 (<0.001)	370.187 (11.611)	0.11279
k	h	$SAR(3)_4$ with trend process for $\phi_1 = 0.10$, $\phi_2 = 0.15$, $\phi_3 = 0.20$		
4.50	2.091	370.486 (<0.001)	369.727 (11.293)[a]	0.20487
5.00	1.577	370.501 (<0.001)	369.925 (10.835)	0.15547
5.50	1.068	370.470 (<0.001)[a]	370.187 (11.077)	0.07639

[a.] The values in parentheses are CPU times in numerical integration method (minutes).

In Tables 3 to 4, ARL results show the performance for detection change in processes between explicit formula and numerical integration on $m = 500$ subintervals with variety shift in mean. In in-control state, the value of parameter $\alpha_0 = 1$ and out of control state parameter values $\alpha_1 = \alpha_0(1+\delta)$ where shift size δ = 0.1 to 1.0. The comparison results for the ARL_0 = 370 for $SAR(2)_4$ and $SAR(3)_4$ in different parameters are shown in Tables 3 and 4.

Table 3 Comparison of ARL between using explicit formulas and numerical integration for $SAR(2)_4$ with trend process for parameter (ϕ_1, ϕ_2) = (0.3, 0.2) and (0.3, 0.5) with γ =0.2 and h = 3.0

ϕ_2	Shift size	Explicit Formulas	Numerical Integration	APRE (%)
0.2 (k=4.1191)	0.0	370.037 (<0.001) [a]	368.978 (10.136)[a]	0.286
	0.1	210.502 (<0.001)	209.967 (9.830)	0.254
	0.2	131.521 (<0.001)	131.221 (9.792)	0.228
	0.3	88.378 (<0.001)	88.197 (10.022)	0.205
	0.4	62.911 (<0.001)	62.794 (9.475)	0.186
	0.5	46.909 (<0.001)	46.830 (10.141)	0.168
	0.6	36.326 (<0.001)	36.270 (10.448)	0.154
	0.7	29.024 (<0.001)	28.983 (9.960)	0.140
	0.8	23.801 (<0.001)	23.770 (9.758)	0.130
	0.9	19.951 (<0.001)	19.927 (10.009)	0.120
	1.0	17.037 (<0.001)	17.019 (9.968)	0.106
0.5 (k=4.4200)	0.0	370.408 (<0.001) [a]	369.348 (11.080)[a]	0.286
	0.1	210.502 (<0.001)	210.161 (10.667)	0.162
	0.2	131.634 (<0.001)	131.334 (10.435)	0.228
	0.3	88.449 (<0.001)	88.268 (10.502)	0.205
	0.4	62.958 (<0.001)	62.842 (11.401)	0.184
	0.5	46.942 (<0.001)	46.863 (10.924)	0.168
	0.6	36.350 (<0.001)	36.294 (10.718)	0.154
	0.7	29.042 (<0.001)	29.001 (10.278)	0.141
	0.8	23.815 (<0.001)	23.784 (10.014)	0.130
	0.9	19.962 (<0.001)	19.938 (10.870)	0.120
	1.0	17.047 (<0.001)	17.028 (10.323)	0.111

[a.] The values in parentheses are CPU times in numerical integration method (minutes).

Table 4 Comparison of ARL values between using explicit formulas and numerical integration for $SAR(3)_4$ with trend process for parameter $(\phi_1, \phi_2, \phi_3) = (0.27, 0.12, 0.15)$ with $\gamma = 0.2$, 0.5 and $k = 4.5$

γ	Shift size	Explicit Formulas	Numerical Integration	APRE (%)
	0.0	370.396 (<0.001)[a]	369.459 (10.788)[a]	0.255
	0.1	212.724 (<0.001)	212.242 (10.632)	0.227
	0.2	133.931 (<0.001)	133.657 (10.697)	0.205
	0.3	90.541 (<0.001)	90.374 (11.158)	0.185
0.2	0.4	64.750 (<0.001)	64.641 (11.550)	0.169
(h=2.615)	0.5	48.447 (<0.001)	48.372 (10.842)	0.155
	0.6	37.610 (<0.001)	37.557 (11.072)	0.141
	0.7	30.099 (<0.001)	30.061 (10.945)	0.126
	0.8	24.709 (<0.001)	24.680 (10.808)	0.118
	0.9	20.723 (<0.001)	20.700 (10.971)	0.111
	1.0	17.699 (<0.001)	17.681 (11.427)	0.102
	0.0	370.126 (<0.001)[a]	369.227 (10.720)[a]	0.243
	0.1	213.043 (<0.001)	212.578 (10.993)	0.218
	0.2	134.377 (<0.001)	134.112 (10.852)	0.197
	0.3	90.977 (<0.001)	90.814 (10.952)	0.179
0.5	0.4	65.136 (<0.001)	65.031 (10.980)	0.161
(h=2.505)	0.5	48.779 (<0.001)	48.707 (10.650)	0.148
	0.6	37.894 (<0.001)	37.842 (10.942)	0.137
	0.7	30.342 (<0.001)	30.304 (11.208)	0.125
	0.8	24.916 (<0.001)	24.888 (10.536)	0.112
	0.9	20.901 (<0.001)	20.879 (11.137)	0.105
	1.0	17.854 (<0.001)	17.836 (11.258)	0.101

