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Generalized Order Statistics from Power-Linear Hazard Rate Distribution and Characterizations

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Abstract

The goal of this article is to investigate the moment aspects of generalized order statistics (GOS) via power linear hazard rate distribution. The explicit formulation and relations between moments of GOS are derived. In addition, various deductions and related results are reviewed. Some numerical computations are accomplished. The characterization results are also presented by several techniques at the end.

Keywords: Single, product, conditional and truncated moments.

1. Introduction

Kamps (1995) instigated the concept of generalized order statistics (GOS) as a combined procedure of observing random variables (RV) arranged in ascending order. It is defined in the following.

Let $b \in N$, $b \geq 2,\ a \geq 1$, $\tilde{c} = (c_1, c_2, \ldots, c_{b-1}) \in \Re^{b-1}$ and $C_g = \sum_{m=g}^{b-1} c_m$, such that $\phi_g = a + (b-g) + C_g > 0$ for all $g \in \{1, 2, \ldots, b-1\}$ are the parameters of GOS. Further, let $V_{(1:b,\tilde{c},a)}, \ldots, V_{(g:b,\tilde{c},a)}$ be b GOS having continuous distribution function $T(.)$ and probability density function $t(.)$. If it assumes the following joint probability density function (pdf) as

$$a\left(\prod_{m=1}^{b-1}\phi_m\right)\left(\prod_{n=1}^{b-1}[1-T(v_n)]^{c_n}\, t(v_n)\right)[1-T(v_b)]^{a-1}\, t(v_b), \tag{1}$$

for $T^{-1}(0+) < v_1 \leq v_2 \leq \cdots \leq v_n < T^{-1}(1).$ Consider two cases:

Case I: $c_1 = \cdots = c_{b-1} = c$. In view of Equation (1), density of g^{th} GOS is

$$t_{g,b,c,a}(v) = \frac{K_{g-1}}{(g-1)!} t(v)[\overline{T}(v)]^{\phi_g - 1}\, p_c^{g-1}[T(v)],\ \ -\infty < v < \infty \tag{2}$$

where $\overline{T}(v)=1-T(v)$, $K_{g-1}=\prod_{n=1}^{g}\phi_n$, $\phi_n=a+(b-n)(c+1)$, $n=1,\ldots,b$. The density of $(g^{\text{th}}-h^{\text{th}})$ GOS is $(1\le g<h\le b)$

$$t_{g,h:b,c,a}(v_1,v_2)=\frac{K_{h-1}}{(g-1)!(h-g-1)!}[\overline{T}(v_1)]^m t(v_1)\,p_c^{g-1}[T(v_1)]$$
$$\times[q_m(T(v_2))-q_m(T(v_1))]^{h-g-1}[\overline{T}(v_2)]^{\phi_h-1}t(v_2),\ -\infty<v_1<v_2<\infty, \tag{3}$$

where

$$q_c(v)=\begin{cases}-\frac{1}{c+1}(1-v)^{c+1}, & c\ne -1,\\ -\ln(1-v), & c=-1,\end{cases}\quad \text{and}\quad p_c(v)=q_c(v)-q_c(0)$$

The pdf of $(h^{\text{th}}\mid g^{\text{th}})$GOS is

$$f(v_2\mid v_1)=\frac{K_{h-1}}{K_{g-1}(h-g-1)!}\frac{[(T(v_1))^{c+1}-(T(v_2))^{c+1}]^{h-g-1}[\overline{T}(v_2)]^{\phi_h-1}}{(c+1)^{h-g-1}[\overline{T}(v_1)]^{\phi_{g+1}}}t(v_2), \tag{4}$$

Case II: $\phi_n\ne\phi_m$, $n\ne m$, $n,m=1,2,\ldots,b-1$. Considering Equation (1), the density of g^{th}GOS is

$$t_{g:b,\tilde{c},a}(v)=K_{g-1}\,t(v)\sum_{n=1}^{g}w_n(g)[\overline{T}(v)]^{\phi_n-1}, \tag{5}$$

The pdf of $(g^{\text{th}}-h^{\text{th}})$GOS is

$$t_{g,h:b,\tilde{c},a}(v_1,v_2)=K_{h-1}\sum_{m=g+1}^{h}w_m^{(g)}(h)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\left[\sum_{n=1}^{g}w_n(g)[\overline{T}(v_1)]^{\phi_n}\right]\frac{t(v_1)}{\overline{T}(v_1)}\frac{t(v_2)}{\overline{T}(v_2)}, \tag{6}$$

where
$$\phi_n=a+(b-n)+\sum_{m=n}^{b-1}c_m,$$

$$w_n(g)=\prod_{\substack{m=1\\ m\ne i}}^{g}\frac{1}{(\phi_m-\phi_n)},\quad 1\le n\le g\le b,$$

and
$$w_m^{(g)}(h)=\prod_{\substack{m=g+1\\ m\ne n}}^{h}\frac{1}{(\phi_m-\phi_n)},\quad g+1\le n\le h\le b.$$

When $\phi_n\ne\phi_m$ but $c_1=\cdots=c_{b-1}=c$, then

$$w_n(g)=\frac{(-1)^{g-n}}{(c+1)^{g-1}}\frac{1}{(b-1)!(g-1)!}\quad\text{and}\quad w_m^{(g)}(h)=\frac{(-1)^{h-n}}{(c+1)^{h-g-1}}\frac{1}{(m-g-1)!(h-n)!}.$$

Therefore, Equation (5) reduces to Equation (2) and Equation (6) reduces to Equation (3), (Khan et al. 2006). Special cases of GOS are listed in Table 1 (Cramer, 2002).

