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On The Burr III-Moment Exponential Distribution

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Abstract

In this study, we introduce a new lifetime model derived from T-X family called Burr III moment exponential (BIII-ME) distribution. The shapes of the probability density and hazard rate functions for this model are obtained. The BIII-ME density function has bimodal, arc, symmetrical, left-skewed, right-skewed, J and reverse-J shapes. The proposed model can produce monotone and non-monotone failure rates shapes. To illustrate the importance of the proposed distribution, various mathematical properties of it are established such as ordinary moments, order statistics, conditional moments, reliability measures and Rényi entropy. The BIII-ME distribution is characterized via innovative techniques. The maximum likelihood estimates (MLE) for the model parameters are studied. The precision of the MLEs is estimated via a simulation study. We consider two applications to two real data sets to demonstrate the potentiality and utility of the BIII-ME model. Then, we establish empirically that the proposed model is suitable for strength of glass fibers and fracture toughness applications. Finally, the goodness of fit statistics and graphical tools are used to examine the adequacy of the BIII-ME distribution.

Keywords: Moment exponential, characterizations, estimation, Mills ratio, moments, reliability.

1. Introduction

In the recent decades, many continuous distributions have been introduced in statistical literature. These distributions, however, are not flexible enough to be suitable for the data sets from survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. Hence, the applications of the generalized models to these fields are clear requisite. Generalization of the distribution is the only way to increase the applicability of the parent distribution. The generalizations are derived either by inserting a shape parameter or by transforming into the parent distribution. So, the generalized distributions will be more suitable than the competing model and sub-models.

The moment exponential (ME) distribution was established by Dara and Ahmad (2011). The probability density function (pdf) and cumulative distribution function (cdf) of the ME distribution are given, respectively, by

$$g_\lambda(x)=\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}, x>0, \tag{1}$$

and

$$G_\lambda(x)=1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}, x\geq 0, \tag{2}$$

where the r^{th} moment about the origin is $\mu_r'=E\left(X^r\right)=\lambda^r\mathbf{\Gamma}(2+r).$ The odds ratio for the ME random variable X is given by $W\left(G(x)\right)=\frac{G_\lambda(x)}{1-G_\lambda(x)}=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right].$ Various forms of the ME distribution have been studied by different authors. The exponentiated ME (EME) distribution is obtained by Hasnain (2013), generalized exponentiated ME (GEME) distribution by Iqbal et al. (2014) and Weibull ME (WME) distribution by Hashmi et al. (2019). However, new flexible generalizations of the ME distribution are still needed.

This study focuses on the following motivations: (i) to generate distributions with bimodal, arc, symmetrical, left-skewed, right-skewed, J and reverse-J shaped as well as high kurtosis; (ii) to have monotone and non-monotone failure rate function; (iii) to study numerically descriptive measures for the BIII-ME distribution based on the parameter values; (iv) to derive mathematical properties such as random number generator, sub-models, ordinary moments and conditional moments, reliability measures, Rényi entropy and characterizations; (v) to estimate the precision of the maximum likelihood estimators via simulation study; (vi) to reveal the potentiality and utility of the BIII-ME model; (vii) to work as the preeminent substitute model to other existing models to discover and model the real data in finances, survival analysis, manufacturing, reliability, life testing and new zones of research; (viii) to deliver better fits model than the existing models and (ix) to infer empirically from goodness of fit statistics (GOFs) and graphical tools.

The contents of the article are structured as follows: Section 2 derives the BIII-ME model. We study basic structural properties, random number generator and sub-models for the BIII-ME model. We highlight the nature of density and failure rate functions. Section 3 presents certain mathematical properties such as the ordinary moments, conditional moments, reliability measures, Rényi entropy and some other properties. Section 4 characterizes the BIII-ME distribution. Section 5 addresses the MLE (maximum likelihood estimation) for the BIII-ME model. In Section 6, we evaluate the precision of the MLEs via a simulation study. In Section 7, two applications are considered to illustrate the potentiality and utility of the BIII-ME model. We test the competency of the BIII-ME distribution using goodness of fit statistics. In Section 8, the article is concluded.

2. The BIII-ME Distribution

In this section, we derive the BIII-ME distribution from the T-X family technique. The BIII-ME model from link concerning the exponential and gamma variables is also obtained. Basic structural properties are studied. Then, we highlight the nature of the density and failure rate functions.

2.1. T-X family technique

To obtain a wider family of distributions, Alzaatreh et al. (2016) derived the cdf for the T-X family as follows

$$F(x)=\int_{a}^{W[G(x;\xi)]} r(t)\,dt\,|_{x\in\mathbb{R}}, \tag{3}$$

where $r(t)$ is the pdf of the random variable (rv) T, where $T\in[a,b]$ for $-\infty\le a<b<\infty$ and $W[G(x;\xi)]$ is a function of the baseline cdf of a rv X, subject to the vector parameter ξ and satisfying: i) $W[G(x;\xi)]\in[a,\ b]$, ii) $W[G(x;\xi)]$ is differentiable and monotonically non-decreasing and iii) $\lim_{x\to-\infty} W[G(x;\xi)]\to a$ and $\lim_{x\to\infty} W[G(x;\xi)]\to b$. For the T-X family of distributions, the pdf of X is given by

$$f(x)=\left\{\frac{\partial}{\partial x}W[G(x;\xi)]\right\}r\{W[G(x;\xi)]\}\Big|_{x\in\mathbb{R}}. \tag{4}$$

We derive the cdf of the BIII-ME distribution in the T-X family technique by setting

$$r(t)=\alpha\beta t^{-\beta-1}\left(1+t^{-\beta}\right)^{-\alpha-1}\,|_{t>0,\alpha>0,\beta>0},$$

and

$$W(G(x))=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right].$$

Then, the cdf of the BIII-ME distribution can be written as

$$F(x;\alpha,\beta,\lambda)=\alpha\beta\int_{0}^{\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]} t^{-\beta-1}\left(1+t^{-\beta}\right)^{-\alpha-1}dt,$$

or

$$F(x)=\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}|_{x\ge0}, \tag{5}$$

where $\alpha>0$ and $\beta>0$ are shape parameters and $\lambda>0$ is scale parameter.

The pdf corresponding to (5) is given by

$$f(x)=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1}|_{x>0}. \tag{6}$$

Hereafter, the random variable (rv) with pdf (6) is denoted by $X\sim$ BIII-ME(α,β,λ). For $\alpha=1$, the BIII-ME distribution reduces to log-logistic-ME (LL-ME) distribution and for $\beta=1$, the BIII-ME distribution reduces to inverse Lomax-ME (IL-ME) distribution.

2.2. Nexus between gamma and exponential variables

We derive the BIII-ME distribution by linking the exponential and gamma rvs, i.e., $W_1\sim\exp(1)$ and $W_2\sim gamma(\alpha,1)$. W_1 and W_2 independently distributed.

