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A New Compound Exponentiated Weibull Distribution with Statistical Properties and Applications

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Abstract

This paper is intended to design a new distribution focused on Poisson zero truncated that also accommodates many significant failure rates. Several of its characteristics are derived mathematically. The actual model's density is being interpreted as a mixture of exponentiated Weibull densities. The maximum likelihood method is considered to estimate model parameters. We demonstrated the significance and versatility of the compound exponentiated Weibull distribution through simulation of three data sets.

Keywords: Truncated Poisson model, moments, generating function, maximum likelihood, simulation.

1. Introduction

A random variable (r.v.) X is stated to follow an exponentiated Weibull (EW) distribution (Mudholkar and Srivastava 1993) if its probability density function (PDF) and cumulative distribution function (CDF) are specified by

$$g_{\alpha,\beta}(x) = \alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha-1},$$

and

$$G_{\alpha,\beta}(x) = \left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha},$$

respectively, for $x>0,\ \alpha>0$ and $\beta>0$. For, $\alpha=1$ we have the standard one parameter Weibull model (Weibull 1951). For $\beta=1,$ we have the one parameter exponentiated exponential model (Gupta and Kundu 2001). For $\beta=2,$ we have the standard one parameter Burr type X distribution. Within the paper, we have suggested and analyzed a new distribution with a strong physical inspiration using zero truncated Poisson (ZTP) model. Suppose a process has N components that work independently at a time when N has ZTP distribution of parameter Z. It is also the conditional probability distribution of a random variable having Poisson distribution so that the r.v. value is not zero. Therefore the probability mass function (PMF) of N is determined by

$$PMF_{ZTP}^{(\lambda)}\left(N=n\right)=\left[\exp\left(-\lambda\right)\lambda^{n}\right]/\left\{n!c_{\lambda}\right\}\Big|_{(n=1,2,...)}.$$

Let $E\left(N|\lambda\right)$ and $Var\left(N|\lambda\right)$ are expectation and variance of the ZTP r.v. which are given by, respectively

$$E\left(N|\lambda\right)=\lambda/c_{\lambda} \quad \text{and} \quad Var\left(N|\lambda\right)=\frac{\lambda+\lambda^{2}}{c_{\lambda}}-\frac{\lambda^{2}}{c_{\lambda}^{2}},$$

where $c_{\lambda}=1-\exp\left(-\lambda\right)$. Assume that each subsystem's failure time has the Burr X EW distribution $\left(BXEW\left(\theta,\alpha,\beta\right)\right)$ identified from the cumulative distribution function (CDF) below:

$$H_{BXEW}^{(\theta,\alpha,\beta)}\left(x\right)=\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{2}\right)\right]^{\theta},$$

or $H_{BXEW}^{(\theta,\alpha,\beta)}\left(x\right)=\left[1-\exp\left(-\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right]^{\theta}$, in which $\theta>0$ is a shape parameter. Let Y_i symbolize the i^{th} subsystem failure time and let $X=\min\left\{Y_1,Y_2,...,Y_N\right\}$. The conditional CDF of X given N is then specified by

$$F\left(x|N\right)=1-Pr\left(X>x|N\right)=1-\left[1-H_{BXEW}^{(\theta,\alpha,\beta)}\left(x\right)\right]^{N}.$$

Therefore, the unconditional CDF of the Poisson Burr X exponentiated Weibull (PBXEW) density function, as described in Silva et al. (2015), can be expressed as

$$H_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}\left(x\right)=c_{\lambda}^{-1}\left(1-\exp\left\{-\lambda\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{2}\right)\right]^{\theta}\right\}\right),$$

or

$$H_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}\left(x\right)=c_{\lambda}^{-1}\left(1-\exp\left\{-\lambda\left[1-\exp\left(-\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right]^{\theta}\right\}\right). \tag{1}$$

Along with the respective probability density function (PDF) as

$$f_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}\left(x\right)=2\lambda\theta\alpha\beta c_{\lambda}^{-1}\frac{x^{\beta-1}\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha-1}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3}}\times\frac{\exp\left(-x^{\beta}\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)}{\left[1-\exp\left(-\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right]^{1-\theta}} \tag{2}$$

$$\times\exp\left\{-\lambda\left[1-\exp\left(-\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right]^{\theta}\right\}.$$