In the literature, numerous authors researched recurrence relations between moments of GOS. Reference may be referred to Cramer and Kamps (2000), Bieniek and Szynal (2003), Khan et al. (2008), Khan et al. (2015a, b), Khan and Khan (2016a, b), Faizan and Khan (2017), Singh et al. (2018), Khan (2018), and Saran et al. (2018) and reference therein.

The basic principles of recurrence relations are to reduce the computation, labor, and time. In addition, these are applied in characterizing the distributions too that is the main fields, granting the recognition of population distribution from characteristics of sample.

Table 1 GOS′ variants

	Models	c	ϕ_g	c_g
i	Order statistics (O.S.)	1	$b-g+1$	0
ii	Sequential O.S.	w_b	$(b-g+1)\ w_g$	$\phi_g-(\phi_{g+1}+1)$
iii	Progressively type-II censored O.S.	R_b+1	$b-(g-1)+\sum_{m=g}^{b}R_m$	R_g
iv	Records	1	1	-1

Power-linear hazard rate (P-LHR) distribution was introduced by (Tarvirdizade and Nematollahi 2019). This distribution induces several lifetime distributions (see Table 2).

The PLHRD is very simple and can cover constant, decreasing, increasing, bathtub-shaped and non-monotone hazard rate too. These properties enable this distribution to be used in many applications in several areas, such as reliability, survival analysis, life testing and others. For more details, properties, and application of PLHRD, (see Tarvirdizade and Nematollahi, 2019).

A RV $V\sim \text{PLHRD}(\alpha,\beta,\lambda),$ if its distribution function and density are as follows:

$$T(v)=1-e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}},\ v>0, \tag{7}$$

and

$$t(v)=(\beta\, v+\alpha\, v^{\lambda})e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}}, v>0, \tag{8}$$

where $\alpha,\beta\geq 0,\ \lambda>-1,$ and $\lambda\neq 1.$ The sub-models of the PLHRD are shown in Table 2.

Table 2 Sub-models of PLHRD

Model	α	β	λ	cdf	References
Exponential	-	0	0	$1-e^{-\alpha v}$	Bain (1974)
Rayleigh	0	2μ	0	$1-e^{-\mu v^2}$	Bain (1974)
LHR	-	2μ	0	$1-e^{\{\alpha v+\mu v^2\}}$	Bain (1974)
QHR	0	-	2	$1-e^{\left\{\frac{\beta}{2}v^2+\frac{\alpha}{3}v^2\right\}}$	Bain (1974)
Weibull	$\delta\theta$	0	$\delta-1$	$1-e^{-\theta v^{\delta}}$	Weibull (1951)
PHR	-	0	-	$1-e^{-\left\{\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}}$	Mugdadi (2005)

We note that the characterizing differential equation for PLHRD from Equations (7) and (8) is

$$t(v)=(\beta\, v+\alpha\, v^{\lambda})[1-T(v)]. \tag{9}$$

Equation (9) will be considered to find: (i) explicit expressions, (ii) relations for moments of GOS and (iii) the characterization results. The intent of this research is to exhibit moment properties of GOS derived from PLHRD. Characterization results are given through several techniques. The formulation of the paper is as follows. Outright expression, single moments and numeric computations are addressed in Section 2. Product moments are dealt in Section 3. The characterization outcomes are proved in last section.

2. Single Moments

Firstly, existence of $E[V^m(g:b,c,a)]$ is set up,

$$E[V^m(g:b,c,a)] = \int_0^\infty v^m t_{g:b,c,a}(v)\,dv. \tag{10}$$

When $c \neq -1$, using Equation (2) in (10) and apply binomial expansion. It gives

$$I\int_0^\infty v^m [\bar{T}(v)]^{\phi_{g-x}} \frac{t(v)}{\bar{T}(v)} dv, \tag{11}$$

where $I = \frac{K_{g-1}}{(g-1)!(c+1)^{g-1}} \sum_{x=0}^{g-1} (-1)^x \binom{g-1}{x}$. Further, on using Equation (9) in (11), we obtain $-\frac{I}{\phi_{g-x}} \int_0^\infty v^m \left(\frac{d\,(e^{-\phi_{g-x}(\beta v + \frac{\alpha}{\lambda} v^\lambda)})}{dv} \right) dv.$ Integrating by parts now yields $\frac{(m+1)I}{\phi_{g-x}} \int_0^\infty v^m e^{-\phi_{g-x}(\beta v + \frac{\alpha}{\lambda} v^\lambda)}\,dv.$

Now expanding $\exp(-\phi_{g-x}\,\beta v)$ in Taylor series, it gives

$$I^* \int_0^\infty v^{m+y} e^{-\phi_{g-x}\frac{\alpha}{\lambda} v^\lambda}\,dv, \tag{12}$$

where $I^* = \frac{(m+1)I}{\phi_{g-x}} \sum_{y=0}^{\infty} (-1)^y \frac{(\beta\phi_{g-x})^y}{y!}$. Here, we use

$$\int_0^\infty v^r e^{-\beta v^s}\,dv = \frac{Gamma(r+1)/s}{s\beta^{(r+1)/s}}, \quad \beta, r, s > 0. \text{ (see Gradshetyn and Ryzhik 2007).} \tag{13}$$

