



Thailand Statistician
July 2022; 20(3): 508-517
<http://statassoc.or.th>
Contributed paper

Ratio and Inverse Moments of Marshall-Olkin extended Pareto Distribution through Generalized Record Values and Characterizations

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Received: 22 December 2019

Revised: 12 June 2020

Accepted: 17 August 2020

Abstract

In this paper, exact expressions as well as recurrence relations for ratio and inverse moments of generalized record values of different orders from Marshall-Olkin extended Pareto distribution have been obtained. Further, we characterize the given distribution through recurrence relations and some of its deductions are also discussed.

Keywords: Single moment, product moment, upper record values, Stirling numbers of first kind, harmonic numbers, hyper-geometric function, recurrence relations.

1. Introduction

A random variable X is said to have a Marshall-Olkin extended Pareto distribution, its probability density function (*pdf*) is of the form

$$f(x) = \frac{\lambda \left(\frac{p}{\alpha}\right) \left(\frac{\alpha}{x}\right)^{p+1}}{\left[1 - (1 - \lambda) \left(\frac{\alpha}{x}\right)^p\right]^2}, \quad \alpha < x < \infty, \quad \lambda, \alpha, p > 0 \quad (1)$$

and the corresponding distribution function (*df*)

$$F(x) = \frac{1 - \left(\frac{\alpha}{x}\right)^p}{1 - (1 - \lambda) \left(\frac{\alpha}{x}\right)^p}, \quad \alpha < x < \infty, \quad \lambda, \alpha, p > 0. \quad (2)$$

It is easy to see that

$$p\bar{F}(x) = x \left[1 - (1 - \lambda) \left(\frac{\alpha}{x}\right)^p\right] f(x), \quad (3)$$

where

$$\bar{F}(x) = 1 - F(x).$$

For more details on this distribution refer to Ghitany (2005).

The relation (3) will be used to establish recurrence relations for moments of generalized record values.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identical distributed (*iid*) continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. The j -th order statistic of a sample $X_1, X_2, \dots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k , we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of k -th upper record times of $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of generalized upper record values (k -th upper record values) of $\{X_n, n \geq 1\}$. Note that for $k = 1$, we have $Y_n^{(1)} = X_{U_n}$, $n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ as defined in Ahsanullah (1995).

The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by [Dziubdziela and Kopcoński (1976), Grudziński (1982)]

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x), \quad n \geq 1, \quad (4)$$

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} \\ \times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y), \quad x < y, \quad 1 \leq m < n, \quad n \geq 2. \quad (5)$$

and the conditional *pdf* of $Y_n^{(k)}$ given $Y_m^{(k)} = x$, is

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ \times \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{k-1} \frac{f(y)}{\bar{F}(x)}, \quad x < y. \quad (6)$$

For single and product of moments of generalized record values, we have the recurrence relations [Khan *et al.* (2017)]

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{j k^{n-1}}{(n-1)!} \int_{\alpha}^{\beta} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx$$

and

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^j] = \frac{j k^{n-1}}{(m-1)!(n-m-1)!} \int_{\alpha}^{\beta} \int_x^{\beta} x^i y^{j-1} \\ \times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k dy dx.$$

Thus for Marshall-Olkin extended Pareto distribution, in view of (3)

$$E(Y_n^{(k)})^j = \left(\frac{pk}{pk-j} \right) E(Y_{n-1}^{(k)})^j + \left(\frac{j}{pk-j} \right) \alpha^p (1-\lambda) E(Y_n^{(k)})^{j-p} \quad (7)$$

and

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \left(\frac{pk}{pk-j} \right) E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^j] - \left(\frac{j\alpha^p(1-\lambda)}{pk-j} \right) \times E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}] \quad (8)$$

as obtained by Nayabuddin and Athar (2017).

It may be noted here that for $j < p$, $E(Y_n^{(k)})^{j-p}$ and $E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}]$ will give moments of inverse and ratio generalized record values. Nayabuddin and Athar (2017) could not obtain explicit expressions for $E(Y_n^{(k)})^{j-p}$ and $E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}]$. In this paper, we have given some simple relations for $E(Y_n^{(k)})^{j-p}$ and $E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}]$ which can be used for $j > p$ as well as for $j < p$.