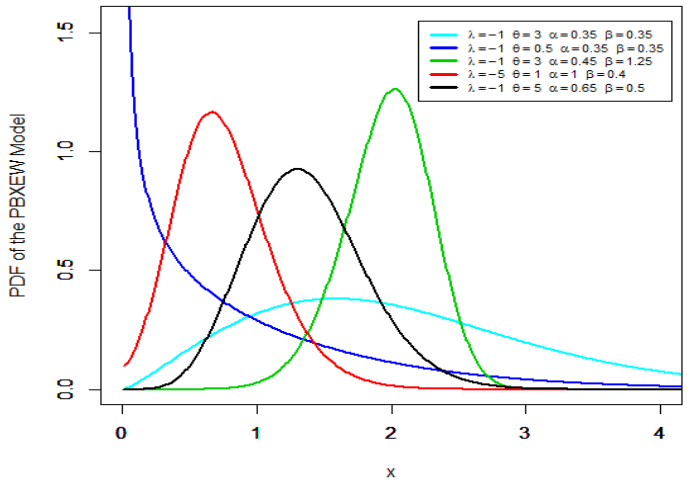


Figure 1 PDF plots for the PBXEW model

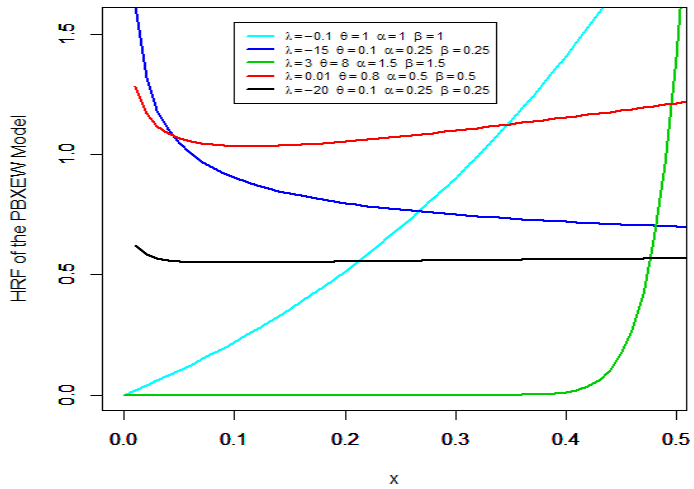


Figure 2 HRF plots for the PBXEW model

One may cite some useful extensions of the W model such as Alizadeh et al. (2018), Almalki and Nadarajah (2014), Ansari and Nofal (2021), Aryal et al (2017), Bebbington et al. (2007), Cordeiro et al. (2014a, 2017), Hamedani et al. (2018, 2019), Hogg et al. (2005), Johnson et al. (1994), Kamal et al. (2020), Lai et al. (2003), Mansour et al. (2020a, 2020b), Marshall and Olkin (2007), Nadarajah and Haghighi (2011), Nofal et al. (2016), Oluyede et al. (2015), Yousof et al. (2017a, 2017b, 2018) among others.

Figure 1 illustrates that the PBXEW model is suitable for modelling right skewed, symmetric, left skewed and unimodal data sets. Figure 2 illustrates that the PBXEW model's HRF may be either increasing or bathtub-shaped or decreasing or constant or J-shaped.

2. Mathematical Properties

2.1. Useful Expansions

The PDF in (2) can also be presented using the power series as

$$f_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}(x)=\sum_{h=0}^{\infty}\left\{\frac{2\lambda^{1+h}\theta\alpha\beta x^{\beta-1}(-1)^{h}\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha-1}}{h!c_{\lambda}\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3}}\times\exp\left(-x^{\beta}\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right.$$
$$\left.\times\left[1-\exp\left(-\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)\right]^{\theta(h+1)-1}\right\}. \tag{3}$$

Consider the following power series

$$(1-s)^{c-1}=\sum_{i=0}^{\infty}\frac{(-s)^{i}\Gamma(c)}{i!\Gamma(c-i)}\Bigg|_{(|s|<1\text{ and }c>0)}. \tag{4}$$

Applying (4) to (3), we have

$$f_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}(x)=\frac{2\theta\alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha-1}}{c_{\lambda}\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3}}$$
$$\times\sum_{h=0}^{\infty}\frac{\lambda^{1+h}(-1)^{h+i}\Gamma\left(\theta(h+1)\right)}{i!\Gamma\left(\left(\theta(h+1)\right)-i\right)}\times\underbrace{\exp\left(-(i+1)\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}-1\right\}^{-2}\right)}_{A}. \tag{5}$$

When the power series is applied to term A, Equation (5) becomes

$$f_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}(x)=\frac{2\theta\alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha-1}}{c_{\lambda}\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3}}$$
$$\times\sum_{h=0}^{\infty}\frac{2\lambda^{1+h}(-1)^{h+i+d}(i+1)^{d}\Gamma\left(\theta(h+1)\right)}{i!d!\left(1-e^{-\lambda}\right)\Gamma\left(\left(\theta(h+1)\right)-i\right)}\times\underbrace{\frac{x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha(d+1)-1}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3+2d}}}_{B}. \tag{6}$$

Let us try the expansion of the series

$$(1-\zeta)^{-c}\Big|_{(|\zeta|<1,c>0)}=\sum_{k=0}^{\infty}\frac{\Gamma(c+k)}{k!\Gamma(c)}\zeta^{k}. \tag{7}$$

If the extension for the term B is extended in (7) to (6), Equation (6) has now become

$$f_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}(x)=\sum_{k=0}^{\infty}b_{d,k}g_{\alpha(2d+k+2),\beta}(x), \tag{8}$$

where

$$b_{d,k}=\frac{2\theta\lambda^{1+h}(-1)^{d}\Gamma(3+2d+h)}{d!k!\mathrm{c}_{\lambda}\Gamma(2d+3)\Gamma 2d+k+2}\sum_{h,i=0}^{\infty}\frac{(-1)^{h+i}(i+1)^{d}\Gamma\left(\theta(h+1)\right)}{i!\Gamma\left(\left(\theta(h+1)\right)-i\right)},$$

and

$$g_{\alpha(2d+k+2),\beta}(x)=\left[2\left(2d+k+2\right)\right]\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha(2d+k+2)-1},$$

which is the PDF of EW model with parameters $\alpha\left(2d+k+2\right)$ and β. Equation (8) shows that density of X can also be represented as linear combination of exp-G densities. Hence numerous mathematical characteristics of the new family will be computed from those of the EW model. Consequently, the CDF of PBXEW model can be given by

$$F_{PBXEW}^{(\lambda,\theta,\alpha,\beta)}(x)=\sum_{d,k=0}^{\infty}b_{d,k}G_{\alpha(2d+k+2),\beta}(x),$$

where $G_{\alpha(2d+k+2),\beta}(x)=\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha(2d+k+2)}$ is the CDF of the EW model parameters $\alpha\left(2d+k+2\right)$ and β.

2.2. Quantile function (QF)

Let $Q(u)$ be the QF of X, where $X\sim PNXEW\left(\lambda,\theta,\alpha,\beta\right)$ then $Q(u)$ is computed by inverting (1) as

$$Q(u)=\left(-\ln\left\{1-\left[1+\left(-\ln\left\{1-\left[\frac{-\ln\left(1-uc_{\lambda}\right)}{\lambda}\right]^{1/\theta}\right\}\right)^{1/2}\right]^{1/\alpha}\right\}\right)^{1/\beta},\quad 0\le u\le 1.$$

Simulation of the PBXEW r.v. is straightforward. Assume that U is a uniform variate on the unit interval $(0,1)$, then the r.v. $X=Q(u)$ follows Equation (2).