Now inserting Equation (13) in (12), one obtains

$$\frac{K_{g-1}(m+1)}{(g-1)!\lambda\,(c+1)^{g-1}} \sum_{y=0}^{\infty} \sum_{x=0}^{g-1} (-1)^{x+y} \binom{g-1}{x} \frac{\beta^y (\phi_{g-y})^{y-1-(m+y+1)/\lambda}}{y!\left(\frac{\alpha}{\lambda}\right)^{(m+y+1)/\lambda}}\, gamma\{(m+y+1)/\lambda\}. \tag{14}$$

When c tending to -1, $E[V^m(g,b,c,a)] = \frac{0}{0}$ as $\sum_{x=0}^{g-1} (-1)^x \binom{g-1}{x} = 0.$ Since Equation (14) holds the form of $\left(\frac{0}{0}\right)$ at $c \to -1$, we simplify the expression as

$$B \sum_{x=0}^{g-1} (-1)^y \binom{g-1}{x} \frac{[a+(b-g+x)(c+1)]^{y-\{(m+y+1)/\lambda\}-1}}{(c+1)^{g-1}}, \tag{15}$$

where $B = \frac{K_{g-1}(m+1)}{(g-1)!\lambda} \sum_{y=0}^{\infty} (-1)^y \frac{\alpha^y\ gamma\{(m+y+1)/\lambda\}}{y!\left(\frac{\alpha}{\lambda}\right)^{(m+y+1)/\lambda}}$. Differentiating Equation (15) $(r-1)$ times about to m and adopting L' Hospital's rule, we come across

$$\lim_{c\to -1} E[V^m(g:b,c,a)] = B \frac{[\{(m+y+1)/\lambda\}+1-y]\ldots[\{(m+y+1)/\lambda\}+g-(1+y)]}{(g-1)!\,a^{\{(m+y+1)/\lambda\}+g-y}} \times \sum_{x=0}^{g-1} (-1)^x \binom{g-1}{x} (g-b-x)^{g-1}. \tag{16}$$

From Ruiz (1996), we have

$$\sum_{z=0}^{b}(-1)^{z}\binom{b}{z}(v-z)^{b}=b!.\text{ (for all integers } b\geq 0 \text{ and for all real numbers } v). \quad (17)$$

Now substituting Equation (17) in (16) and simplifies

$$E(V_{g}^{(a)})^{m}=\frac{m+1}{(g-1)!\lambda}\sum_{y=0}^{\infty}(-1)^{y}\frac{\alpha^{y}gamma\{(m+y+1)/\lambda\}gamma\{((m+y+1)/\lambda)-y+g\}}{y!a^{\{(m+y+1)/\lambda\}-y}\left(\frac{\alpha}{\lambda}\right)^{(m+u+1)/\lambda}gamma\{((m+y+1)/\lambda)-y+1\}}, \quad (18)$$

where $V_{g}^{(a)}$ represents the a^{th} upper record.

Special cases:

(i) The exact expression for O.S. can be derived from Equation (14) as follows

$$E(V_{g:b}^{m})=K_{g:b}B'\sum_{y=0}^{\infty}\sum_{x=0}^{g-1}(-1)^{x+y}\binom{g-1}{x}(b-g+x+1)^{y-1-(m+y+1)/\lambda}, \quad (19)$$

where $K_{g:b}=\frac{b!}{(g-1)!(b-g)!}$ and $B'=\frac{(m+1)}{\lambda}\frac{\beta^{y}}{y!\left(\frac{\alpha}{\lambda}\right)^{(m+y+1)/\lambda}}gamma\{(m+y+1)/\lambda\}$

(ii) From Equation (18), the explicit formula for upper record as follows

$$E[V_{U(g)}^{m}]=\frac{m+1}{(g-1)!\lambda}\sum_{y=0}^{\infty}(-1)^{y}\frac{\beta^{y}\,gamma\{(m+y+1)/\lambda\}\,gamma\{((m+y+1)/\lambda)-y+g\}}{y!\left(\frac{\alpha}{\lambda}\right)^{(m+y+1)/\lambda}gamma\{((m+y+1)/\lambda)-y+1\}}$$

(20)

The expressions presented in (19) and (20) enable to compute the moments of O.S. and upper record values for specific parameters and distinct sample size. Tables 3 and 4 consists of these values respectively.