[a] The values in parentheses are CPU times in numerical integration method (minutes).

From Tables 3 and 4, it can be seen that ARL_0 from explicit solution are close to numerical integration on $m = 500$ subintervals with APRE less than 0.3% and CPU times are 10-11 minutes. However, the CPU time of explicit formula are much less than the CPU time from numerical method.

6. Real-world Application

Application to real-world data was studied to evaluate the ARL by the explicit formula and numerical integration equation, as shown in Table 5. The silver price (USD/oz) was collected monthly from January 2017 to December 2020. We proved that the dataset is an autocorrelated time series suitable for $SAR(1)_{12}$ process with trend. Its statistics were the coefficient of the seasonal first order of autoregressive process $\phi_1 = 0.979,$ slope parameter $\gamma = 0.138.$ The residual of this model was confirmed as exponential white noise with the mean $\alpha_0 = 8.8114$ by applying the Kolmogorov-Smirnov statistic. The results in Table 5 are similar to the results in Tables 3 and 4 in that the numerical integration results approached the explicit formula results. In addition, the ARL results were evaluated by varying the constant $k = 2.5$, 4.0, 5.5, and 7.0. The results found that when $0 \le \delta \le 0.1$ or small shift, the process can detect better because ARL_1 values decrease when k gets larger. For $0.3 \le \delta \le 5.0$ in all k levels, processes can detect change equally because the ARL_1 is not different as shown in Figures 1 and 2.

Table 5. Comparison of ARL values between using explicit formulas and numerical integration for $SAR(1)_{12}$ with trend when $\alpha_0 = 8.8114,\ k = 4,\ \gamma = 0.138, \phi_1 = 0.979, h = 4.4286$ for ARL_0=370

Shift	k = 2.5			k = 4.0		
	Explicit Formulas	Numerical Integration	APRE (%)	Explicit Formulas	Numerical Integration	APRE (%)
0.00	370.922	370.657 (9.127)[a]	0.071	370.602	370.400 (9.116)[a]	0.055
0.01	368.440	368.177 (9.138)	0.071	368.123	367.923 (9.116)	0.054
0.03	363.542	363.283 (9.210)	0.071	363.232	363.035 (9.108)	0.054
0.05	358.731	358.476 (9.596)	0.071	358.427	358.233 (9.128)	0.054
0.07	354.004	353.753 (9.445)	0.071	353.708	353.516 (9.122)	0.054
0.09	349.361	349.114 (9.186)	0.071	349.071	348.882 (9.156)	0.054
0.10	347.070	346.825 (9.334)	0.071	346.783	346.596 (9.112)	0.054
0.30	305.199	304.988 (9.148)	0.069	304.969	304.808 (9.101)	0.053
0.50	269.864	269.681 (9.118)	0.068	269.679	269.54 (9.119)	0.052
0.70	239.855	239.697 (9.446)	0.066	239.708	239.588 (9.155)	0.050
0.90	214.220	214.081 (9.120)	0.065	214.103	213.998 (9.152)	0.049
1.00	202.798	202.668 (9.428)	0.064	202.695	202.596 (9.168)	0.049
3.00	82.3421	82.299 (9.098)	0.052	82.3522	82.320 (9.122)	0.040
5.00	43.3881	43.369 (9.100)	0.044	43.4152	43.401 (9.103)	0.034