Lemma 1 *If* $W_1\sim\exp(1)$ *and* $W_2\sim gamma(\alpha,1)$, *then for* $W_1=\left[\dfrac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}W_2$, *we have*

$$X=-\lambda-\lambda W_{-1}\left[-e-e\left(\frac{W_2}{W_1}\right)^{\frac{1}{\beta}}\right]^{-1}\text{~}BIII\text{-}ME(\alpha,\beta,\lambda),\ \textit{where } W_{-1} \textit{ is inverse Lambert function.}$$

Proof:

If $W_1\sim\exp(1)$, i.e. $f(w_1)=e^{-w_1}, w_1>0,$ and if $W_2\sim gamma(\alpha,1)$, i.e.

$f(w_2)=\dfrac{w_2^{\alpha-1}\mathrm{e}^{-w_2}}{\Gamma(\alpha)}, w_2>0.$

Then, the joint distribution of the two rvs is $f(w_1,w_2)=\dfrac{w_2^{\alpha-1}\mathrm{e}^{-w_2}\mathrm{e}^{-w_1}}{\Gamma(\alpha)}, w_1>0,\ w_2>0.$ The joint density of the rvs X and W_2 has the form

$$f(x,w_2)=\frac{w_2^{\alpha-1}e^{-w_2}e^{-\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}w_2}}{\Gamma(\alpha)}\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}w_2\,|_{x>0,w_2>0}\,.$$

The BIII-ME density of X is obtained as

$$f(x)=\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}\int_0^\infty\frac{w_2^{\alpha}e^{-w_2}e^{-\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}w_2}}{\Gamma(\alpha)}dw_2.$$

After simplifying, we have

$$f(x)=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1}|_{x>0},$$

which is the BIII-ME density.

2.3. Basic structural properties

If $X \sim$ BIII-ME(α,β,λ), the survival, failure rate, cumulative hazard and reverse hazard functions, Mills ratio and elasticity are given, respectively, by (for $x>0$)

$$S(x)=1-\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}, \tag{7}$$

$$h(x)=-\frac{d\ln\left(1-\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}\right)}{dx}, \tag{8}$$

$$H(x)=-\ln\left(1-\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}\right), \tag{9}$$

$$r(x)=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-1}, \tag{10}$$

$$m(x)=-\left[\frac{d}{dx}\ln\left(1-\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}\right)\right]^{-1}, \tag{11}$$

and

$$\eta_F(x)=\frac{dlnF(x)}{d\ln x}=\frac{d}{d\ln x}\left(\ln\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}\right). \tag{12}$$

The quantile function of the BIII-ME distribution for $0<q<1$ is given by

$$x=-\lambda\left(1+\left\{W_{-1}\left[(w-1)/e\right]\right\}\right), \tag{13}$$

where $w=\left[\left(q^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\beta}}+1\right]^{-1}$, and its random number generator is

$$X = -\lambda\left(1+\left\{W_{-1}\left[\left(w-1\right)/e\right]\right\}\right), \tag{14}$$

where $w=\left[\left(Z^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\beta}}+1\right]^{-1}$, where $Z \sim \text{Uniform}(0,1)$.

2.4. Shapes of the BIII-ME density and failure rate functions

Now, we provide shapes of the density and failure rate functions of the BIII-ME distribution for selected values of the parameters. Figure 1 displays that the BIII-ME density can take various shapes such as bimodal, arc, symmetrical, left-skewed, right-skewed and reverse-J. Figure 2 shows that failure rate function can be modified bathtub, bathtub, increasing, decreasing, increasing-decreasing and decreasing-increasing-decreasing shaped. Therefore, the BIII-ME distribution is quite flexible and can be applied for numerous data sets. We plot all figures for the density and failure rate functions using MATHEMATICA 11 (Wolfram Research 2017). All figures are listed in the Appendix.

2.5. Useful expansions

The pdf in (6) can be expressed as

$$f(x)=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}\underbrace{\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1}}_{A(x)}. \tag{15}$$

Applying

$$\left(1+\frac{z_1}{z_2}\right)^{-b}=\sum_{j=0}^{\infty}\binom{-b}{j}\left(\frac{z_1}{z_2}\right)^{j}, \tag{16}$$

for $A(x)$ we get $A(x)=\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-j\beta}$. Then,

$$f(x)=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta^*-1}\underbrace{\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta^*-1}}_{B(x)}, \tag{17}$$

where $\beta^*=\beta(j+1)$. Then, applying

$$\left(1-\frac{z_1}{z_2}\right)^{b}=\sum_{k=0}^{\infty}(-1)^k\binom{b}{k}\left(\frac{z_1}{z_2}\right)^{k}, \tag{18}$$

for $B(x)$ we have $B(x)=\sum_{k=0}^{\infty}(-1)^k\binom{\beta^*-1}{k}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{k}$.

Then the pdf (16) can be written as

$$f(x)=\sum_{j,k=0}^{\infty}\frac{\alpha\beta}{\lambda^{2}}xe^{-\frac{x}{\lambda}}\left(-1\right)^{k}\binom{-\alpha-1}{j}\binom{\beta^{*}-1}{k}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{k-\beta^{*}-1}, \tag{19}$$

then expanding the last term in (17) we arrive at

$$f(x)=\sum_{\ell=0}^{k^{*}}\sum_{m=0}^{\ell}\mathbf{C}_{j,k,\ell,m}x^{m+1}e^{-(\ell+1)\frac{x}{\lambda}}\,|_{k^{*}=k-\beta^{*}-1}, \tag{20}$$

where

$$\mathbf{C}_{j,k,\ell,m}=\alpha\beta\sum_{j,k=0}^{\infty}\frac{\left(-1\right)^{k+\ell}}{\lambda^{m+2}}\binom{-\alpha-1}{j}\binom{\beta^{*}-1}{k}\binom{k-\beta^{*}-1}{l}\binom{l}{m}.$$

Many properties of the BIII-ME distribution can be easily determined using mixture representation given in (20).

3. Mathematical Properties

Now, we present certain mathematical and statistical properties such as the ordinary moments, the Mellin transform, and their related moments, conditional moments, reliability measures and Rényi entropy.