Table 3 First four moments of O.S. for PLHRD

n	r	$\beta=1,\ \lambda=3,\ \alpha=1$				$\beta=1,\ \lambda=3,\ \alpha=2$			
		$m=1$	$m=2$	$m=3$	$m=4$	$m=1$	$m=2$	$m=3$	$m=4$
1	1	0.3867	0.3277	0.3264	0.4564	0.2121	0.1935	0.1401	0.1121
2	1	0.1524	0.0631	0.0204	0.0985	0.1245	0.0234	0.0352	0.0304
	2	0.4179	0.4820	0.5221	0.7141	0.3940	0.2713	0.2045	0.1715
3	1	0.0584	0.0635	0.0112	0.0329	0.0442	0.0352	0.0211	0.0120
	2	0.3322	0.1249	0.1598	0.2257	0.1671	0.0375	0.0568	0.0119
	3	0.6241	0.6506	0.7523	0.9086	0.5012	0.1828	0.1262	0.1109
4	1	0.0074	0.0287	0.0161	0.0052	0.1056	0.0211	0.0086	0.0093
	2	0.1054	0.0873	0.1168	0.0731	0.0202	0.0437	0.0122	0.0202
	3	0.3755	0.2555	0.3928	0.1552	0.2339	0.1262	0.0234	0.0736
	4	0.6453	0.7080	0.0401	0.7587	0.5673	0.4257	0.3225	0.2083
5	1	0.0776	0.0311	0.0147	0.0078	0.0785	0.0274	0.0110	0.0046
	2	0.0264	0.1100	0.0511	0.0440	0.1132	0.0761	0.0367	0.0164
	3	0.3256	0.2882	0.2103	0.1640	0.3798	0.2163	0.1323	0.0860
	4	0.4739	0.5321	0.2144	0.2877	0.3710	0.2028	0.1007	0.1344
	5	0.9137	0.9169	0.9963	0.9776	0.6192	0.5429	0.5079	0.5018

Table 3 (cont.)

n	r	$\beta=2,\ \lambda=3,\ \alpha=1$				$\beta=2,\ \lambda=3,\ \alpha=2$			
		$m=1$	$m=2$	$m=3$	$m=4$	$m=1$	$m=2$	$m=3$	$m=4$
1	1	0.1696	0.1131	0.0862	0.0651	0.1523	0.0433	0.0107	0.0785
2	1	0.0094	0.0465	0.0240	0.0135	0.1063	0.0387	0.0150	0.0131
	2	0.2477	0.2665	0.2414	0.2255	0.1772	0.1355	0.1133	0.0912
3	1	0.0568	0.0116	0.0106	0.0030	0.0284	0.0140	0.0057	0.0032
	2	0.1867	0.1167	0.0652	0.0542	0.1986	0.0872	0.0501	0.0188
	3	0.4667	0.4421	0.4106	0.4420	0.4634	0.2153	0.2110	0.2212
4	1	0.0213	0.0110	0.0022	0.0020	0.0219	0.0101	0.0032	0.0014
	2	0.0987	0.0342	0.0206	0.0165	0.1156	0.0413	0.0121	0.0098
	3	0.2317	0.1927	0.1287	0.0982	0.1764	0.1287	0.0767	0.0872
	4	0.4545	0.4815	0.4923	0.5858	0.5128	0.4030	0.3189	0.3070
5	1	0.0224	0.0066	0.0020	0.0008	0.0242	0.0061	0.0019	0.0006
	2	0.0706	0.0207	0.0127	0.0059	0.0737	0.0190	0.0091	0.0042
	3	0.1712	0.0773	0.0458	0.0341	0.1860	0.1982	0.0231	0.0198
	4	0.3127	0.1874	0.1098	0.1450	0.3128	0.1872	0.1345	0.0785
	5	0.7123	0.6134	0.6753	0.6512	0.6094	0.3452	0.4087	0.6028

In Table 3, the relation $E\sum_{n=1}^{b}(V_{n:b}^{m})=bE(V^{m})$ is satisfied (David and Nagaraja 2003).

Table 4 First four moments of upper record for PLHRD

n	$\beta=1,\ \lambda=3,\ \alpha=1$				$\beta=1,\ \lambda=3,\ \alpha=2$			
	$m=1$	$m=2$	$m=3$	$m=4$	$m=1$	$m=2$	$m=3$	$m=4$
1	0.4387	0.4218	0.4093	0.4321	0.4914	0.3277	0.2542	0.2183
2	0.8123	0.9015	0.9751	0.9342	0.6542	0.5420	0.6177	0.5619
3	0.9864	0.8754	0.9842	1.6757	0.8971	0.9097	0.8976	1.0783
4	0.9456	0.8423	1.7789	2.0375	0.0957	0.8935	0.9571	0.9822
5	0.9753	1.2407	2.5406	3.7487	0.9903	0.9869	0.9004	1.4961
n	$\beta=2,\ \lambda=3,\ \alpha=1$				$\beta=2,\ \lambda=3,\ \alpha=2$			
	$m=1$	$m=2$	$m=3$	$m=4$	$m=1$	$m=2$	$m=3$	$m=4$
1	0.2965	0.1539	0.1037	0.0864	0.2564	0.1013	0.1366	0.1461
2	0.5674	0.5064	0.5105	0.5431	0.5143	0.3583	0.3818	0.3463
3	0.8730	0.8145	0.7865	0.9520	0.6059	0.5212	0.4572	0.6754
4	1.1873	1.3608	1.7572	2.3884	0.9536	0.9724	1.0463	1.1779
5	1.2543	1.6535	2.0349	2.5674	1.0451	1.0137	1.1218	1.8743

The single moments of GOS from PLHRD are presented below.