For related topics, one may refer to Khan et al. (1984), Ali and Khan (1994), Khan and Ali (1995), Khan and Athar (2000), Athar et al. (2007), Afify (2008) and Khan and Khan (2012) among others.

2. Relations for Single Moments

In this section, we derive the exact expressions for single moments of generalized upper record values using hyper-geometric function and recurrence relations in the following theorems. Before coming to the main result, we prove the following lemma.

Lemma 1 For the Marshall-Olkin extended Pareto distribution as given in (1) and non-negative finite integers a and b

$$\Phi_{j-p}(a, b) = \alpha^{j-p} \sum_{u=0}^a (-1)^u \binom{a}{u} \frac{\lambda^{b+u+1}}{(b+u+2-\frac{j}{p})} \times {}_2F_1[b+u+1, b+u+2-\frac{j}{p}; b+u+3-\frac{j}{p}; (1-\lambda)] \quad (9)$$

and

$$\Phi_0(a, b) = \sum_{u=0}^a (-1)^u \binom{a}{u} \frac{\lambda^{b+u+1}}{(b+u+1)} {}_2F_1[b+u+1, b+u+1; b+u+2; (1-\lambda)], \quad (10)$$

where

$$\Phi_{j-p}(a, b) = \int_{\alpha}^{\infty} x^{j-p} [F(x)]^a [\bar{F}(x)]^b f(x) dx \quad (11)$$

and

$${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; x] = \sum_{r=0}^{\infty} \left[\prod_{j=1}^p \frac{\Gamma(a_j + r)}{\Gamma(a_j)} \right] \left[\prod_{j=1}^q \frac{\Gamma(b_j)}{\Gamma(b_j + r)} \right] \frac{x^r}{r!},$$

for $p = q + 1$ and $\sum_{j=1}^q b_j - \sum_{j=1}^p a_j > 0$. [Mathai and Saxena, (1973)].

Proof: We have

$$\Phi_{j-p}(a, b) = \int_{\alpha}^{\infty} x^{j-p} [1 - \bar{F}(x)]^a [\bar{F}(x)]^b f(x) dx. \quad (12)$$

On expanding $[1 - \bar{F}(x)]^a$ binomially in (12) and using (3) in resulting expression, we get

$$\Phi_{j-p}(a, b) = p \sum_{u=0}^a (-1)^u \binom{a}{u} \int_{\alpha}^{\infty} \frac{x^{j-p-1}}{\left[1 - (1-\lambda) \left(\frac{\alpha}{x} \right)^p \right]} [\bar{F}(x)]^{b+u+1} dx$$

$$= p \sum_{u=0}^a (-1)^u \binom{a}{u} \int_{\alpha}^{\infty} x^{j-p-1} \frac{\lambda^{b+u+1} \left(\frac{\alpha}{x}\right)^{p(b+u+1)}}{\left[1 - (1-\lambda) \left(\frac{\alpha}{x}\right)^p\right]^{b+u+2}} dx. \quad (13)$$

Making the substitution $t = \left(\frac{\alpha}{x}\right)^p$ in (13), we find that

$$\Phi_{j-p}(a, b) = \alpha^{j-p} \sum_{u=0}^a (-1)^u \binom{a}{u} \lambda^{b+u+1} \int_0^1 \frac{t^{b+u+1-\frac{j}{p}}}{[1 - (1-\lambda)t]^{b+u+1}} dt. \quad (14)$$

On using the following expression in (14)

$$\int_0^x \frac{z^{a-1}}{(1+\beta z)^b} dz = \frac{x^a}{a} {}_2F_1[b, a; 1+a; -\beta x] \quad (\text{Gradshteyn and Ryzhik, 2007}),$$

we have the result given in (9).

To prove (10), put $j = p$ in (9).