3.1. Moments of the BIII-ME distribution

The moments are significant tools for statistical analysis in pragmatic sciences. The r^{th} moment about the origin is

$$\mu_{r}'=E(X^{r})=\int_{0}^{\infty}x^{r}f(x;\alpha,\beta,\lambda)dx,$$

Let $w=\left(\ell+1\right)\dfrac{x}{\lambda}$, then $E(X^{r})\,|_{(r=1,2,3,\ldots)}=\sum_{\ell=0}^{k^{*}}\sum_{m=0}^{\ell}\mathbf{C}_{j,k,\ell,m}\dfrac{\lambda^{r+m+2}}{\left(\ell+1\right)^{r+m+2}}\int_{0}^{\infty}w^{r+m+1}e^{-w}dw.$

Here, we have

$$E(X^{r})\,|_{(r=1,2,3,\ldots)}=\sum_{\ell=0}^{k^{*}}\sum_{m=0}^{\ell}\mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,r+m+2)}\mathbf{\Gamma}\left(r+m+2\right), \tag{21}$$

where $\mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,\blacktriangledown)}=\dfrac{\lambda^{\blacktriangledown}}{\left(\ell+1\right)^{\blacktriangledown}}\mathbf{C}_{j,k,\ell,m}.$

The Mellin transformation is applied to find the moments of X is

$$\mathrm{M}\left\{f\left(x\right);r\right\}=\int_{0}^{\infty}f\left(x\right)x^{r-1}dx=\int_{0}^{\infty}x^{r-1}\sum_{\ell=0}^{k^{*}}\sum_{m=0}^{\ell}\mathbf{C}_{j,k,\ell,m}x^{m+1}e^{-(\ell+1)\frac{x}{\lambda}}dx,$$

$$\mathrm{M}\left\{f\left(x\right);r\right\}=\sum_{\ell=0}^{k^{*}}\sum_{m=0}^{\ell}\mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,r+m+1)}\mathbf{\Gamma}\left(r+m+1\right). \tag{22}$$

The r^{th} central moment (μ_r), coefficients of skewness (γ_1) and kurtosis (γ_2) for the BIII-ME model are attained from the well-known relationships. The mean (μ_1') median ($\tilde{\mu}$), standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) of the BIII-ME distribution for selected values of α,β,λ are listed in Table 1. We also depict that BIII-ME model can be effective to model data sets in terms of the descriptive measures.

Table 1 $\tilde{\mu}$, μ_1', σ, γ_1 and γ_2 of the BIII-ME distribution

Parameter (α, β, λ)	$\tilde{\mu}$	μ_1'	σ	γ_1	γ_2
(0.5, 1, 0.5)	0.4799	0.6712	0.6427	1.8092	7.8357
(1, 1, 0.5)	0.8379	0.9987	0.7067	1.4318	6.1934
(1, 2, 0.5)	1.2351	1.3737	0.7393	1.2219	5.4842
(3, 1, 0.5)	1.4750	1.6052	0.7476	1.1748	5.9798
(4, 1, 0.5)	1.6465	1.7723	0.7498	1.1418	5.9500
(5, 1, 0.5)	1.7797	1.9027	0.7505	1.1255	5.9114
(1, 2, 0.5)	0.8390	0.8869	0.3599	1.0431	5.3479
(1, 3, 0.5)	0.8390	0.8613	0.2406	0.7959	4.9707
(1, 4, 0.5)	0.8391	0.8520	0.1807	0.6370	4.7125
(1, 5, 0.5)	0.8391	0.8474	0.1448	0.5318	4.5851
(1, 1, 1)	1.6756	1.9974	1.4152	1.5729	14.1902
(1, 1, 2)	3.3512	3.9947	2.8264	1.4284	6.1639
(1, 1, 3)	5.0266	5.9919	4.2394	1.4273	6.1495
(1, 1, 4)	6.7025	7.9889	5.6520	1.4279	6.1668
(1, 1, 5)	8.3775	9.9862	7.0668	1.4379	6.4186
(0.5, 5, 0.5)	0.7546	0.7504	0.1746	0.0437	3.7792
(0.49, 5.25,0.22)	0.3324	0.3302	0.0741	0.0002	3.8077
(0.25, 5, 0.5)	0.6424	0.6261	0.2201	−0.1154	2.9275
(0.3, 5, 0.5)	0.6759	0.6618	0.2075	−0.1145	3.1674
(0.25, 6, 0.1)	0.1345	0.1304	0.0392	−0.3025	3.0281
(0.75, 0.75,0.75)	0.9532	1.3719	1.3306	1.7360	7.1846

3.2. Conditional moments

Life expectancy, mean waiting time and inequality measures can be obtained from the incomplete moments. The r^{th} conditional moment $E\left(X^r \middle| X > z\right)$ is

$$E\left(X^r \middle| X > z\right) = \frac{1}{S(z)}\left[\mu_r' - E_{X\le z}\left(X^r\right)\right]. \tag{23}$$

The r^{th} lower incomplete moment $E_{X\le z}\left(X^r\right)$ is

$$E_{X\le z}\left(X^r\right) = \int_0^z x^r \sum_{\ell=0}^{k^*}\sum_{m=0}^{\ell} \mathbf{C}_{j,k,\ell,m} x^{m+1} e^{-(\ell+1)\frac{x}{\lambda}} dx.$$

Let $w = (\ell+1)\frac{x}{\lambda}$ then $\frac{\lambda}{(\ell+1)} dw = dx,$ we have

$$E\left(X^r\right) = \sum_{\ell=0}^{k^*}\sum_{m=0}^{\ell} \mathbf{C}_{j,k,\ell,m} \int_0^x \left[\frac{\lambda w}{(\ell+1)}\right]^{r+m+1} e^{-w} \frac{\lambda}{(\ell+1)} dw.$$

Finally,

$$E(X^r) = \sum_{\ell=0}^{k^*}\sum_{m=0}^{\ell} \mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,r+m+2)} \gamma\left(x; r+m+2\right). \tag{24}$$

The r^{th} conditional moment is

$$E\left(X^r \middle| X > z\right) = \frac{1}{S(z)}\left\{\sum_{\ell=0}^{k^*}\sum_{m=0}^{\ell} \mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,r+m+2)} \left[\mathbf{\Gamma}\left(r+m+2\right) - \gamma\left(x; r+m+2\right)\right]\right\},$$

where $\gamma(x;..)$ is lower incomplete gamma function. Similarly, the r^{th} reversed conditional moment $E\left(X^r \middle| X \le z\right)$ for $X \sim$ BIII-ME(α,β,λ), is given by

$$E\left(X^r \middle| X \le z\right) = \frac{1}{F(z)} \sum_{\ell=0}^{k^*} \sum_{m=0}^{\ell} \mathbf{C}_{j,k,\ell,m}^{(\ell,\lambda,r+m+2)} \gamma(x; r+m+2), \tag{25}$$

where $\gamma(x;..)$ is the lower gamma function.

3.3. Reliability estimation of multicomponent stress-strength model

Consider a system with identical elements, out of which s elements are operative. Let represent strengths of elements with the cdf F while, the stress Y enforced on the elements has the cdf G. The strengths and stress Y are independently distributed. The probability that system operates properly, is the reliability of the system, i.e.