Theorem 1 *For PLHRD reported in* (7) *and* $b\in N,\ b\geq 2$,

$$E[V^{m}(1:b,\tilde{c},a)]=\frac{\alpha\phi_{g}}{m+\lambda+1}\Big[E[V^{m+\lambda+1}(1:b,\tilde{c},a)]\Big]+\frac{\beta\phi_{g}}{m+2}\Big[E[V^{m+2}(1:b,\tilde{c},a)]\Big] \tag{21}$$

and for $2\leq g\leq b$,

$$E[V^m(g:b,\tilde{c},a)] = \frac{\alpha}{m+\lambda+1}\Big[\phi_g E[V^{m+\lambda+1}(g:b,\tilde{c},a)]\Big] - E[V^{m+\lambda+1}(g-1:b,\tilde{c},a)]$$
$$+\frac{\beta}{m+2}\Big[\phi_g E[V^{m+2}(g:b,\tilde{c},a)] - E[V^{m+2}(g-1:b,\tilde{c},a)]\Big]. \quad (22)$$

Proof: Making use of Equation (5) and (9), one gets,

$$E[V_1^m(g:b,\tilde{c},a)] = A_1(v) + A_2(v), \quad (23)$$

where $A_1(v) = K_{g-1}\alpha\int_0^\infty v^{m+\lambda}\sum_{n=1}^{g} w_n(g)[\bar{T}(v)]^{\phi_n}\,dv$ and $A_2(v) = K_{g-1}\beta\int_0^\infty v^{m+1}\sum_{n=1}^{g} w_n(g)[\bar{T}(v)]^{\phi_n}\,dv.$

Integrating $A_1(v)$ by parts, it gives

$$A_1(v) = \frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1}\sum_{n=1}^{g} w_n(g)\phi_n[\bar{T}(v)]^{\phi_n-1}t(v)\,dv$$
$$= \frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1}\sum_{n=1}^{g-1} w_n(g)\phi_n[\bar{T}(v)]^{\phi_n-1}t(v)\,dv + \frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1} w_g(g)\phi_g[\bar{T}(v)]^{\phi_g-1}t(v)dv$$
$$= -\frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1}\sum_{n=1}^{g-1} w_n(g)[\phi_g-(\phi_n+\phi_g)][\bar{T}(v)]^{\phi_n-1}t(v)\,dv$$
$$+\frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1} w_g(g)\phi_g[\bar{T}(v)]^{\phi_g-1}t(v)dv$$
$$= -\frac{K_{g-1}\alpha}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1}\sum_{n=1}^{g-1} w_n(g-1)[\bar{T}(v)]^{\phi_n-1}t(v)\,dv$$
$$+\frac{K_{g-1}\alpha\phi_g}{m+\lambda+1}\int_0^\infty v_1^{j+\lambda+1}\sum_{n=1}^{g-1} w_n(g)[\bar{T}(v)]^{\phi_n-1}t(v)\,dv + \frac{K_{g-1}\alpha\phi_g}{m+\lambda+1}\int_0^\infty v^{m+\lambda+1} w_g(g)[\bar{T}(v)]^{\phi_g-1}t(v)\,dv.$$

Using Equation (5), we have

$$A_1(v) = \frac{\alpha}{m+\lambda+1}\Big[\phi_g E[V^{m+\lambda+1}(g:b,\tilde{c},a)] - E[V^{m+\lambda+1}(g-1:b,\tilde{c},a)]\Big].$$

Similarly,

$$A_2(v) = \frac{\beta}{m+2}\Big[\phi_g E[V^{m+2}(g:b,\tilde{c},a)] - E[V^{m+2}(g-1:b,\tilde{c},a)]\Big].$$

Inserting the value of $A_1(v)$ and $A_2(v)$ in (23), we attain the required result. Relation (21) follows from relation (22) by proceeding $V(0:b,\tilde{c},a) = 0.$

Corollary 1 *For Case I, term for single moments of PLHRD has the form*

$$E\left(V^m_{g:b,c,a}\right) = \frac{\alpha}{m+\lambda+1}\Big[\phi_g E\left(V^{m+\lambda+1}_{g:b,c,a}\right) - E\left(V^{m+\lambda+1}_{g-1:b,c,a}\right)\Big] + \frac{\beta}{m+2}\Big[\phi_g E\left(V^{m+2}_{g:b,c,a}\right) - E\left(V^{m+2}_{g-1:b,c,a}\right)\Big].$$

Other model of GOS can be extracted from Theorem 1 at the different values of parameters. Furthermore, several author's works can be extracted as a remark from Equation (22) for different values of parameters satisfying the term for single moments via GOS as follows.

Remarks:

(i) For $\lambda=\beta=0$, the Equation (22) reduces to exponential distribution (Saran and Nain 2014).

(ii) Putting $\lambda=0$ in Equation (22), we get linear exponential distribution (Ahmad 2008).

(iii) Setting $\alpha=\lambda=0,\ \beta=2\theta$ in Equation (22), we have Rayleigh distribution (Mohsin et al. 2010).

(iv) Setting $\beta=0$ in Equation (22), we obtain PHR distribution (Khan 2017).

(v) Setting $\beta=2\theta$ in Equation (22), we get a^{th} upper record valued from PHR distribution (Khan and Khan 2019).

(vi) Setting $\alpha=1,\ \beta=0$ and $\lambda=\delta-1$ in Equation (22), we get Weibull distribution (Saran and Nain 2014).

3. Product Moments

This section contains the product moments for PLHRD through GOS.