2.3. Moments

Let μ_r' be the r^{th} ordinary moment of X, comes from (8) as

$$\mu_r'=\boldsymbol{E}(X^r)=\Gamma\left(1+\frac{r}{\beta}\right)\sum_{d,k,h=0}^{\infty}\boldsymbol{v}_{d,k,h}^{\left\{r,\left[\alpha(2d+k+2)\right]\right\}}\Bigg|_{(r>-\beta)},\tag{9}$$

where

$$\boldsymbol{v}_{d,k,h}^{\left\{r,\left[\alpha(2d+k+2)\right]\right\}}=b_{d,k}\boldsymbol{v}_{h}^{\left\{r,\left[\alpha(2d+k+2)\right]\right\}},$$

and

$$\boldsymbol{v}_{c_1}^{\{r,c_2\}}=c_2\left(-1\right)^{c_1}\left(1+c_1\right)^{-\left(1+\frac{r}{\beta}\right)}\binom{c_2-1}{c_1}.$$

We obtain the mean of X, $\boldsymbol{E}(X^2)$, $\boldsymbol{E}(X^3)$, $\boldsymbol{E}(X^4)$ by placing $r=1, 2, 3$ and 4 in (9),

$$\mu_1'=\boldsymbol{E}\left(X\right)=\Gamma\left(1+\frac{1}{\beta}\right)\sum_{d,k,h=0}^{\infty}\boldsymbol{v}_{d,k,h}^{\left\{1,\left[\alpha(2d+k+2)\right]\right\}}\Bigg|_{(1>-\beta)},$$

$$\mu_2'=\boldsymbol{E}(X^2)=\Gamma\left(1+\frac{2}{\beta}\right)\sum_{d,k,h=0}^{\infty}\boldsymbol{v}_{d,k,h}^{\left\{2,\left[\alpha(2d+k+2)\right]\right\}}\Bigg|_{(2>-\beta)},$$

$$\mu_3'=\boldsymbol{E}(X^3)=\Gamma\left(1+\frac{3}{\beta}\right)\sum_{d,k,h=0}^{\infty}\boldsymbol{v}_{d,k,h}^{\left\{3,\left[\alpha(2d+k+2)\right]\right\}}\Bigg|_{(3>-\beta)},$$

and

$$\mu_4' = \boldsymbol{E}(X^4) = \Gamma\left(1+\frac{4}{\beta}\right)\sum_{d,k,h=0}^{\infty} \boldsymbol{v}_{d,k,h}^{\left\{4,\left\lfloor \alpha(2d+k+2)\right\rfloor\right\}}\Bigg|_{(4>-\beta)}.$$

The above equations are used to get the mean, variance, skewness and kurtosis of the current model.

2.4. Incomplete moments

The r^{th} incomplete moment of X, is defiend as $\tau_r(y) = \int_{-\infty}^{y} x^r f(x)\,dx$. From (8), we should write

$$\tau_r(y) = \Gamma_{\left[\left(\frac{1}{t}\right)^{\beta}\right]}\left(1+\frac{r}{\beta}\right)\sum_{d,k,h=0}^{\infty} \boldsymbol{v}_{d,k,h}^{\left\{r,\left\lfloor \alpha(2d+k+2)\right\rfloor\right\}}\Bigg|_{(r>-\beta)},$$

where

$$\Gamma_{[z]}(\zeta) = \gamma(\zeta,\ldots) = \int_0^z t^{\zeta-1} exp(-t)\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n z^{\zeta+n}}{n!(\zeta+n)}.$$

2.5. Generating functions

Let $M(t) = \boldsymbol{E}\left(\exp(tX)\right)$ be the moment generating function (mgf) of X which is computed from (8) as

$$M(t) = \Gamma\left(1+\frac{r}{\beta}\right)\sum_{d,k,h,r=0}^{\infty} \boldsymbol{v}_{d,k,h,r}^{\left\{r,\left\lfloor \alpha(2d+k+2)\right\rfloor\right\}}\Bigg|_{(r>-\beta)},$$

where $\boldsymbol{v}_{d,k,h,r}^{\left\{r,\left\lfloor \alpha(2d+k+2)\right\rfloor\right\}} = \left[t^r / r!\right]\boldsymbol{v}_{d,k,h}^{\left\{r,\left\lfloor \alpha(2d+k+2)\right\rfloor\right\}}$.

2.6. Skewness and kurtosis: numerical analysis

The effects on the mean μ_1', variance $(V(X))$ of the parameters λ,θ,α and β for the PBXEW model are described in Table 1.

- The latest added shape parameters λ and θ have their impact on μ_1', $V(X)$, $S(X)$ and $K(X)$.

- For the PBXEW model, $S(X)$ can be both positive and negative and can range from -0.146 to $11{,}057.9$.

- For the PBXEW model, $K(X)$ can be less than three and also more than three and can range from 2.8 and $124{,}652{,}267$.

3. Parameter Estimation

Let $x_1, x_2, \ldots, x_n$ be the random sample (r.s.) from PBXEW distribution with parameter $\boldsymbol{\Theta} = (\lambda,\theta,\alpha,\beta)$. We now have following log-likelihood function to determine the maximum likelihood estimators (MLEs) of $\boldsymbol{\Theta}$:

$$l = l(\boldsymbol{\Theta}) = n\log 2 + n\log\lambda + n\log\theta + n\log\alpha + n\log\beta - n\log c_{\lambda}$$
$$+(\beta-1)\sum_{i=1}^{n}\log x_i + (2\alpha-1)\sum_{i=1}^{n}\log s_i - 3\sum_{i=1}^{n}\log\left(1-s_i^{\alpha}\right) + \sum_{i=1}^{n}\left(-x_i^{\beta} - z_i\right)(1$$
$$+(\theta-1)\sum_{i=1}^{n}\log\left[1-exp\left(-z_i\right)\right] - \lambda\sum_{i=1}^{n}\log\left[1-exp\left(-z_i\right)\right]^{\theta}, \tag{10}$$

where $s_i = \left[1-\exp\left(-x_i^{\beta}\right)\right]$ and $z_i = \left(\frac{s_i^{\alpha}}{1-s_i^{\alpha}}\right)^2$.