$R_{s,\kappa} = P[\text{strengths}(X_i, i=1,2...\kappa) > \text{stress}(Y)] = P[\text{at least "s" of } (X_i, i=1,2...\kappa) \text{exceed } Y].$

Then, we can write this probability (Bhattacharyya and Johnson 1974) as follows:

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{l} \int_{-\infty}^{\infty} [1-F(y)]^{\ell} [F(y)]^{\kappa-\ell} dG(y). \tag{26}$$

Let $X \sim$ BIII-ME(α_1,β,λ), $X \sim$ BIII-ME(α_2,β,λ), with unknown α_1 and α_2, common β,λ where X and Y are independently distributed. The reliability in multicomponent stress-strength for the BIII-ME distribution is given by

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{l} \int_0^1 \left(1-u^{v}\right)^{\ell} u^{v(\kappa-\ell)} du,$$

where $v = \dfrac{\alpha_1}{\alpha_2}$ and $u = \left\{1 + \left[\dfrac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha_2}$. Let $w = u^{v}, u = w^{\frac{1}{v}}, du = \frac{1}{v} w^{\frac{1}{v}-1} dw,$ then we have

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{l} \int_0^1 (1-w)^{\ell} w^{(\kappa-\ell)} \frac{1}{v} w^{\frac{1}{v}-1} dw = \frac{1}{v} \sum_{\ell=s}^{\kappa} \binom{\kappa}{l} B\left(\ell+1, \kappa-\ell+\frac{1}{v}\right). \tag{27}$$

The probability in (27) is known as the reliability in multicomponent stress-strength model. For $s = \kappa = 1$, the multicomponent stress-strength model reduces to the stress-strength model (Kotz et al. 2003) as

$$R_{1,1} = P(Y < X) = \frac{\alpha_1}{(\alpha_1 + \alpha_2)},$$

where $\alpha_1 + \alpha_2 > 0$.

3.4. Uncertainty measures

The measure of uncertainty of a random variable is called entropy. Rényi entropy generalizes Hartley, Min, Shannon and collision entropies. Entropies are useful to study daily temperature instabilities (climatic), abnormal diffusion, DNA structures, information content gestures, heart rate variability (HRV) and cardiac autonomic neuropathy (CAN). Here, we study Rényi, Q, Havrda, Chavrat and Tsallis-Entropies. For X ~ BIII-ME$\left(\alpha,\beta,\lambda\right)$, Rényi entropy is given by

$$H_{\nu}\left(X\right)=\frac{1}{1-\nu}\log\left\{I\left(\nu\right)\right\}, \nu>0, \nu\neq 1. \tag{28}$$

Taking $I\left(\nu\right)=\int_{0}^{\infty}f^{\nu}\left(x\right)dx,$ we have

$$f^{\nu}\left(x\right)=\alpha^{\nu}\beta^{\nu}\frac{x^{\nu}}{\lambda^{2\nu}}e^{-\frac{\nu x}{\lambda}}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\nu\left(\beta+1\right)}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\nu\left(\beta-1\right)}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\nu\left(\alpha+1\right)}.$$

Using (16) to expand $\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\nu\left(\alpha+1\right)}$, we obtain

$$f^{\nu}\left(x\right)=\alpha^{\nu}\beta^{\nu}\frac{x^{\nu}}{\lambda^{2\nu}}e^{-\frac{\nu x}{\lambda}}\sum_{j=0}^{\infty}\binom{-\nu\left(\alpha+1\right)}{j}\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\nu\left(\beta-1\right)+\beta j}\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\nu\left(\beta+1\right)-\beta j}.$$

Again, expanding $\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\nu\left(\beta+1\right)-\beta j}$ and simplification, we get

$$f^{\nu}\left(x\right)=\frac{\alpha^{\nu}\beta^{\nu}}{\lambda^{2\nu+\ell}}\sum_{j,k=0}^{\infty}\sum_{\ell=0}^{\nu^{*}}\delta_{j,k,\ell}x^{\nu+\ell}e^{-\left[\nu\left(\beta-1\right)+\beta j+k+\nu\right]\frac{x}{\lambda}},$$

where $\nu^{*}=\nu\left(\beta-1\right)+\beta j+k$ and

$$\delta_{j,k,\ell}=\frac{\left(-1\right)^{k}}{\lambda^{2\nu+\ell}}\binom{-\nu\left(\alpha+1\right)}{j}\binom{\left(j+\nu\right)\beta+\nu+k-1}{k}\binom{\nu^{*}}{l}.$$

Consider

$$I\left(\nu\right)=\alpha^{\nu}\beta^{\nu}\sum_{j,k=0}^{\infty}\sum_{\ell=0}^{\nu^{*}}\delta_{j,k,\ell}\int_{0}^{\infty}x^{\nu+\ell}e^{-\left(\nu\beta+\beta j+k\right)\frac{x}{\lambda}}dx,$$

and let $w=\left(\nu\beta+\beta j+k\right)\frac{x}{\lambda},$ then $\frac{\lambda}{\left(\nu\beta+\beta j+k\right)}dw=dx,$ then we arrive at

$$I\left(\nu\right)=\alpha^{\nu}\beta^{\nu}\sum_{j,k=0}^{\infty}\sum_{\ell=0}^{\nu^{*}}\delta_{j,k,\ell}\frac{\lambda^{\ell+\nu+1}\mathbf{\Gamma}\left[\ell+\nu+1\right]}{\left(\nu\beta+\beta j+k\right)^{\ell+\nu+1}}.$$

Then,

$$H_{\nu}\left(X\right)=\frac{1}{1-\nu}\log\left\{\alpha^{\nu}\beta^{\nu}\sum_{j,k=0}^{\infty}\sum_{\ell=0}^{\nu^{*}}\delta_{j,k,\ell}\frac{\lambda^{\ell+\nu+1}\mathbf{\Gamma}\left[\ell+\nu+1\right]}{\left(\nu\beta+\beta j+k\right)^{\ell+\nu+1}}\right\}. \tag{29}$$

Note that Rényi entropy tends to the following entropies under certain conditions:

i) For $v \to 0,$ Rényi entropy $H_{\text{v}}(X),$ tends to max entropy $H_0(X),$ $\lim\limits_{\text{v}\to 0} H_{\text{v}}(X) = H_0(X)$

ii) For $v \to 1,$ Rényi entropy $H_{\text{v}}(X),$ tends to Shannon entropy $H_1(X),$

iii) For $v \to 2,$ Rényi entropy $H_{\text{v}}(X),$ tends to quadratic entropy $H_2(X),$ $\lim\limits_{v\to 2} H_v(X) = H_2(X)$

iv) For $v \to \infty,$ Rényi entropy $H_{\text{v}}(X),$ tends to min entropy $H_\infty(X),$ $\lim\limits_{\text{v}\to\infty} H_{\text{v}}(X) = H_\infty(X).$