Theorem 2 *For PLHRD reported in* (7) *and* $1\le g<h\le b,\ n,m\ge 0,$

$$E[V^{n,m}(g,h:b,\tilde{c},a)]=\frac{\alpha}{m+\lambda+1}\left\{\phi_h E[V^{n,m+\lambda+1}(g,h:b,\tilde{c},a)]-E[V^{n,m+\lambda+1}(g,h-1:b,\tilde{c},a)]\right\}$$
$$+\frac{\beta}{m+2}\left\{\phi_h E[V^{n,m+2}(g,h:b,\tilde{c},a)]-E[V^{n,m+2}(g,h-1:b,\tilde{c},a)]\right\}. \tag{24}$$

Proof: From Equation (6),

$$E[V^{n,m}(g,h:b,\tilde{c},a)]=\int_0^\infty v_1{}^n\sum_{n=1}^{g}w_n(g)[\bar{T}(v_1)]^{\phi_n}\frac{t(v_1)}{\bar{T}(v_1)}A_1(v_1)dv_1, \tag{25}$$

where

$$A'(v_1)=K_{h-1}\int_{v_1}^\infty v_2{}^m\left\{\sum_{m=g+1}^{h}w_m^{(g)}(h)\left(\frac{\bar{T}(v_2)}{\bar{T}(v_1)}\right)^{\phi_m}\right\}\frac{t(v_2)}{\bar{T}(v_2)}dv_2.$$

Using Equation (9) in $A'(v_1)$, one gets

$$A'(v_1)=A_1'(v_1)+A_2'(v_1), \tag{26}$$

as

$$A_1'(v_1)=K_{h-1}\alpha\int_{v_1}^\infty v_2{}^{m+\lambda}\left\{\sum_{m=g+1}^{h}w_m^{(g)}(h)\left(\frac{\bar{T}(v_2)}{\bar{T}(v_1)}\right)^{\phi_m}\right\}dv_2,$$

$$A_2'(v_1)=K_{h-1}\beta\int_{v_1}^\infty v_2{}^{m+1}\left\{\sum_{m=g+1}^{h}w_m^{(g)}(h)\left(\frac{\bar{T}(v_2)}{\bar{T}(v_1)}\right)^{\phi_m}\right\}dv_2.$$

Integrating $A_1'(v_1)$ by parts, yields

$$A_1'(v_1)=\frac{K_{h-1}\alpha}{m+\lambda+1}\int_{v_1}^\infty v_2{}^{m+\lambda+1}\sum_{m=g+1}^{h}w_m^{(g)}(h)\phi_m\left(\frac{\bar{T}(v_2)}{\bar{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\bar{T}(v_2)}dv_2$$
$$+\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^\infty v_2{}^{m+\lambda+1}w_m^{(g)}(h)\phi_h\left(\frac{\bar{T}(v_2)}{\bar{T}(v_1)}\right)^{\phi_h}\frac{t(v_2)}{\bar{T}(v_2)}dv_2$$

$$= -\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{x}^{\infty} v_2^{m+\lambda+1}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)[\phi_h-(\phi_m+\phi_h)]\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv_2$$

$$+\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}a_m^{(g)}(h)\phi_h\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_h}\frac{t(v_2)}{\overline{T}(v_2)}dv_2$$

$$= -\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)(h-1)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv$$

$$+\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)\phi_h\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv_2$$

$$+\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}w_m^{(g)}(h)\phi_h\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_h}\frac{t(v_2)}{\overline{T}(v_2)}dv_2$$

$$A_1'(v_1) = -\frac{K_{s-1}\alpha}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)(h-1)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv$$

$$+\frac{K_{h-1}\alpha\phi_h}{m+\lambda+1}\int_{v_1}^{\infty} v_2^{m+\lambda+1}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv_2.$$

Similarly

$$A_2'(v_1) = -\frac{K_{s-1}\beta}{m+2}\int_{v_1}^{\infty} v_2^{m+2}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)(h-1)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv$$

$$+\frac{K_{h-1}\beta\phi_h}{m+2}\int_{v_1}^{\infty} v_2^{m+2}\sum_{m=g+1}^{h-1} w_m^{(g)}(h)\left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_m}\frac{t(v_2)}{\overline{T}(v_2)}dv_2.$$

Using the values of $A_1'(v_1)$ and $A_2'(v_1)$ in Equation (26), then Equation (26) putting into Equation (25), we attain the required result.

Note: At $n=0$, product moments correspond to single moments.

Corollary 2 *For Case I, the term for product moments of PLHRD is*

$$E\left(V_{g,h:b,\tilde{c},a}^{n,m}\right) = \frac{\alpha}{m+\lambda+1}\left[\phi_h E\left(V_{g,h:b,\tilde{c},a}^{n,m+\lambda+1}\right) - E\left(V_{g,h-1:b,\tilde{c},a}^{n,m+\lambda+1}\right)\right] + \frac{\beta}{m+2}\left[\phi_h E\left(V_{g,h:b,\tilde{c},a}^{n,m+2}\right) - E\left(V_{g,h-1:b,\tilde{c},a}^{n,m+2}\right)\right].$$

On the choice of parameter, several model of GOS can be set up from Theorem 2. In addition, several remarks can be produced from Equation (24) for product moments of GOS as mentioned below.