Theorem 1 For Marshall-Olkin extended Pareto distribution as given in (1) and $1 \leq k \leq n$

$$E(Y_n^{(k)})^{j-p} = k^n \sum_{r \geq n-1} \frac{c(r, n-1)}{r!} \Phi_{j-p}(r, k-1), \quad (15)$$

where

$$\begin{aligned} \Phi_{j-p}(r, k-1) &= \alpha^{j-p} \sum_{u=0}^r (-1)^u \binom{r}{u} \frac{\lambda^{k+u}}{(k+u+1-\frac{j}{p})} \\ &\quad \times {}_2F_1[k+u+1, k+u+1-\frac{j}{p}; k+u+2-\frac{j}{p}; (1-\lambda)]. \end{aligned}$$

Proof: We have

$$E(Y_n^{(k)})^{j-p} = \frac{k^n}{(n-1)!} \int_{\alpha}^{\infty} x^{j-p} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \quad (16)$$

For any real number s and $|x| < 1$, we have Comtet (1974, p-212)

$$[-\ln(1-x)]^s = s! \sum_{r \geq s} \frac{c(r, s)}{r!} x^r, \quad (17)$$

where $c(r, s)$ is the number of permutations of r elements with s cycles (absolute Stirling numbers), hence $c(r, s) = 0$ if not $1 \leq s \leq r$ except $c(0, 0) = 1$ and

$$\begin{aligned} c(r+1, 1) &= r!, & c(r+1, 2) &= r! H_r^{(1)}, & c(r+1, 3) &= \frac{r!}{2} [(H_r^{(1)})^2 - H_r^{(2)}], \\ c(r+1, 4) &= \frac{r!}{6} [(H_r^{(1)})^3 - 3 H_r^{(1)} H_r^{(2)} + 2 H_r^{(3)}], \end{aligned}$$

where $H_r^{(s)}$, $s \in \mathbb{N}$, $r \in \mathbb{N}$, be the harmonic number of order s ,

$$H_r^{(s)} = \sum_{q=1}^r \frac{1}{q^s}, \quad s \geq 1.$$

On substituting (17) in (16), we find that

$$E(Y_n^{(k)})^{j-p} = k^n \sum_{r \geq n-1} \frac{c(r, n-1)}{r!} \int_{\alpha}^{\infty} x^{j-p} [F(x)]^r [\bar{F}(x)]^{k-1} f(x) dx,$$

in view of (11) we have the result given in (15).

Corollary 1 For $1 \leq n \leq r$

$$\sum_{r \geq n-1} \sum_{u=0}^r (-1)^u \binom{r}{u} \frac{\lambda^u c(r, n-1)}{r! (k+u)} {}_2F_1[k+u+1, k+u; k+u+1; (1-\lambda)] = \frac{1}{\lambda^k k^n}. \quad (18)$$

Proof: (18) can be proved by setting $j-p=0$ in (15).

Remark 1 Setting $k=1$ in (15), we get the single moments of upper records from the Marshall-Olkin extended Pareto distribution as

$$E(X_{U_n})^{j-p} = \sum_{r \geq n-1} \frac{c(r, n-1)}{r!} \Phi_{j-p}(r, 0),$$

where

$$\Phi_{j-p}(r, 0) = \alpha^{j-p} \sum_{u=0}^r (-1)^u \binom{r}{u} \frac{\lambda^{u+1}}{(u+2-\frac{j}{p})} {}_2F_1[u+2, u+2-\frac{j}{p}; u+3-\frac{j}{p}; (1-\lambda)].$$

The following theorem gives the recurrence relations for single moments of generalized record values from df (2).

Theorem 2 For the distribution given in (1), fix a positive integer $k \geq 1$, for $n > 1$, $n \geq k$ and $j = 0, 1, \dots$

$$\left(1 - \frac{j-p}{pk}\right) E(Y_n^{(k)})^{j-p} = E(Y_{n-1}^{(k)})^{j-p} - \left(\frac{\alpha^p (j-p)(1-\lambda)}{pk}\right) E(Y_n^{(k)})^{j-2p}. \quad (19)$$

Proof: In view of Khan et al. (2017), note that

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{j k^{n-1}}{(n-1)!} \int_{\alpha}^{\beta} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx. \quad (20)$$

Using (3) in (20), we have

$$\begin{aligned} E(Y_n^{(k)})^{j-p} - E(Y_{n-1}^{(k)})^{j-p} &= \frac{(j-p) k^{n-1}}{(n-1)! p} \int_{\alpha}^{\infty} x^{j-p} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \\ &\quad - \frac{\alpha^p (j-p)(1-\lambda) k^{n-1}}{(n-1)! p} \int_{\alpha}^{\infty} x^{j-2p} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \end{aligned} \quad (21)$$

The recurrence relations in (19) is derived simply by rewriting the above relation.