The components of the score vector

$$U(\boldsymbol{\Theta}) = \frac{\partial}{\partial\boldsymbol{\Theta}}l(\boldsymbol{\Theta}) = \left(\frac{\partial}{\partial\lambda}l(\boldsymbol{\Theta}), \frac{\partial}{\partial\theta}l(\boldsymbol{\Theta}), \frac{\partial}{\partial\alpha}l(\boldsymbol{\Theta}), \frac{\partial}{\partial\beta}l(\boldsymbol{\Theta})\right),$$

can be obtained easily. Equation (10) can either be explicitly maximized, using the SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS) or R (optim function) or by differentiating Equation (10) to solve the nonlinear likelihood equations derived from it. Assume that the nonlinear equation system $U_{\lambda} = 0,\ \ U_{\theta} = 0,\ U_{\alpha} = 0,\ U_{\beta} = 0$ and simultaneously solving them yields MLE $\hat{\boldsymbol{\Theta}} = (\hat{\lambda}, \hat{\theta}, \hat{\alpha}, \hat{\beta})$.

4. Simulation

Graphically, the simulation experiments can be conducted to determine the finite sample activity of the MLE's. The evaluation was focused on the algorithm below:

- Use the QF $Q(u)$, 1,000 samples for the PBXEW distribution of size m are generated.

- Determine the MLEs for the 1,000 samples, say $\hat{\lambda}_h, \hat{\theta}_h, \hat{\alpha}_h, \hat{\beta}_h\big|_{(h=1,2,...,1000)}$.

- Determine the SEs of the MLEs for 1,000 samples, say $S_{\hat{\lambda}_h}, S_{\hat{\theta}_h}, S_{\hat{\alpha}_h}, S_{\hat{\beta}_h}\big|_{(h=1,2,...,1000)}$.

The standard errors (SEs) were determined by inverting the matrix of observed information. Determine the biases and mean squared errors given for $\Psi = \lambda, \theta, \alpha, \beta.$ Those steps we repeated for $m = 50,100,...,1000$ with $\lambda = \theta = \alpha = \beta = 1,$ so evaluating biases $\left(B_{\Psi}(m)\right),$ mean squared errors $\left(MSE_{\Psi}(m)\right)$ for $\lambda, \theta, \alpha, \beta$ and $m = 50,100,...,1000$ where

$$B_{\Psi}\big|_{(\Psi=\lambda,\theta,\alpha,\beta)} = \frac{1}{1,000}\sum_{h=1}^{1,000}\left(\hat{\Psi}_h - \Psi\right) \text{ and } MSE_{\Psi}\big|_{(\Psi=\lambda,\theta,\alpha,\beta)} = \frac{1}{1,000}\sum_{h=1}^{1,000}\left(\hat{\Psi}_h - \Psi\right)^2.$$

Figure 3 (left panels) demonstrate how distinct the four biases are in relation to $m.$ Figure 3 (right panels) reveals how distinct the four MSEs differ in relation to $m.$ The broken line refers to 0 for biases. Based on Figure 3, the biases are usually negative and decrease to zero as $m \to \infty$ for each parameter, the MSEs decrease to zero as $m \to \infty$ for each parameter.

5. Modeling Real Data

We include three applications in this section to empirically demonstrate the potentialities of the PBXEW model. We consider Cramer-Von Mises (W^*) and Anderson-Darling (A^*) for evaluating the PBXEW distribution fits with many other competitive distributions. These two statistics are often

used to specify how close a specific CDF fits the empirical distribution for a given set of data. These statistics are defines by

$$A^* = \left(1+1/2n\right)\left[\left(1/12n\right)+\sum_{d=1}^{n}\left[z_i-\left(2d-1\right)/2n\right]^2\right],$$

and

$$W^* = \left[n+\frac{1}{n}\sum_{d=1}^{n}\left(2d-1\right)\,\log\left[z_i\left(1-z_{n-d+1}\right)\right]\right]\left(1+\frac{9}{4n^2}+\frac{3}{4n}\right),$$

respectively, where $z_i = F(y_d)$ and the y_d's are ordered observations. A better fit means lower values of the statistics. The R software is used for the requisite calculations.

Table 1 Numerical results for $\mu_1', V(X), S(X), K(X)$ for the PBXEW model

λ	α	β	θ	μ_1'	$V(X)$	$S(X)$	$K(X)$
−15	2	1	1	1.081441	0.01094069	0.1874222	3.196878
−10				1.041460	0.01368599	0.02777449	3.325186
−5				0.959483	0.02232738	−0.3534373	3.758993
−1				0.799412	0.03871115	−0.2040497	2.911872
1				0.687222	0.03924141	0.1246622	2.83593
5				0.524522	0.02359394	0.3317759	3.481975
10				0.439776	0.01447703	0.0667729	3.082955
50				0.300303	0.00616999	−0.1459851	2.824816
150				0.232961	0.00372534	−0.168445	2.804012
500				0.176162	0.00216850	−0.1677585	2.788833
10	0.1	2	1.1	0.014398	0.001642284	9.038191	146.1937
	0.5			0.35162	0.033787420	0.7140117	3.547585
	1			0.643065	0.034980490	0.05803876	2.910273
	5			1.167212	0.010397160	−0.2570168	3.444327
	25			1.441001	0.003163861	−0.07347154	3.454363
	50			1.520001	0.002124347	−0.00233779	3.478043
	200			1.641251	0.001120774	0.1055236	3.558249
	500			1.703986	0.000799697	0.1565948	3.617884
	1,000			1.745204	0.000640236	0.1874305	3.661593
15	15	1	1.5	0.860899	0.002356695	−0.2916031	3.351838
		2		1.300658	0.002616336	−0.3223008	3.39222
		5		1.894054	0.002594853	−0.3371329	3.413269
		10		2.335263	0.002478051	−0.3405368	3.41836
		50		3.315228	0.002178266	−0.3407365	3.419107
		100		3.718528	0.002068431	−0.3387504	3.288472
		500		4.618672	0.001864174	−0.3380841	3.416854
		1,000		4.993056	0.001793830	−0.3374026	3.414874
−3	2	15	0.15	$7.223476\,e^{-8}$	$6.060405\,e^{-7}$	11057.89	124,652,267
			0.25	$2.612059\,e^{-5}$	0.000228099	587.3260	349,291.5
			0.35	0.003424655	0.0303889	51.50644	2,680.587
			0.50	1.561727	11.720490	1.764314	4.176411
			0.75	5.040145	0.2119882	−0.6001655	3.96153
			1	3.361101	0.0540751	−0.6978863	4.224569

