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Alpha Power Transformation of Lomax Distribution: Properties and Applications

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Abstract

In this paper, we present a new three-parameter alpha power transformation of Lomax distribution (APTLx). Some statistical properties of the APTLx distribution are obtained including moments, quantiles, entropy, order statistics, and stress-strength analysis and its explicit expressions are derived. Maximum likelihood estimation method is used to estimate the parameters of the distribution. The goodness-of-fit of the proposed model show that the new distribution performs favorably when compare with existing distributions. The application of APTLx distribution is emphasized using a real-life data.

Keywords: Alpha power family, probability weighted moments, stochastic ordering, order statistics, MLE.

1. Introduction

Lifetime data plays an important role in a wide range of applications such as medicine, engineering, biological science, and public health. Statistical distributions are used to model the life of an item in order to study its important properties. The most popular traditional distributions often not able to characterize and predict most of the interesting data sets. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. Generated family of continuous distribution is a new improvement for creating and extending the usual classical distributions. Zografos and Balakrishnan (2009) suggested a gamma generated (gamma-G) family using gamma distribution. Its cumulative distribution function (cdf) is defined as

$$F_{ZB}(x)=\frac{1}{\Gamma(\delta)}\int_{0}^{-\log(1-G(x))} x^{\delta-1}e^{-x}dx;\quad \text{for } \delta>0. \tag{1}$$

Cordeiro and de Castro (2011) proposed a Kumaraswamy generalized (Kw-G) family of distribution. The cdf of Kw-G distribution is defined as follows:

$$F_{KW-G}(x)=1-\left\{1-G(x)^{a}\right\}^{b};\quad \text{for } a,b>0. \tag{2}$$

The Lomax distribution, also known as Pareto type II distribution was proposed by Lomax in 1954. It is most commonly used for analyzing business failure life time data, actuarial science, medical and biological sciences, lifetime and reliability modelling. Hassan and Al-Ghamdi (2009) mentioned that it used for reliability modelling and life testing. Bryson (1974) had suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed. Atkinson and Harrison (1978) used it for modelling the business failure data. Corbellini et al. (2007) used it to model firm size. Lomax distribution is used as the basis of several generalizations, e.g., Ghitany and Al-Awadhi (2001) used Lomax distribution as a mixing distribution for the Poisson parameter and derived a discrete Poisson-Lomax distribution.

A random variable X has the Lomax distribution with two parameters β and λ if it has cdf given by

$$F(x) = 1 - \left\{1 + \frac{x}{\lambda}\right\}^{-\beta}; \quad \text{for } x > 0, \tag{3}$$

where $\beta > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The corresponding probability density function (pdf) is

$$f(x) = \frac{\beta}{\lambda}\left\{1 + \frac{x}{\lambda}\right\}^{-(\beta+1)}; \quad \text{for } x > 0, \beta > 0, \lambda > 0. \tag{4}$$

In the literature, some extensions of the Lomax distribution are available such as follows: Marshall-Olkin extended-Lomax distribution by Ghitany et al. (2007), Kumaraswamy-Generalized Lomax distribution by Shams (2013), Gumbel-Lomax distribution by Tahir et al. (2016), Exponential Lomax distribution by El-Bassiouny (2015), half-logistic Lomax distribution by Anwar (2018) and power Lomax distribution by El-Houssainy (2016).

The alpha power transformation (APT) is proposed by Mahadavi and Kundu (2015) in the paper "A new method of generating distribution with an application to exponential distribution".

The cdf of APT is given by

$$F_{APT}(x;\alpha) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}; & \text{if } \alpha > 0,\ \alpha \neq 1 \\ F(x); & \text{if } \alpha = 1. \end{cases} \tag{5}$$

The corresponding pdf is

$$f_{APT}(x;\alpha) = \begin{cases} \dfrac{\log\alpha}{\alpha - 1} f(x)\alpha^{F(x)}; & \text{if } \alpha > 0,\ \alpha \neq 1 \\ f(x); & \text{if } \alpha = 1. \end{cases} \tag{6}$$

Based on APT, many new distributions are like alpha power Weibull distribution by Nassar et al. (2017), denoted by APW with $\lambda > 0,\ \alpha > 0,\ \beta > 0.$ Its cdf is expressed as

$$F_{APW}(x;\alpha) = \begin{cases} \dfrac{1}{1-\alpha}\left(1 - \alpha^{1-e^{-\lambda x^{\beta}}}\right); & \text{if } \alpha > 0,\ \alpha \neq 1 \\ 1 - e^{-\lambda x^{\beta}}; & \text{if } \alpha = 1. \end{cases} \tag{7}$$

Malik and Ahmad (2017) introduced two-parameter alpha power Rayleigh distribution (APR), with $\alpha > 0,\ \theta > 0$ if its cdf is given by

$$F_{APR}(x;\alpha)=\begin{cases}\dfrac{\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}-1}{1-\alpha}; & \text{if } \alpha>0,\ \alpha\neq 1\\ 1-e^{-\frac{x^2}{2\theta^2}}; & \text{if } \alpha=1.\end{cases} \tag{8}$$