Remark 2 Setting $k=1$ in (19), we get the recurrence relations between single moments of upper records from the Marshall-Olkin extended Pareto distribution in the form

$$\left(1 - \frac{j-\beta}{p}\right) E(X_{U_n})^{j-p} = E(X_{U_{n-1}})^{j-p} - \frac{\alpha^p (j-p)(1-\lambda)}{p} E(X_{U_n})^{j-2p}.$$

3. Relations for Product Moments

This section contains the recurrence relations for product moments of generalized upper record values from the Marshall-Olkin extended Pareto distribution.

Theorem 3 For the distribution given in (1) and $1 \leq k \leq m$, $i, j = 0, 1, \dots$

$$\begin{aligned} \left(1 - \frac{j-p}{pk}\right) E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^{j-p}] &= E[(Y_m^{(k)})^{i+j-p}] \\ &\quad - \frac{\alpha^p (j-p)(1-\lambda)}{pk} E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^{j-2p}]. \end{aligned} \quad (22)$$

and for $1 \leq m \leq n-2$, $i, j = 0, 1, \dots$

$$\begin{aligned} \left(1 - \frac{j-p}{pk}\right) E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}] &= E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^{j-p}] \\ &\quad - \frac{\alpha^p (j-p)(1-\lambda)}{pk} E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-2p}]. \end{aligned} \quad (23)$$

Proof: Khan *et al.* (2017) have shown that for $1 \leq m \leq n-1$

$$\begin{aligned} E[(Y_m^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^j] &= \frac{j k^{n-1}}{(m-1)!(n-m-1)!} \\ &\quad \times \int_{\alpha}^{\beta} \int_x^{\beta} x^i y^{j-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k dy dx. \end{aligned} \quad (24)$$

On using relation (3) in (24), we find that

$$\begin{aligned} E[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p}] - E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^{j-p}] &= \frac{(j-p) k^{n-1}}{(m-1)!(n-m-1)! p} \\ &\quad \times \int_{\alpha}^{\infty} \int_x^{\infty} x^i y^{j-p-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ &\quad \times \left[y \left\{ 1 - (1-\lambda) \left(\frac{\alpha}{y} \right)^p \right\} f(y) \right] [\bar{F}(y)]^{k-1} dy dx \end{aligned} \quad (25)$$

and hence rearranging the resulting expression, yields the result given in (23). Now putting $n = m+1$ in (23) and noting that $E[(Y_m^{(k)})^i (Y_m^{(k)})^j] = E[(Y_m^{(k)})^{i+j}]$, the recurrence relation given in (22) can be easily be established.

Remark 3 Setting $i = 0$ in (23), Theorem 3 reduces to Theorem 2.

Remark 4 Setting $k = 1$ in (23), we get the recurrence relations for the product moments of upper records from the Marshall-Olkin extended Pareto distribution.

4. Characterization

This section contains the characterization results using the recurrence relations for single and product moments and conditional moment of function of generalized record values.

Theorem 4 Fix a positive integer $k \geq 1$ and let j be a non negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$\left(1 - \frac{j-p}{pk}\right) E(Y_n^{(k)})^{j-p} = E(Y_{n-1}^{(k)})^{j-p} - \left(\frac{\alpha^p (j-p)(1-\lambda)}{pk}\right) E(Y_n^{(k)})^{j-2p}. \quad (26)$$

if and only if

$$F(x) = \frac{1 - \left(\frac{\alpha}{x}\right)^p}{1 - (1-\lambda)\left(\frac{\alpha}{x}\right)^p}, \quad \alpha < x < \infty, \quad \lambda, \alpha, p > 0.$$