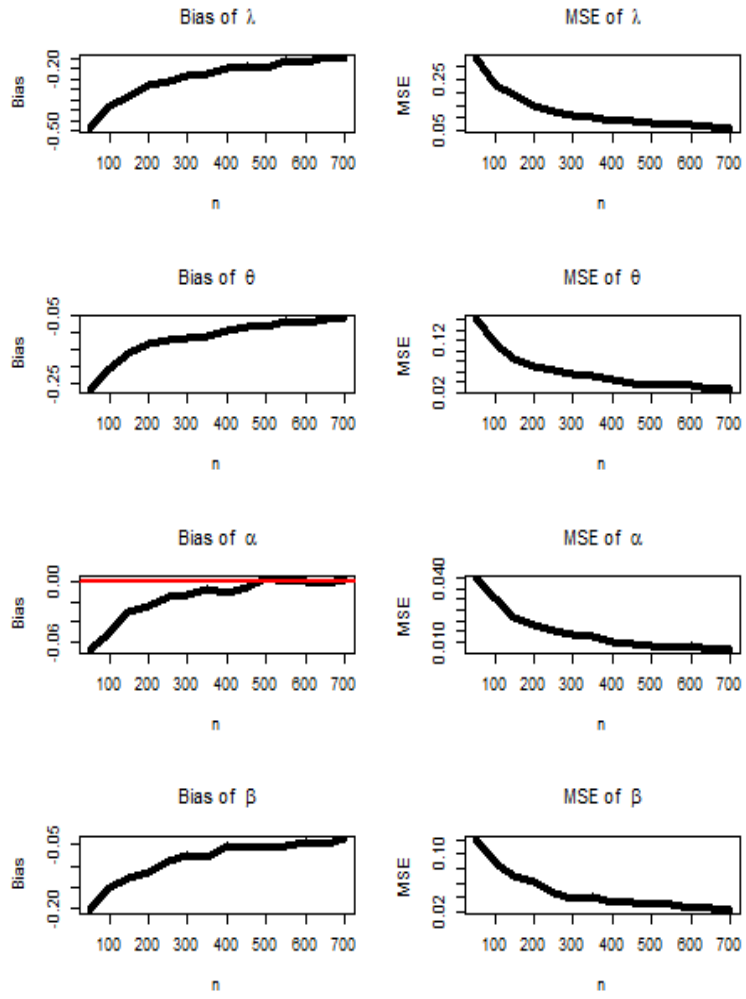


Figure 3 Biases and MSEs for the parameters $\lambda, \theta, \alpha, \beta$

Tables 3, 5 and 7 indicate the MLEs and their related standard errors (SEs) (in parentheses) for model parameters. The calculated values of statistics W^* and A^* are reported in Tables 2, 4 and 6. Figures 4, 5 and 6 respectively show the histograms, P-P plot, TTT plot, and estimated CDF for the three data sets for proposed model.

5.1. Modeling failure times

The first data set is the data of 84 aircraft windshield failure time which are displayed in Appendix A. We can then associate the PBXEW distribution fits with some competitive models. We infer from Table 2 and Figure 3 that the current model offers appropriate fits compared to other Weibull models. The new model is considerably better than the MOEW, GaW, KwW, WFr, BW, TMW, KwTW, MBW, McW and TExGW distributions whose PDFs are provided in Appendix B.

Table 2 A^* and W^* for the 1st data set

Distribution	A^*	W^*
PBXEW	0.627	0.064
MOEW	4.448	0.399
GaW	1.949	0.255
KwW	1.506	0.185
WFr	1.957	0.254
BW	3.220	0.465
TMW	11.205	0.806
KwTW	1.363	0.164
MBW	3.266	0.472
McW	1.591	0.199
TExGW	6.233	1.008

Table 3 MLEs (SEs) for aircraft windshield data

Distribution	Estimates				
MOEW(γ,β,α)	488.90 (189.34)	0.28 (0.01)	1,261.97 (351.07)		
GaW(α,β,γ)	2.38 (0.38)	0.85 (0.0005)	3.53 (0.67)		
PBXEW$(\lambda,\theta,\alpha,\beta)$	0.03 (1.50)	1.19 (0.71)	2.84 (1.10)	0.43 (0.09)	
KwW(α,β,a,b)	14.43 (27.10)	0.20 (0.04)	34.66 (17.53)	81.85 (52.01)	
WFr(α,β,a,b)	630.94 (697.94)	0.30 (0.032)	416.10 (232.36)	1.17 (0.36)	
BW(α,β,a,b)	1.36 (1.002)	0.30 (0.06)	34.18 (14.838)	11.50 (6.73)	
TMW$(\alpha,\beta,\gamma,\lambda)$	0.27 (0.014)	1.00 (5.2×10^{-5})	4.6×10^{-6} (1.9×10^{-4})	0.47 (0.17)	
TExGW(α,β,λ,b)	4.26 (33.40)	0.15 (0.017)	0.098 (0.61)	1,173.33 (9.79)	
KwTW$(\alpha,\beta,\lambda,a,b)$	27.7912 (33.40)	0.178 (0.02)	0.445 (0.61)	29.53 (9.79)	168.10 (129.2)
MBW(α,β,a,b,c)	10.15 (18.70)	0.16 (0.02)	57.42 (14.06)	19.39 (10.02)	2.004 (0.66)
McW(α,β,a,b,c)	1.94 (1.01)	0.31 (0.05)	17.69 (6.2)	33.64 (19.99)	16.72 (9.72)

Table 4 A^* and W^* for the 2nd data set

Distribution	A^*	W^*
PBXEW	0.124	0.019
W	0.663	0.106
TMW	0.760	0.125
MBW	0.720	0.107
TAW	0.703	0.113
ETGR	2.361	0.398