For X ~ BIII-ME$(\alpha,\beta,\lambda),$ the Q-entropy is $H_q(f) = \frac{1}{1-q}\log\{1 - I(q)\}, q > 0, q \neq 1,$

$$H_q = \frac{1}{1-q}\log\left\{\alpha^q\beta^q \sum_{j,k=0}^{\infty} \sum_{\ell=0}^{[q(\beta-1)+\beta j+k]} \delta_{j,k,\ell} \frac{\lambda^{\ell+q+1}\Gamma[\ell+q+1]}{(q\beta+\beta j+k)^{\ell+q+1}}\right\}. \tag{30}$$

For X ~ BIII-ME$(\alpha,\beta,\lambda),$ Havrda and Chavrat entropy is $S_{HC}(f) = \frac{1}{v-1}\log\{I(\text{v})\}, v > 0, v \neq 1,$

$$I_{HC} = \frac{1}{v-1}\log\left\{\alpha^v\beta^v \sum_{j,k=0}^{\infty} \sum_{\ell=0}^{[v(\beta-1)+\beta j+k]} \delta_{j,k,\ell} \frac{\lambda^{\ell+v+1}\Gamma[\ell+v+1]}{(v\beta+\beta j+k)^{\ell+v+1}}\right\}. \tag{31}$$

For X ~ BIII-ME$(\alpha,\beta,\lambda),$ Tsallis-entropy is $S_q(f) = \frac{1}{q-1}\log\{1 - I(q)\}, q \neq 1, q > 0,$

$$S_q(f) = \frac{1}{q-1}\log\left(1 - \left\{\alpha^q\beta^q \sum_{j,k=0}^{\infty} \sum_{\ell=0}^{[q(\beta-1)+\beta j+k]} \delta_{j,k,\ell} \frac{\lambda^{\ell+q+1}\Gamma[\ell+q+1]}{(q\beta+\beta j+k)^{\ell+q+1}}\right\}\right), q \neq 1, q > 0. \tag{32}$$

4. Characterizations

In this section, we characterize the BIII-ME distribution through conditional expectation and truncated moments.

4.1. Conditional expectation

Here, we characterize the BIII-ME distribution via conditional expectation.

Proposition 4.1.1 *Let* $X : \Omega \to (0,\infty)$ *be a continuous rv with the cdf* $F(x)$*. Then for* $\alpha > 1,$ X *has the pdf (6) iff*

$$E\left\{\left[\frac{1-\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}{\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}\right]^{-\beta} \middle| X < t\right\} = \frac{1}{(\alpha-1)}\left\{1 + \alpha\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta}\right\}, t > 0. \tag{33}$$

Proof: If X has the pdf (6), then we have

$$E\left\{\left[\frac{1-\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}{\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}\right]^{-\beta}\middle| X<t\right\}=\left(F(t)\right)^{-1}\int_0^t\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta} f(x)\,dx$$

$$=\left(F(t)\right)^{-1}\int_0^t\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\frac{\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}}{\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta+1}}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1} dx.$$

Upon integration by parts and simplification, we obtain

$$E\left\{\left[\frac{1-\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}{\left(1+\frac{X}{\lambda}\right)e^{-\frac{X}{\lambda}}}\right]^{-\beta}\middle| X<t\right\}=\frac{1}{(\alpha-1)}\left\{1+\alpha\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta}\right\}, t>0.$$

Conversely, if (33) holds, then

$$\int_0^t\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta} f(x)\,dx=\frac{F(t)}{(\alpha-1)}\left\{1+\alpha\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta}\right\}. \tag{34}$$

Differentiating (34) with respect to t, we obtain

$$\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta} f(t)=\frac{f(t)}{(\alpha-1)}\left\{1+\alpha\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta}\right\}$$

$$-\frac{F(t)}{(\alpha-1)}\left\{\alpha\beta\frac{t}{\lambda^2}e^{-\frac{t}{\lambda}}\frac{\left[1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}\right]^{-\beta-1}}{\left[\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}\right]^{-\beta+1}}\right\}.$$

After simplification and integration, we obtain $F(t)=\left\{1+\left[\frac{1-\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}{\left(1+\frac{t}{\lambda}\right)e^{-\frac{t}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha}, t\geq 0.$

4.2. Truncated moment

We characterize the BIII-ME distribution via truncated moment.

Proposition 4.2.1 *Let* $X:\Omega\to(0,\infty)$ *be a continuous rv. Let*

$$h_1(x)=\frac{1}{\alpha}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{\alpha+1},x>0,$$

and

$$h_2(x)=2\alpha^{-1}\left[e^{\frac{x}{\lambda}}\left(1+\frac{x}{\lambda}\right)^{-1}\right]^{-\beta}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{\alpha+1},x>0.$$

The random variable X *has the pdf (6) iff, the function* $p(x)=\frac{E\left[h_1(X)\middle|X\geq x\right]}{E\left[h_2(X)\middle|X\geq x\right]}$ *has the form*

$$p(x)=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{\beta},x>0.$$

Proof: If X has the pdf (6), then

$$\left(1-F(x)\right)E\left(h_1(X)\middle|X\geq x\right)=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta},x>0,$$

$$\left(1-F(x)\right)E\left(h_2(X)\middle|X\geq x\right)=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-2\beta},x>0,$$

$$\frac{E[h_1(x)|X\geq x]}{E[h_2(x)|X\geq x]}=p(x)=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{\beta},x>0.$$

Conversely, if $p(x)$ has the given form, then $p'(x)=\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\frac{\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}}{\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta+1}},x>0.$

The differential equation $s'(x)=\dfrac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)}=\dfrac{2\beta\dfrac{x}{\lambda^2}e^{-\frac{x}{\lambda}}}{\left[\left(1+\dfrac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^2}\left[\dfrac{1-\left(1+\dfrac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\dfrac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-1}$ has solution

$$s(x)=\ln\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{2\beta},x>0.$$

Therefore, in light of Theorem G (Glänzel 1990), X has the pdf (6).

Corollary 4.2.1 *Let $X:\Omega\to(0,\infty)$ be a continuous rv and let*

$$h_2(x)=2\alpha^{-1}\left[e^{\frac{x}{\lambda}}\left(1+\frac{x}{\lambda}\right)^{-1}\right]^{-\beta}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{\alpha+1},x>0.$$

The pdf of X is (6) if and only if there exist functions $p(x)$ and $h_1(x)$ satisfying the differential equation

$$\frac{p'(x)}{p(x)h_2(x)-h_1(x)}=\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\frac{\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta-1}}{\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{\beta+1}}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1}. \quad (35)$$

Remark 4.2.1 *The general solution of (35) is given by*

$$p(x)=\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{2\beta}\times$$

$$\left[\int\left(-\alpha\beta\frac{x}{\lambda^2}e^{-\frac{x}{\lambda}}\frac{\left[1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta-1}}{\left[\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}\right]^{-\beta+1}}\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}^{-\alpha-1}h_1(x)\right)dx+D\right],$$

where D is a constant.