Remarks:

(i) Setting $\lambda=\beta=0$ in Equation (24), we obtain exponential distribution (Saran and Nain 2014).

(ii) Setting $\lambda=0$ in Equation (24), we get L.E. distribution (Ahmad, 2008).

(iii) Setting $\alpha=\lambda=0,\ \beta=2\theta$ in Equation (24), we attain Rayleigh distribution (Mohsin et al. 2010)

(iv) Setting $\beta = 0$ in Equation (24), we get PHR distribution (Khan 2017).

(v) Setting $\beta = 2\theta$ in Equation (24), we get a^{th} record values from PHR distribution (Khan and Khan 2019).

(vi) Setting $\alpha = 0$, $\beta = 0$ and $\lambda = \delta - 1$ in Equation (24), we get Weibull distribution (Saran and Nain 2014).

4. Characterizations

We discuss the characterizations of PLHRD using the different technique namely (i) recurrence relations (ii) minimal O.S. (iii) conditional moments based on GOS and (iv) truncated moment.

Theorem 3 *If $V \sim PLHRD(\alpha, \beta, \lambda)$, then the necessary and sufficient condition for RV, V is given as*

$$E[V^m(g:b,\tilde{c},a)] = \frac{\alpha}{m+\lambda+1}\Big[\phi_g E[V^{m+\lambda+1}(g:b,\tilde{c},a)]\Big] - E[V^{m+\lambda+1}(g-1:b,\tilde{c},a)] + \frac{\beta}{m+2}\Big[\phi_g E[V^{m+2}(g:b,\tilde{c},a)] - E[V^{m+2}(g-1:b,\tilde{c},a)]\Big]. \tag{27}$$

Proof: From Corollary 1, necessary part follows, if the expression in (27) is fulfilled, then RHS of Equation (27) can be simplified as

$$\frac{K_{g-1}}{(g-1)!}\int_0^\infty v^m [\overline{T}(v)]^{\phi_{g-1}} f(v)\, p_c^{g-1}[T(v)]dv = \frac{\alpha}{(m+\lambda+1)}\frac{K_{g-1}}{(g-1)!}\int_0^\infty v^{m+\lambda+1} I'(v)\,dv + \frac{\beta}{(m+2)}\frac{K_{g-1}}{(g-1)!}\int_0^\infty v^{m+2} I'(v)\,dv. \tag{28}$$

where $$I(v) = -[\overline{T}(v)]^{\phi_g}\, p_c^{g-1}[T(v)]. \tag{29}$$

Integrating RHS of (28), by parts and employing value of $I(v)$ from (29), it gives

$$\frac{K_{g-1}}{(g-1)!}\int_0^\infty v^{m-1} [\overline{T}(v)]^{\phi_{g-1}}\, p_c^{g-1}[T(v)][t(v) - (\beta v + \alpha v^\lambda)[1-T(v)]dv = 0. \tag{30}$$

Applying the Müntz-Szász' generalized Theorem to (30), one gets,

$$t(v) = (\beta v + \alpha v^\lambda)[1-T(v)].$$

Hence, Theorem 3 is proved.

Theorem 4 *Let k be a non-negative integer and conditions stated in Theorem 3. Then relation for minimal O.S. is given as,*

$$E[V_{1:b}^k] = \frac{1}{k+\lambda+1} b\alpha E[V_{1:b}^{k+\lambda+1}] + \frac{1}{k+2} b\beta E[V_{1:b}^{k+2}].$$

Proof: Theorem 4 can be justified in the same way as in Theorem 3.

Theorem 5 *Let $V(b,g,c,a)$ be the g^{th}GOS fixed on continuous DF and expectation exists.* Then $1 \le g < h \le b,\ g = 1,\ldots,b$

$$E\Big[\psi\Big(V_{h|l:b,c,a} = v\Big)\Big] = e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} \prod_{n=1}^{h-l}\left(\frac{\phi_{l+n}}{\phi_{l+n}+1}\right),\ l = g, g+1, \tag{31}$$

if and only if Equation (7) holds, where $\psi(v_2) = e^{-\left\{\frac{\beta}{2}v_2^2 + \frac{\alpha}{\lambda+1}v_2^{\lambda+1}\right\}}$.

Proof: From Equation (4),

$$E\left[\psi\left(V_{h|l:b,c,a} = v\right)\right] = D\int_{v_1}^{\infty} e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} \left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{\phi_h} \left\{1 - \left(\frac{\overline{T}(v_2)}{\overline{T}(v_1)}\right)^{c+1}\right\}^{h-g-1} \frac{t(v_2)}{\overline{T}(v_2)} dv_2, \quad (32)$$

where $D = \dfrac{K_{h-1}}{K_{g-1}(h-g-1)!(c+1)^{h-g-1}}$. Setting $z = \dfrac{\overline{T}(v_2)}{\overline{T}(v_1)} = \dfrac{e^{-\left\{\frac{\beta}{2}v_2^2 + \frac{\alpha}{\lambda+1}v_2^{\lambda+1}\right\}}}{e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}}}$ in Equation (32),

$$E\left[\psi\left(V_{h|l:b,c,a} = v\right)\right] = De^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} \int_0^1 z^{\phi_h} (1 - z^{c+1})^{h-g-1} dz. \quad (33)$$