Table 5 MLEs (SEs) for the remission times data

Distribution	Estimates				
W$(\alpha,\beta,\gamma,\lambda)$	9.56 (0.85)	1.05 (0.07)	0.05 (0.04)		
ETGR$(\alpha,\beta,\delta,\lambda)$	7.38 (5.39)	0.05 (3.96×10^{-3})	1.24 (0.18)	0.118 (0.26)	
PBXEW$(\lambda,\theta,\alpha,\beta)$	4.31 (2.45)	4.812 (5.03)	0.0002 (0.011)	0.11 (0.02)	
TMW$(\alpha,\beta,\gamma,\lambda)$	0.121 (0.024)	0.90 (0.63)	57.42 (37.32)	0.25 (0.41)	
MBW(α,β,a,b,c)	0.150 (22.44)	0.16 (0.044)	3.094×10^{-5} (6.11×10^{-3})	19.39 (13.49)	2.004 (0.79)
TAW$(\alpha,\beta,\gamma,\theta,\lambda)$	0.114 (0.03)	0.97 (0.13)	0.05 (0.04)	1.007 (0.035)	0.16 (0.28)

Table 6 A^* and W^* for the 3rd data set

Distribution	A^*	W^*
PBrX-W	0.474	0.077
BrX-EW	0.567	0.091
W-W	0.781	0.143
GaEE	1.721	0.315
EEGc	0.579	0.105

Table 7 MLEs (SEs) for survival times data

Distribution	Estimates			
BrXEW(θ,α,β)	3.18 (2.12)	5.54 (2.44)	0.17 (0.02)	
WW(β,γ,λ)	2.66 (0.71)	0.69 (0.17)	0.03 (0.02)	
GaEE(λ,α,θ)	2.114 (1.33)	2.60 (0.56)	0.008 (0.005)	
EEGc(α,θ,p)	2.59 (0.48)	0.0004 (0.004)	0.99 (0.104)	
PBXEW $(\lambda,\theta,\alpha,\beta)$	13.84 (0.00)	0.46 (0.00)	4.06 (1.34)	0.15 (0.02)

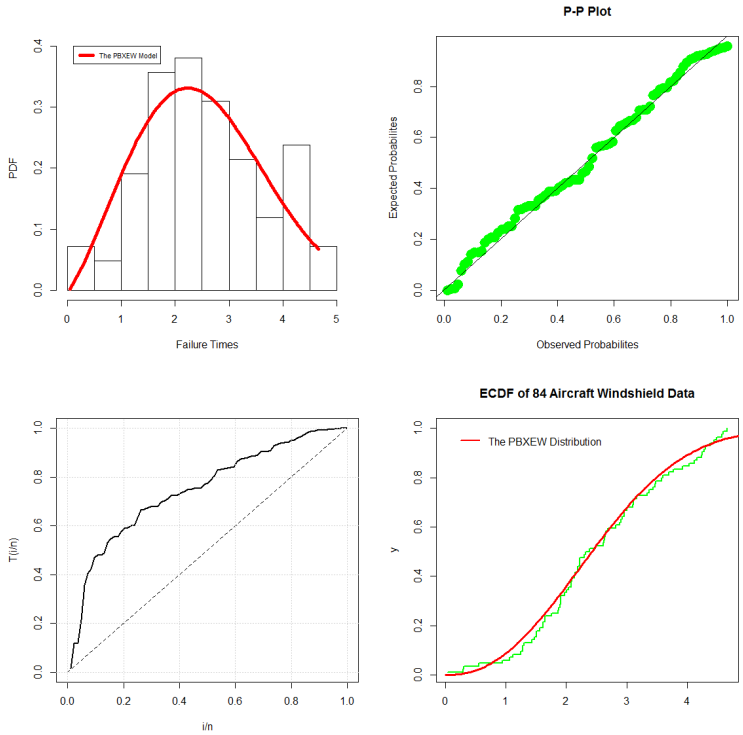


Figure 4 Estimated PDF, P-P plot, TTT plot and estimated CDF and for the 1st data set

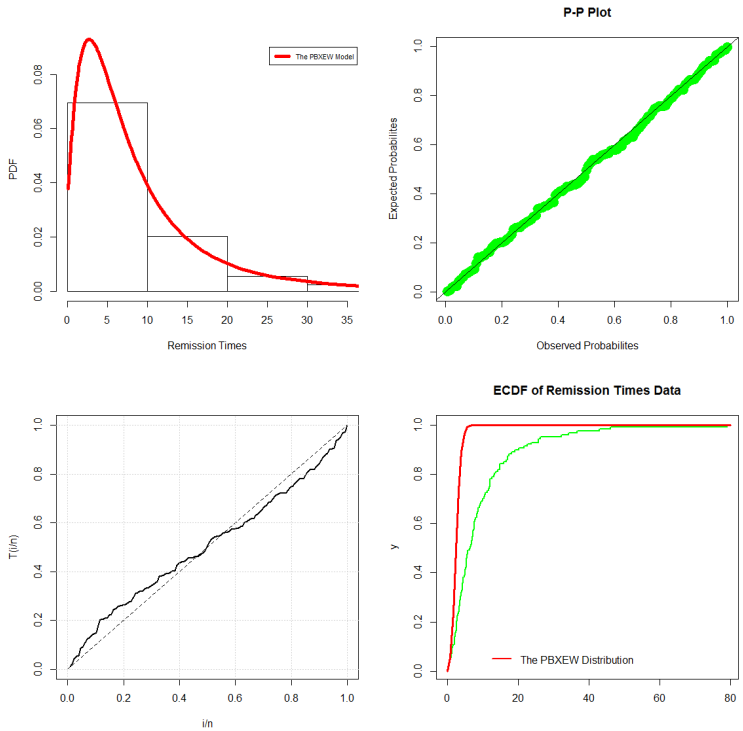


Figure 5 Estimated PDF, P-P plot, TTT plot and estimated CDF and for the 2nd data set