5. Estimation

Here, we adopt MLE technique for estimating the BIII-ME parameters. Let $X_1,X_2,\ldots,X_n$ be a random sample from the BIII-ME distribution with observed values $x_1,x_2,\ldots,x_n$ and $\xi=(\alpha,\beta,\lambda)^T$

be the vector of the model parameters. The log likelihood function $\ell(\xi)$ for the BIII-ME distribution is given by

$$\ell=\ell(\xi)=n\ln\alpha+n\ln\beta-2n\ln\lambda+\sum_{i=1}^{n}\ln x_i-\frac{1}{\lambda}\sum x_i-(\beta+1)\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{x_i}{\lambda}\right)e^{-\frac{x_i}{\lambda}}\right]+$$
$$(\beta-1)\sum_{i=1}^{n}\ln\left[\left(1+\frac{x_i}{\lambda}\right)e^{-\frac{x_i}{\lambda}}\right]-(\alpha+1)\sum_{i=1}^{n}\ln\left\{1+\left[\frac{1-\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}{\left(1+\frac{x}{\lambda}\right)e^{-\frac{x}{\lambda}}}\right]^{-\beta}\right\}. \quad (36)$$

We can compute the MLEs of α,β and λ by solving equations $\frac{\partial\ell}{\partial\alpha}=0,$ $\frac{\partial\ell}{\partial\beta}=0,$ and $\frac{\partial\ell}{\partial\lambda}=0$ either directly or using quasi-Newton procedures in computer software such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

6. Simulation Study

In this section, we evaluate the behavior of the MLEs of the BIII-ME parameters regarding sample size n. We generate 10,000 samples of sizes $n=30$, 50, 100, 200, 300, 500 from the inverse cdf of the BIII-ME with true parameter values $(\alpha,\beta,\lambda)=(0.75,0.50,0.25)$, $(0.90,0.60,0.30)$ and $(1.25,0.75,0.50)$. We estimate the MLEs $(\hat{\alpha},\hat{\beta},\hat{\lambda})$ for 10,000 samples from non-linear optimization technique. We also compute the means, biases and mean squared errors (MSEs) of MLEs. From simulation results given in Table 2, we infer that as sample size n increases, the means approach to true parameter value, the estimated MSEs decrease and estimated biases drop to zero. We observe that as shape parameter increases, MSE of estimated parameters increases. Finally, we infer that MLEs for the BIII-ME distribution are consistent. We obtain the simulation results via R package "maxLik function" (Henningsen and Toomet 2011).

7. Applications

We consider applications to two data sets such as strength of glass fibers and fracture toughness to verify the BIII-ME distribution. We compare the BIII-ME distribution with competitive models such as Weibull-Moment exponential (WME), generalized exponentiated moment exponential (GEME), generalized moment exponential (GME), exponentiated moment exponential (EME), moment exponential (ME) and BIII distributions. For selection of the best distribution, we compute the estimate of likelihood ratio statistics $(-2\hat{\ell})$, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises (W*), Anderson Darling (A*), and Kolmogorov- Smirnov [K-S] statistics with p-values for all competing and sub distributions. We compute the MLEs and their standard errors (SEs) (in parentheses). We also compute goodness of fit statistics (GOFs) values for the BIII-ME, W-ME, GEME, GME, EME, ME and BIII models.

Table 2 Means, bias and MSEs of the BIII-ME distribution

n		$\alpha=0.75$	$\beta=0.5$	$\lambda=0.25$	$\alpha=0.9$	$\beta=0.6$	$\lambda=0.3$	$\alpha=1.25$	$\beta=0.75$	$\lambda=0.5$
30	Means	0.8989	0.5830	0.2600	1.1080	0.6510	0.2991	1.5877	0.7791	0.4809
	Bias	0.1489	0.0830	0.0100	0.2080	0.0510	−9e-04	0.3377	0.0291	−0.0191
	MSE	0.2098	1.6343	0.0249	0.3392	0.8596	0.0263	0.7300	0.6734	0.0518
50	Means	0.8449	0.5424	0.2643	1.0441	0.6317	0.3065	1.5170	0.7570	0.4870
	Bias	0.0949	0.0424	0.0143	0.1441	0.0317	0.0065	0.2677	0.0070	−0.0130
	MSE	0.1425	0.1955	0.0203	0.2440	0.2233	0.0222	0.5823	0.1153	0.0436
100	Means	0.7874	0.5393	0.2709	0.9776	0.6266	0.3121	1.4092	0.7637	0.5020
	Bias	0.0374	0.0393	0.0209	0.0776	0.0266	0.0121	0.1592	0.0137	0.002
	MSE	0.0836	0.0441	0.0154	0.1458	0.0495	0.0168	0.3710	0.0648	0.0344
200	Means	0.7621	0.5283	0.2674	0.9289	0.6302	0.3166	1.3401	0.7679	0.5104
	Bias	0.0121	0.0283	0.0174	0.0289	0.0302	0.0166	0.0901	0.0179	0.0104
	MSE	0.0468	0.0249	0.0098	0.0893	0.0351	0.0125	0.2403	0.0471	0.0268
300	Means	0.7532	0.5244	0.2649	0.9134	0.6282	0.3161	1.3057	0.7722	0.5145
	Bias	0.0032	0.0244	0.0149	0.0134	0.0282	0.0161	0.0557	0.0222	0.0145
	MSE	0.0344	0.0185	0.0073	0.0680	0.0267	0.0098	0.1843	0.0381	0.0224
500	Means	0.7522	0.5149	0.2594	0.9023	0.6222	0.3131	1.2752	0.7728	0.5164
	Bias	0.0022	0.0149	0.0094	0.0023	0.0222	0.0131	0.0252	0.0228	0.0164
	MSE	0.0215	0.0104	0.0042	0.0447	0.0175	0.0066	0.1311	0.0284	0.0171

7.1. Data set I: Strength of glass fibers

The data about measurement of the strengths of 1.5 cm glass fibers (Smith and Naylor 1987) are 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58,1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13,1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

A descriptive summary for the strengths of 1.5 cm glass fibers data set provides the following values: 63 (sample size), 0.55 (minimum), 2.24 (maximum), 1.59 (median), 1.506825 (mean), 0.3241257 (standard deviation), 21.5105 (coefficient of variation), -0.89993 (coefficient of skewness) and 3.92376 (coefficient of kurtosis). The boxplot (Figure 3(left)) for strengths of glass fibers data is negatively skewed. The TTT (total time on test) plot (Figure 3 (right)) for strengths of glass fibers data is concave, which infers increasing failure rate. So, the BIII-ME distribution is suitable to model these data. Table 3 reports the MLEs (SEs in parentheses) and measures W*, A*, KS (p-values). Table 4 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC. From the Tables 3 and 4, it is clear that our proposed model is the best fitted, with the smallest values for all statistics and maximum p-value. Figure 4 infers that the proposed model is closely fitted to glass fiber data.