Again, by setting $a = z^{c+1}$ in Equation (33), gives

$$E\left[\psi\left(V_{h|l:b,c,a} = v\right)\right] = De^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} \int_0^1 a^{\frac{\phi_h+1}{c+1}-1} (1-a)^{h-g-1} da.$$

On simplification, the necessary part is determined, as $\dfrac{K_{h-1}}{K_{g-1}} = \prod_{n=1}^{h-g} \phi_{g+n}$. For sufficiency part. Assume Equation (4) and (7),

$$D\int_{v_1}^{\infty} e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} [(\overline{T}(v_1))^{c+1} - (\overline{T}(v_2))^{c+1}]^{h-g-1} \times [\overline{T}(v_2)]^{\phi_h - 1} t(v_2) dv_2 = G_{h|g}(v_1)[\overline{T}(v_1)]^{\phi_{g+1}}, \quad (34)$$

where $P_{h|g}(v_1) = e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}} \prod_{n=1}^{h-g} \left(\dfrac{\phi_{g+n}}{\phi_{g+n}+1}\right)$. Differentiating Equation (34) about v_1, obtains

$$P_{h|g}(v_1)[\overline{T}(v_1)]^{\phi_{g+1}-1} t(v_1) - P_{h|g+1}(v_1)[\overline{T}(v_1)]^{\phi_{g+2}+c} t(v_1) = \frac{P'_{h|g}(v_1)[\overline{T}(v_1)]^{\phi_{g+1}}}{\phi_{g+1}},$$

where

$$P'_{h|g}(x) = -\prod_{n=1}^{h-g} \left(\frac{\phi_{g+n}}{\phi_{g+n}+1}\right)(\beta v_1 + \alpha v_1^{\lambda}) e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}}$$

and

$$P_{h|g+1}(v_1) = \left(\frac{\phi_{g+1}+1}{\phi_{g+1}}\right) \prod_{n=1}^{h-g} \left(\frac{\phi_{g+n}}{\phi_{g+n}+1}\right) e^{-\left\{\frac{\beta}{2}v_1^2 + \frac{\alpha}{\lambda+1}v_1^{\lambda+1}\right\}}.$$

Therefore, $\dfrac{t(v_1)}{\overline{T}(v_1)} = -\dfrac{1}{\phi_{g+1}}\left[\dfrac{P'_{h|g}(v_1)}{P_{h|g+1}(v_1) - P_{h|g}(v_1)}\right]$ (Khan et al. 2006). After simplification, $\dfrac{t(v_1)}{\overline{T}(v_1)} = \beta v_1 + \alpha v_1^{\lambda}$. Hence, it is the result.

Theorem 6 *Let* V *be a continuous RV with df* $T(v)$ *and pdf* $t(v)$. *Further assume that* $t'(v)$ *and* $E(V \mid V \le v)$ *exists. Then*

$$E(V \mid V \le v) = \xi(v)\psi(v), \quad 0 < v < \infty, \quad (35)$$

where

$$\xi(v)=\frac{1}{(\beta v+\alpha v^{\lambda})}\left[-v+e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}}\int_0^v e^{-\left\{\frac{\beta}{2}u^2+\frac{\alpha}{\lambda+1}u^{\lambda+1}\right\}}du\right] \text{ and } \psi(v)=\frac{t(v)}{T(v)}$$

if and only if Equation (7) holds.

Proof: Since

$$E(V\mid V\le v)=\frac{1}{T(v)}\int_0^v u\,t(u)\,du=\frac{1}{T(v)}\int_0^v u(\beta u+\alpha u^{\lambda})e^{-\left\{\frac{\beta}{2}u^2+\frac{\alpha}{\lambda+1}u^{\lambda+1}\right\}}du. \tag{36}$$

Integrating Equation (38) by parts treating $(\beta u+\alpha u^{\lambda})e^{-\left\{\frac{\beta}{2}u^2+\frac{\alpha}{\lambda+1}u^{\lambda+1}\right\}}$ for integration, one obtains,

$$E(V\mid V\le v)=\frac{1}{T(v)}\left\{-v\,e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}}+\int_0^v e^{-\left\{\frac{\beta}{2}u^2+\frac{\alpha}{\lambda+1}u^{\lambda+1}\right\}}du\right\}. \tag{37}$$

Divide and multiply by $t(v)$ in Equation (37), the relation (35) holds. For the sufficiency part. From Ahsanullah et al. (2016), we have the following

$$\frac{1}{T(v)}\int_0^v u\,t(u)\,du=\frac{\xi(v)f(v)}{T(v)}. \tag{38}$$

Differentiating above equation both sides regarding v, it gives

$$\frac{t'(v)}{t(v)}=\frac{v-\xi'(v)}{\xi(v)}. \tag{39}$$

Performing integration both sides in Equation (39) concerning v, one gets,

$$t(v)=K(\beta v+\alpha v^{\lambda})e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}}.$$

Using the fact $\int_0^\infty t(v)\,dv=1$ and $K=1$. This implies

$$t(v)=(\beta v+\alpha v^{\lambda})e^{-\left\{\frac{\beta}{2}v^2+\frac{\alpha}{\lambda+1}v^{\lambda+1}\right\}},\ \ v>0,\ \alpha,\beta\ge 0,\ \lambda>-1,\ \ \lambda\ne 1.$$

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