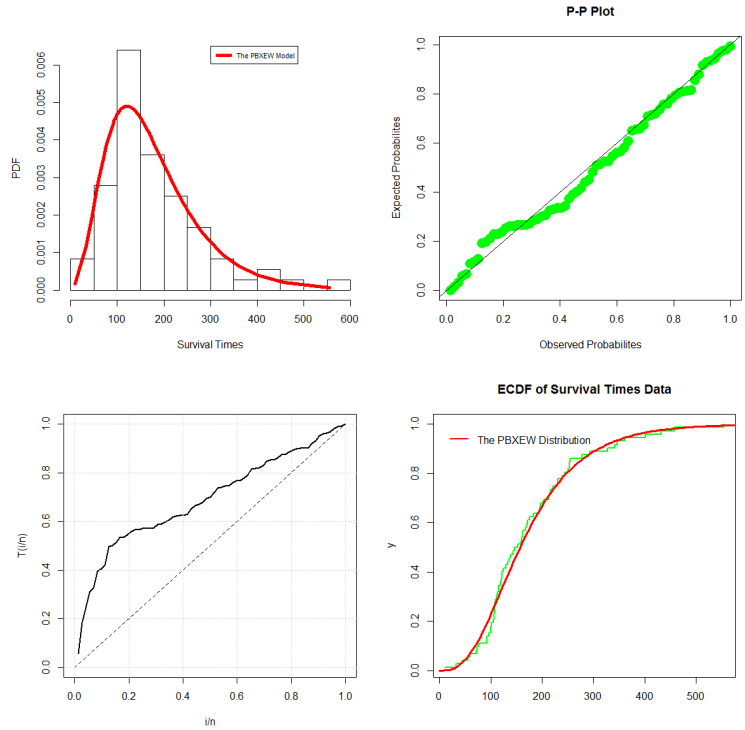


Figure 6 Estimated PDF, P-P plot, TTT plot and estimated CDF and for the 3rd data set

5.2. Modeling remission times

The data set appears in Appendix C shows a random sample of 128 patients with bladder cancer with the remission periods (in months). We compare the PBXEW distribution fits with other competing models, whose PDFs in the literature are available with their authors. We reach the conclusion that proposed model is more preferable than distribution W, TMW distribution, MBW distribution, TAW distribution and ETGR distribution (whose PDFs are provided in Appendix D) with small A^* and W^* values.

5.3. Modeling survival times

The third data set refers to 72 guinea pigs, contaminated with bacterial tuberculosis, survival times (in days). The data appears in Appendix E. We can compare the PBXEW distribution fits with other competing models which PDFs are present in the literature along with their authors. On basis of Figure 5 & Table 6, we reach the conclusion that the new model is substantially better than BrXEW distribution, WW distribution, OWW distribution, GaEE distribution and EEGc distribution (whose PDFs are provided in Appendix F) with small A^* and W^* values in the modeling for the above data set.

6. Concluding Remarks

We established a new continuous distribution based on the Poisson truncated zero that accommodates several important failure rates. The paper also discussed several computational characteristics of the current model. The density of the new model is being represented as a

combination of Weibull exponentiated densities. The effects on mean, variance, skewness and kurtosis of the parameters λ, θ, α and β for the new model are studied and note that:

- The latest added shape parameters have an impact on mean, variance, skewness, and kurtosis.
- For the PBXEW model, skewness may be positive and negative and can range from 0.146 to 11,057.9.
- For the PBXEW model, kurtosis may be less than three and also more than three and can range from 2.8 to 124,652,267.

The method of the maximum likelihood is graphically assessed and used to estimate the model parameters. Three real data applications are provided for the importance and flexibility the new distribution with other competitive models.

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Appendix A: 0.040, 1.866, 2.224, 3.1170, 2.823, 4.449, 1.505, 2.154, 2.964, 4.278, 1.912, 2.632, 3.595, 1.303, 2.089, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 3.344, 4.602, 1.615, 4.485, 1.652, 2.229, 1.506, 2.190, 3.000, 4.305, 1.070, 1.899, 1.757, 3.578, 0.943, 3.166, 2.688, 3.924, 1.281, 2.038, 4.035, 1.281, 2.085, 2.890, 1.619, 2.097, 2.625, 4.570, 1.652, 2.300, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 2.902, 4.167, 1.432, 4.376, 2.610, 3.478, 0.557, 1.911, 2.223, 3.114, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 4.121, 1.568, 2.194, 3.103, 2.324, 3.376, 4.663.

Appendix B: The Gamma-Weibull (GaW) (Provost et al. 2011)

$$f_{GaW}^{(\alpha,\beta,\gamma)}(x)=\frac{\beta\alpha^{(\gamma/\beta)+1}}{x^{1-\beta-\gamma}exp\left(\alpha x^{\beta}\right)}\Gamma\left((\gamma/\beta)+1\right).$$

Transmuted modifed-Weibull (TMW) (Khan and King 2013)

$$f_{TMW}^{(\alpha,\beta,\gamma,\lambda)}(x)=\frac{\left(\gamma\beta x^{\beta-1}+\alpha\right)\left\{\left[-\lambda+2\lambda\,exp\left(-\alpha x-\gamma x^{\beta}\right)\right]+1\right\}}{exp\left(\gamma x^{\beta}+\alpha x\right)}\Bigg|_{(|\lambda|\le 1)}.$$

Beta-Weibull (BW) (Lee et al. 2007)

$$f_{BW}^{\alpha,\beta,a,b}(x)=\frac{\beta\alpha^{\beta}\left\{1-\exp\left[-a\left(\alpha x\right)^{\beta}\right]\right\}^{a-1}}{B(a,b)x^{1-\beta}\exp\left[-a\left(\alpha x\right)^{\beta}\right]}.$$

Transmuted exponentiated generalized Weibull (TExGW) (Yousof et al. 2015)

$$f_{TExGW}^{(\alpha,\beta,\lambda,a,b)}(x)=\frac{ab\beta\alpha^{\beta}\left\{1-\exp\left[-a\left(\alpha x\right)^{\beta}\right]\right\}^{b-1}\exp\left[-a\left(\alpha x\right)^{\beta}\right]}{x^{1-\beta}\left(-2\lambda\left\{1-\exp\left[-a\left(\alpha x\right)^{\beta}\right]\right\}^{b}+1+\lambda\right)^{-1}}\Bigg|_{(|\lambda|\le 1)}.$$

Weibull-Fréchet (WFr) (Afify et al. 2016a)

$$f_{WFr}^{(\alpha,\beta,a,b)}(x)=\frac{ab\beta\alpha^{\beta}\left\{1-\left\{1-\exp\left[-a\left(\alpha x\right)^{\beta}\right]\right\}^{a}\right\}^{b-1}}{x^{1-\beta}exp\left[\left(\alpha x\right)^{\beta}\right]\left\{1-\exp\left[-\left(\alpha x\right)^{\beta}\right]\right\}^{1-1}}.$$