Proof: The necessary part is proved in (19). Now suppose the recurrence relations (26) is satisfied, then on rearranging the terms in (26) and using Khan *et al.* (2017), we have

$$\begin{aligned} \frac{k^{n-1}}{(n-1)!} \int_{\alpha}^{\infty} x^{j-p-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx &= \frac{k^{n-1}}{(n-1)!p} \int_{\alpha}^{\infty} x^{j-p} [-\ln \bar{F}(x)]^{n-1} \\ &\times [\bar{F}(x)]^{k-1} f(x) dx - \frac{\alpha^p (1-\lambda) k^{n-1}}{(n-1)!} \int_{\alpha}^{\infty} x^{j-2p} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \end{aligned} \quad (27)$$

which implies

$$\int_{\alpha}^{\infty} x^{j-p} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} \left\{ \frac{\bar{F}(x)}{x} - \frac{f(x)}{p} + \frac{\alpha^p (1-\lambda) f(x)}{p x^p} \right\} dx = 0. \quad (28)$$

Now applying a generalization of the Müntz-Szász Theorem [see for example Hwang and Lin (1984)] to (28), we get

$$p \bar{F}(x) = x \left[1 - (1-\lambda) \left(\frac{\alpha}{x} \right)^p \right] f(x)$$

which proves the sufficiency part.

Remark 5 If $k = 1$ in (26), we obtain the characterizing result based on upper records for the Marshall-Olkin extended Pareto distribution.

Theorem 5 For a positive integer k , i and j be a non-negative integers, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) and for $1 \leq m \leq n-2$, is that

$$\begin{aligned} \left(1 - \frac{j-p}{p k} \right) E \left[(Y_m^{(k)})^i (Y_n^{(k)})^{j-p} \right] &= E \left[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^{j-p} \right] \\ &- \frac{\alpha^p (j-p)(1-\lambda)}{p k} E \left[(Y_m^{(k)})^i (Y_n^{(k)})^{j-2p} \right] \end{aligned} \quad (29)$$

if and only if

$$F(x) = \frac{1 - \left(\frac{\alpha}{x} \right)^p}{1 - (1-\lambda) \left(\frac{\alpha}{x} \right)^p}, \quad \alpha < x < \infty, \quad \lambda, \alpha, p > 0.$$

Proof: The necessary part follows from (23). On the other hand if the relations in (29) is satisfied, then on rearranging the terms in (29) and using Khan *et al.* (2017), we have

$$\begin{aligned} \frac{k^{n-1}}{(m-1)!(n-m-1)!} \int_{\alpha}^{\infty} \int_x^{\infty} x^i y^{j-p-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ \times [\bar{F}(y)]^k dy dx &= \frac{k^{n-1}}{(m-1)!(n-m-1)!p} \int_{\alpha}^{\infty} \int_x^{\infty} x^i y^{j-p} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} \\ &\times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx - \frac{\alpha^p (1-\lambda) k^{n-1}}{(m-1)!(n-m-1)!p} \\ &\times \int_{\alpha}^{\infty} \int_x^{\infty} x^i y^{j-2p} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx, \end{aligned}$$

which implies that

$$\int_0^{\infty} \int_x^{\infty} x^i y^{j-p-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}$$

$$\times [\bar{F}(y)]^{k-1} f(y) \left\{ \frac{\bar{F}(y)}{f(y)} - \frac{y}{p} + \frac{\alpha^p (1-\lambda) y^{1-p}}{p} \right\} dy dx = 0. \quad (30)$$

Now applying the extension of Müntz-Szász Theorem [see for example Hwang and Lin (1984)] in (30), we get

$$p\bar{F}(y) = x \left[1 - (1-\lambda) \left(\frac{\alpha}{y} \right)^p \right] f(y).$$

Hence, the sufficiency part.

Remark 6 At $k = 1$ in (29), we obtain the characterization results from the Marshall-Olkin extended Pareto distribution on record values.