Shift	k = 5.5			k = 7		
	Explicit Formulas	Numerical Integration	APRE (%)	Explicit Formulas	Numerical Integration	APRE (%)
0.00	370.576	370.437 (9.076)[a]	0.038	370.052	369.976 (9.134)[a]	0.021
0.01	368.099	367.960 (9.125)	0.038	367.578	367.503 (9.061)	0.020
0.03	363.209	363.073 (9.205)	0.037	362.698	362.624 (9.080)	0.020
0.05	358.406	358.272 (9.075)	0.037	357.903	357.830 (9.053)	0.020
0.07	353.688	353.556 (9.061)	0.037	353.193	353.122 (9.064)	0.020
0.09	349.053	348.923 (9.104)	0.037	348.566	348.496 (9.061)	0.020
0.10	346.765	346.637 (9.133)	0.037	346.283	346.213 (9.102)	0.020
0.30	304.965	304.854 (9.085)	0.036	304.556	304.495 (9.078)	0.020
0.50	269.684	269.589 (9.057)	0.035	269.337	269.285 (9.068)	0.019
0.70	239.722	239.639 (9.068)	0.035	239.424	239.379 (9.078)	0.019
0.90	214.123	214.050 (9.020)	0.034	213.867	213.827 (9.074)	0.019
1.00	202.717	202.649 (9.041)	0.034	202.479	202.442 (9.072)	0.018
3.00	82.3894	82.367 (9.026)	0.028	82.3259	82.314 (9.050)	0.015
5.00	43.4472	43.437 (9.613)	0.023	43.4275	43.422 (9.099)	0.013

[a.] The values in parentheses are CPU times in numerical integration method (minutes).

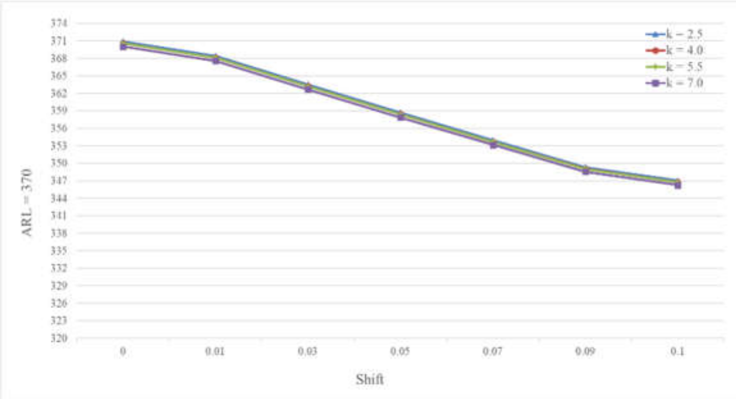

Figure 1 Comparison of ARL values using explicit formula for SAR(1)$_{12}$ with trend when $0 \le \delta \le 0.1$

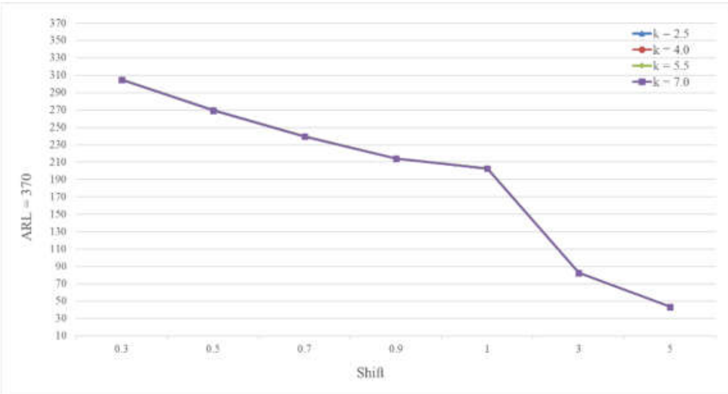

Figure 2 Comparison of ARL values using explicit formula for SAR(1)$_{12}$ with trend when $0.3 \le \delta \le 5.0$

7. Conclusions

The paper proposes an explicit formula and numerical integration for the Cumulative Sum control chart's average run length. Additionally, the existence and uniqueness of the explicit ARL have been proved. The comparison results indicate that the ARL from proposed explicit formulas are closed to numerical integration with an absolute percentage relative error of less than 0.3%. In addition, the explicit formula takes less than one second to compute, while the numerical integration method takes around 9-11 minutes in the case of SAR(P)$_L$ with trend process. As a conclusion, the explicit formulas can decrease the computational times much better than the numerical integration method. In addition, the silver price (USD/oz) is studied to evaluate the ARL by the explicit formula and numerical integration; the results agree with simulation data.

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