Kumaraswamy-Weibull (KwW) (Cordeiro et al. 2010)

$$f_{KwW}^{(\alpha,\beta,a,b)}(x)=\frac{ab\beta\alpha^{\beta}\left\{1-\left\{1-\exp\left[-a\left(\alpha x\right)^{\beta}\right]\right\}^{a}\right\}^{b-1}}{x^{1-\beta}exp\left[\left(\alpha x\right)^{\beta}\right]\left\{1-\exp\left[-\left(\alpha x\right)^{\beta}\right]\right\}^{1-\alpha}}.$$

Marshall Olkin extended-Weibull (MOEW) (Ghitany et al. 2005)

$$f_{MOEW}^{(\gamma,\beta,\alpha)}(x)=\frac{\alpha\beta\gamma^{\beta}}{x^{1-\beta-\gamma}\exp\left[\left(\gamma x\right)^{\beta}\right]}\left(\left\{-(1-\alpha)\exp\left[\left(\gamma x\right)^{\beta}\right]\right\}+1\right)^{-2}..$$

Kumaraswamy transmuted-Weibull (KwTW) (Afify et al. 2016b)

$$f_{KwTW}^{(\alpha,\beta,\lambda,a,b)}(x)=\frac{ab\beta\alpha^{\beta}\left(\lambda-2\lambda\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}+1\right)\exp\left[-(\alpha x)^{\beta}\right]}{\left[\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}\left(\lambda-\lambda\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}+1\right)\right]^{1-a}}$$
$$\times\left[1-\left(\frac{\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}}{(1+\lambda)^{-1}}-\frac{\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{2}}{\lambda^{-1}}\right)^{a}\right]^{b-1}.$$

Modifed beta-Weibull (MBW) (Khan 2015)

$$f_{MBW}^{(\alpha,\beta,a,b,c)}(x)=\frac{\beta\gamma^{a}\alpha^{-\beta}\left\{1-\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{a-1}}{B(a,b)\exp\left[b\left(\frac{x}{\alpha}\right)^{\beta}\right]}\left[-\left(\left\{1-\exp\left[b\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}(1-\gamma)\right)+1\right]^{-(a+b)}.$$

Mcdonald-Weibull (McW) (Cordeiro et al. 2014b)

$$f_{McW}^{(\alpha,\beta,a,b,c)}(x)=\frac{\beta c\alpha^{\beta}\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1}\exp\left[-(\alpha x)^{\beta}\right]}{x^{1-\beta}B(a/c,b)\left(-\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{c}+1\right)^{1-b}}.$$

Appendix C: 0.08, 5.85, 8.26, 11.908, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50, 2.09, 3.48, 4.87, 0.81, 2.62, 11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90 , 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93, 5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.

Appendix D: The transmuted additive Weibull distribution (TAW) (Elbatal and Aryal 2013)

$$f_{TAW}^{(\alpha,\beta,\gamma,\theta,\lambda)}(x)=\left(\gamma\beta x^{\beta-1}+\alpha\theta x^{\theta-1}\right)\left\{2\lambda\,\exp\left[-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)\right]+1-\lambda\right\}\exp\left[-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)\right].$$

The exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al. 2015)

$$f_{ETGR}^{(\alpha,\beta,\gamma,\theta,\lambda)}(x)=\left.\frac{2\alpha\delta\beta^{2}x\left\{1+\lambda-2\lambda\left(1-exp\left[-(\beta x)^{2}\right]\right)^{\alpha}\right\}\exp\left[-(\beta x)^{2}\right]}{\left\{1+\lambda-\lambda\left(1-\exp\left[-(\beta x)^{2}\right]\right)^{\alpha}\right\}^{1-\delta}\left\{1-\exp\left[-(\beta x)^{2}\right]\right\}^{-(\alpha\delta-1)}}\right|_{(|\lambda|\le 1)}.$$

Appendix E: 10, 33, 44, 74, 100, 361, 402, 432, 77, 92, 93, 96, 100, 112, 102, 105, 116, 213, 215, 216, 222, 56, 59, 72, 230, 278, 120, 113, 115, 195, 196, 121, 122, 122, 124, 231, 240, 245, 251, 253, 254, 255, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 107, 107, 108, 108, 108, 109, 197, 202, 293, 327, 342, 347, 458, 555.

Appendix F: The Burr X Exponentiated Weibull (BrXEW)

$$f_{BrXEW}^{(\theta,\alpha,\beta,)}(x)=\frac{2\theta\alpha\beta X^{\beta-1}\left[1-\exp\left(-x^{\beta}\right)\right]^{2\alpha-1}\exp\left(-\left\{\frac{\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{2}\right)}{\exp\left(-x^{\beta}\right)\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{3}\left[1-\exp\left(-\left\{\frac{\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{2}\right)\right]^{1-\theta}}.$$

The Weibull-Weibull (WW) (Tahir et al. 2016)

$$f_{WW}^{(\beta,\gamma,\lambda)}(x)=\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}\right).$$

The gamma exponentiated-exponential (GaEE) (Ristic and Balakrishnan 2012)

$$f_{GaEE}^{(\lambda,\alpha,\theta)}(x)=\frac{\alpha\theta\left\{-\alpha\ \log\left[1-\exp\left(-\theta x\right)\right]\right\}^{\lambda-1}}{\Gamma(\lambda)\exp\left(-\theta x\right)\left[1-\exp\left(-\theta x\right)\right]^{1-\alpha}}.$$

Exponential exponential-geometric (EEGc) (Rezaei et al. 2013)

$$f_{EEGc}^{(\alpha,\theta,p)}(x)=\frac{\alpha\theta(1-p)\left[1-\exp\left(-\theta x\right)\right]^{1-\alpha}\left\{-\alpha\ \log\left[1-\exp\left(-\theta x\right)\right]\right\}^{\lambda-1}}{\exp\left(-\theta x\right)\left\{1-p+p\left[1-\exp\left(-\theta x\right)\right]^{2}\right\}}.$$