Theorem 6 Let X be a non-negative random variable having an absolutely continuous df $F(x)$ and $F(0) = 0$ and $0 \leq F(x) \leq 1$ for all $x > 0$, then

$$E[\xi(Y_n^{(k)}) | (Y_l^{(k)}) = x] = \xi(x) \left(\frac{k}{k+1} \right)^{n-l}, \quad l = m, m+1, m \geq k, \quad (31)$$

if and only if

$$F(x) = \frac{1 - \left(\frac{\alpha}{x} \right)^p}{1 - (1-\lambda) \left(\frac{\alpha}{x} \right)^p}, \quad \alpha < x < \infty, \quad \lambda, \alpha, p > 0.$$

where

$$\xi(y) = \frac{\lambda \left(\frac{\alpha}{y} \right)^p}{\left[1 - (1-\lambda) \left(\frac{\alpha}{y} \right)^p \right]}.$$

Proof: From (6), we have

$$\begin{aligned} E[\xi(Y_n^{(k)}) | (Y_m^{(k)}) = x] &= \frac{k^{n-m}}{(n-m-1)!} \int_x^\infty \xi(y) [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ &\quad \times \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{k-1} \frac{f(y)}{\bar{F}(x)} dy. \end{aligned} \quad (32)$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{\xi(y)}{\xi(x)}$ from (2) in (32), we have

$$E[\xi(Y_n^{(k)}) | (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!} \xi(x) \int_0^1 u^k [-\ln u]^{n-m-1} du. \quad (33)$$

We have Gradshteyn and Ryzhik ((2007), $p - 551$)

$$\int_0^1 [-\ln x]^{\mu-1} x^{\nu-1} dx = \frac{\Gamma(\mu)}{\nu^\mu}, \quad \mu > 0, \nu > 0. \quad (34)$$

On using (34) in (33), we have the result given in (31).

To prove sufficient part, we have

$$\begin{aligned} &\frac{k^{n-m}}{(n-m-1)!} \int_x^\infty \xi(y) [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy \\ &= [\bar{F}(x)]^k g_{n|m}(x), \end{aligned} \quad (35)$$

where

$$g_{n|m}(x) = \xi(x) \left(\frac{k}{k+1} \right)^{n-m}.$$

Differentiating (35) both sides with respect to x , we get

$$\begin{aligned} & -\frac{k^{n-m}}{(n-m-2)!} \frac{f(x)}{\bar{F}(x)} \int_x^\infty \xi(y) [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} [\bar{F}(y)]^{k-1} f(y) dy \\ & = g'_{n|m}(x) [\bar{F}(x)]^k - k g_{n|m}(x) [\bar{F}(x)]^{k-1} f(x) \end{aligned}$$

or

$$-k g_{n|m+1}(x) [\bar{F}(x)]^{k-1} = g'_{n|m}(x) [\bar{F}(x)]^k - k g_{n|m}(x) [\bar{F}(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = -\frac{\xi'(x)}{\xi(x)}, \quad (36)$$

where

$$\begin{aligned} g'_{n|m}(x) &= \xi'(x) \left(\frac{k}{k+1} \right)^{n-m}, \\ g_{n|m+1}(x) - g_{n|m}(x) &= \xi(x) \frac{1}{k} \left(\frac{k}{k+1} \right)^{n-m}, \end{aligned}$$

Integrating both the sides (36) with respect to x between $(0, y)$ and hence the sufficiency part.

Remark 7 Setting $l = n - 1$ in (31) we obtain the following characterization of the Marshall-Olkin extended Pareto distribution based on the left truncated moment of generalized upper record values.

$$E[\xi(Y_n^{(k)}) | (Y_{n-1}^{(k)}) = x] = E[\xi(Z_n^{(k)}) | X \geq x] = \xi(x) \left(\frac{k}{k+1} \right).$$

Remark 8 Setting $n = m$ and $l = m + 1$ in (31) we obtain the following characterization of the Marshall-Olkin extended Pareto distribution based on the right truncated moment of generalized upper record values.

$$E[\xi(Y_m^{(k)}) | (Y_{m+1}^{(k)}) = y] = E[\xi(Z_m^{(k)}) | Y \leq y] = \xi(y) \left(\frac{k}{k+1} \right)^{-1}.$$

Acknowledgements

The authors are thankful to the anonymous reviewers and the Editor-in-Chief for their valuable suggestions and comments which led to considerable improvement in the manuscript.

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