Table 3 MLEs (standard errors) and W*, A*, KS (p-values) for glass fiber data

Model	α	β	λ	W*	A*	K-S(p-value)
BIII-ME	0.2457(0.0744)	9.5897(2.183)	1.0595(0.0278)	0.1043	0.5785	0.1238(0.289)
W-ME	0.0564(0.1374)	2.1159(0.866)	0.5544(0.3314)	0.1762	0.9919	0.1369(0.189)
EME	12.926(3.6417)		0.3126(0.0258)	0.7475	4.0836	0.2271(0.003)
GEME	0.3753(0.1443)	6.6027(1.5957)	23.908(28.136)	0.2063	1.1459	0.1479(0.127)
GME	3.7323(0.3735)		2.7968(0.5928)	0.3296	1.8054	0.1713(0.050)
EME			0.7534(0.0671)	0.5691	3.1209	$0.4\,(5.2\times10^{-7})$
BIII	3.44171(0.4505)	4.0886(0.3357)		0.9554	5.1997	0.2462(0.001)

Table 4 $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for glass fiber data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BIII-ME	23.37236	29.37237	29.77915	35.80177	31.90108
W-ME	28.95490	34.95490	35.36168	41.38430	37.48361
EME	60.16142	64.16143	64.36143	68.44770	65.84724
GEME	29.71776	35.71777	36.12455	42.14717	38.24649
GME	34.84620	38.84621	39.04621	43.13248	40.53202
EME	132.6345	134.6345	134.7001	136.7777	135.4774
BIII	73.76724	77.76725	77.96725	82.05352	79.45306

7.2 Data set II: Fracture toughness (Nadarajah and Kotz 2007)

The data about fracture toughness MPa m1/2 from the material Alumina (2 3 Al O) are 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4,4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35,5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3,5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95,3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

A descriptive summary for the fracture toughness data set provides the following values: 119 (sample size), 1.68 (minimum), 6.81(maximum), 4.38 (median), 4.325378 (mean), 1.018495 (standard deviation), 23.54696 (coefficient of variation), −0.42 (coefficient of skewness) and 3.093 (coefficient of kurtosis). The boxplot (Figure 5(left)) for fracture toughness data is negatively skewed. The TTT plot (Figure 5 (right)) for fracture toughness data is concave, which infers increasing failure rate. So, the BIII-ME distribution is suitable to model these data. Table 5 reports the MLEs (SEs in parentheses) and measures W*, A*, KS (p-values). Table 6 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC. From the Tables 5 and 6, it is clear that our proposed model is the best fitted, with the smallest values for all statistics and maximum p-value. Figure 6 infers that the proposed model is closely fitted to fracture toughness data.

Table 5 MLEs (standard errors) and W*, A*, KS (p-values) for fracture toughness data

Model	α	β	λ	W*	A*	K-S(p-value)
BIII-ME	0.3258(0.0904)	6.5304(1.2468)	3.0591(0.1052)	0.0487	0.2977	0.051(0.9158)
W-ME	4.7411(25.6625)	2.5466(0.4864)	3.8138(3.9112)	0.0865	0.5235	0.0703(0.5995)
EME	12.4391(2.6143)		0.901(0.0560)	0.6321	3.7406	0.1524(0.0080)
GEME	0.4693(0.0578)	4.9913(0.2112)	1783.06(664.0)	0.0879	0.5317	0.0713(0.5804)
GME		3.2628(0.2311)	71.0850(26.449)	0.1558	0.9660	0.0861(0.3414)
EME			2.1627(0.1402)	0.3892	2.3547	$0.31\,(2.9\times10^{-10})$
BIII	52.051(11.2354)	3.0605(0.1802)		1.3660	7.6631	0.1963(0.0002)

Table 6 $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for fracture toughness data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BIII-ME	335.4878	341.4877	341.6964	349.8251	344.8733
W-ME	337.2478	343.2477	343.4564	351.5851	346.6333
EME	371.0412	375.0413	375.1447	380.5995	377.2983
GEME	337.3738	343.3738	343.5825	351.7112	346.7593
GME	341.0136	345.0136	345.1171	350.5719	347.2706
EME	502.3618	504.3617	504.3959	507.1409	505.4903
BIII	419.5350	423.5350	423.6384	429.0932	425.7920

8. Conclusions

In this study, we propose the BIII-ME distribution from (i) the T-X family procedure and (ii) the nexus concerning exponential and gamma variables. The BIII-ME density can be bimodal, arc, symmetrical, left-skewed, right-skewed, J, reverse-J and arc shapes. The BIII-ME failure rate can take monotone and non- monotone shapes. Certain mathematical properties such as random number generator, sub-models, ordinary moments, conditional moments, density functions of record values and reliability measures are derived. The BIII-ME distribution is characterized via innovative techniques. We address the maximum likelihood estimation for the BIII-ME parameters. Then, the precision of the MLES is evaluated via simulation study. We consider applications to two real data sets to illustrate the potentiality of the new model. We compute GOFs for testing the adequacy and competency of the BIII-ME model. We show empirically that the proposed model is suitable for strength of glass fibers and fracture toughness data analysis.

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Appendix

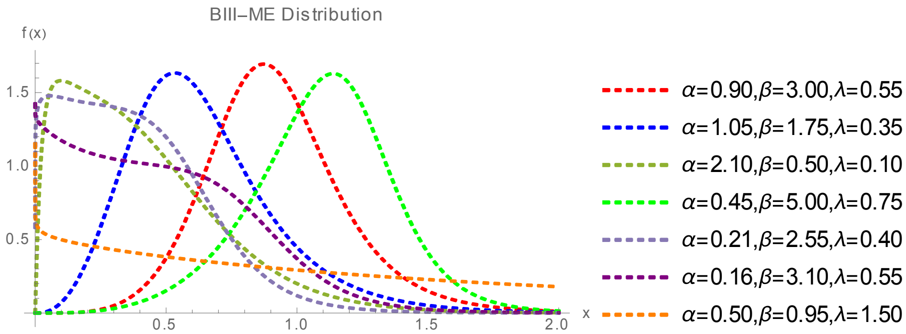

Figure 1 Plots of the BIII-ME density

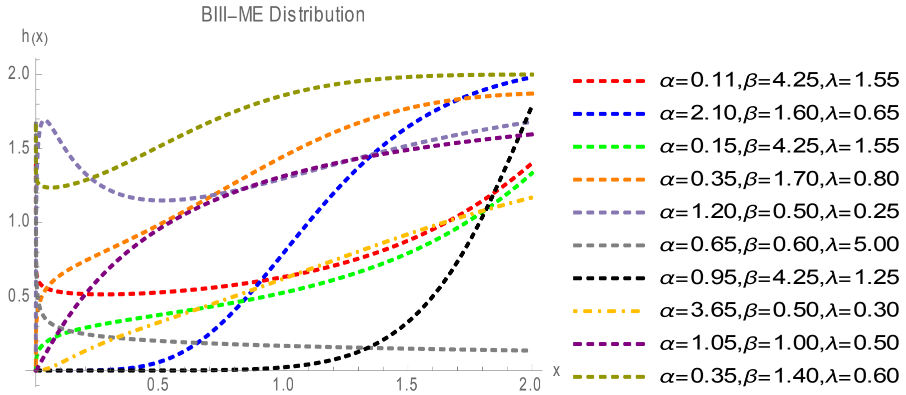

Figure 2 Plots of the failure rate function of the BIII-ME distribution

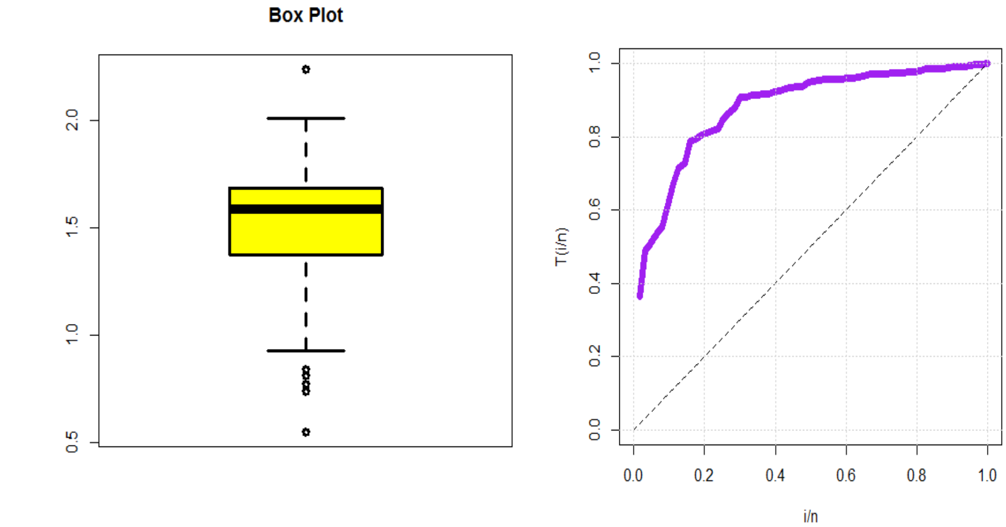

Figure 3 Box plot (left) and TTT plot (right) for glass fiber data

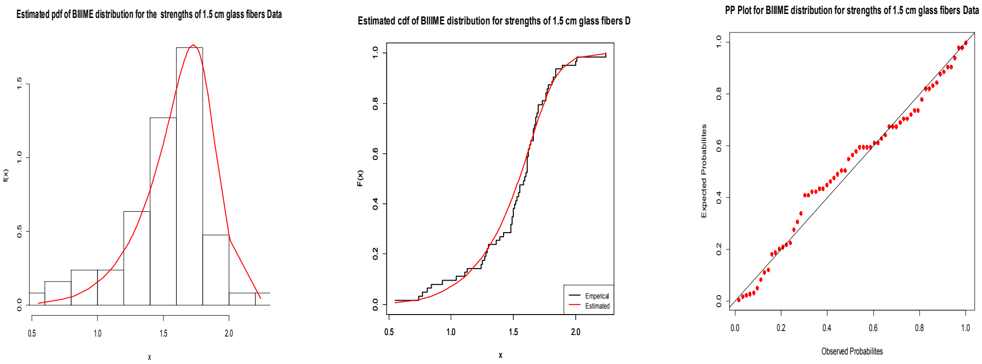

Figure 4 Fitted (left) pdf, (center) cdf, (right) PP plots for the BIII-ME distribution to glass fiber data

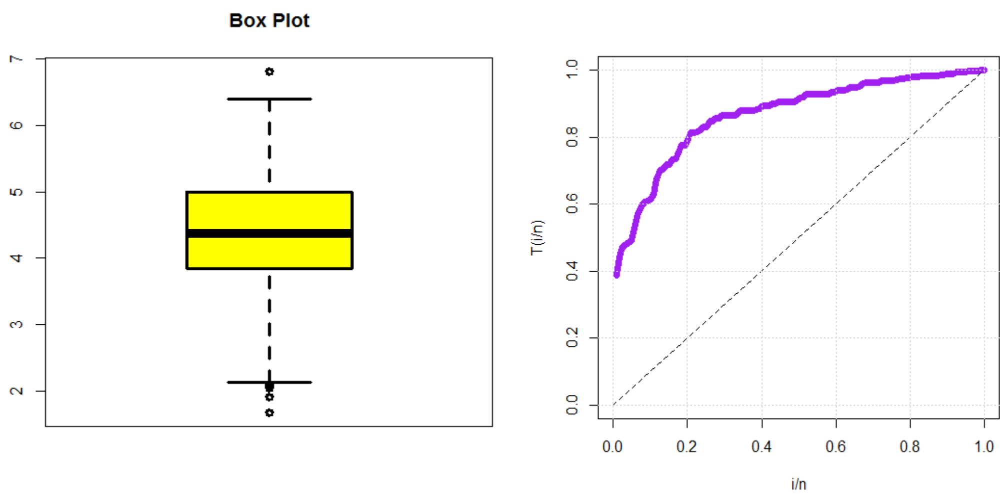

Figure 5 Boxplot (left) and TTT plot (right) for fracture toughness data

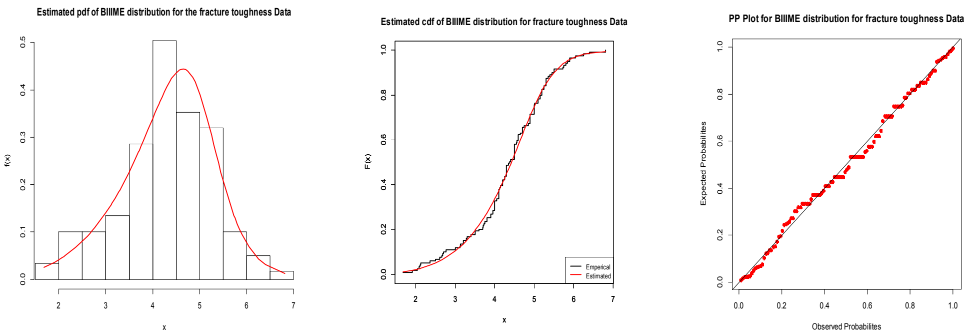

Figure 6 Fitted (left) pdf, (center) cdf and (right) P-P plots for the BIII-ME model to fracture toughness