Hassan and Elgarhy (2019) presented a three-parameter alpha power transformed power Lindley distribution (APTPL) with $\theta>0,\ \alpha>0,\ \beta>0$ if its cdf is

$$F_{APTPL}(x;\alpha)=\begin{cases}\dfrac{\alpha^{1-e^{-\theta x^{\beta}}\left[1+\frac{\theta x^{\beta}}{\theta+1}\right]}-1}{1-\alpha}; & \text{if } \alpha>0,\ \alpha\neq 1\\ 1-e^{-\theta x^{\beta}}\left[1+\dfrac{\theta x^{\beta}}{\theta+1}\right]; & \text{if } \alpha=1.\end{cases} \tag{9}$$

Some other distributions are available such as alpha power inverted exponential by Unal et al. (2018), Alpha power transformed Fréchet by Nasiru et al. (2019), alpha-power transformed Lindley by Dey et al (2018).

The aim of this paper is to propose and study a new lifetime model called alpha power Lomax (APTLx) distribution based on APT. The new distribution is very flexible in the sense that it can be skewed depending upon the special choices of the parameters. This paper is organized as follows: In Section 2, we introduced the APTLx distribution and presented some illustrations. In Section 3, we studied some of its structural properties including quantile function, moments, moment generating function, entropy, order statistics, and stress strength parameter. In Section 4, we discussed the maximum likelihood estimates (MLEs) of the model parameter. In Section 5, the analysis of real data sets was illustrated the potentiality of the new model. In Section 6, we concluded the study.

2. APTLx Distribution

The random variable X is said to follow the three-parameter APTLx distribution with the shape parameter $\beta>0$ and scale parameter $\lambda>0$, if the cdf of X is given by:

$$F_{APTLx}(x)=\begin{cases}\dfrac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}{\alpha-1}; & \text{if } \alpha>0,\ \alpha\neq 1\\ 1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta}; & \text{if } \alpha=1.\end{cases} \tag{10}$$

The corresponding pdf of APTLx distribution is

$$f_{APTLx}(x)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\left(\dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}; & \text{if } \alpha>0,\ \alpha\neq 1\\ \dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}; & \text{if } \alpha=1.\end{cases} \tag{11}$$

The hazard rate function of APTLx distribution is given by

$$h_{APTLx}(x)=\begin{cases}\dfrac{\log\alpha\left(\dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}; & \text{if } \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{\beta}{\lambda+x}; & \text{if } \alpha=1.\end{cases} \quad (12)$$

The survival function of APTLx distribution is given by

$$S_{APTLx}(x)=\begin{cases}\dfrac{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}{\alpha-1}; & \text{if } \alpha>0,\ \alpha\neq 1\\[2ex] \left(1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta}\right); & \text{if } \alpha=1.\end{cases} \quad (13)$$

The reversed hazard rate function of APTLx distribution is given by

$$\tau_{APTLx}(x)=\begin{cases}\dfrac{\log\alpha\left(\dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}{\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1}; & \text{if } \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{\dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}}{1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta}}; & \text{if } \alpha=1.\end{cases} \quad (14)$$

Figures 1 and 2 displays the graph of pdf of APTLx distribution which is right skewed and the hazard rate function. They can be increasing and decreasing.

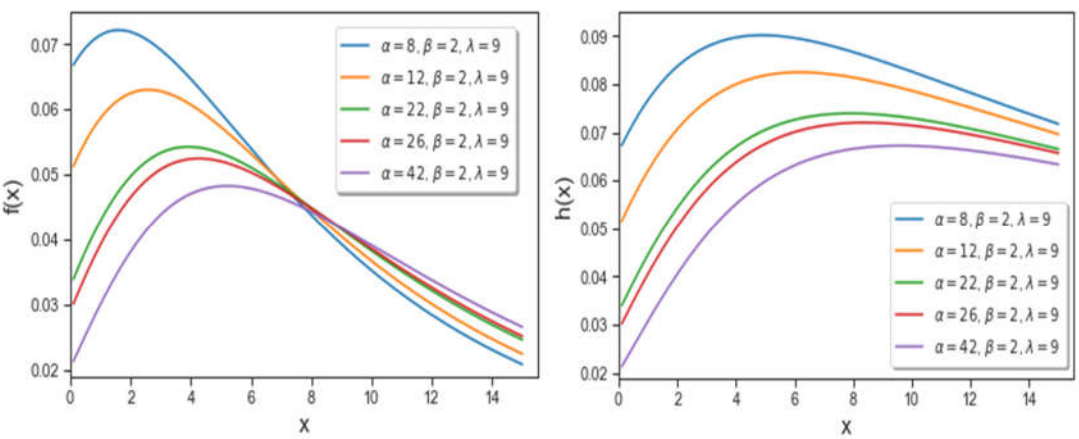

Figure 1 The pdf and hazard function plot of APTLx distribution for $\beta=2,\lambda=9$ and different values of α

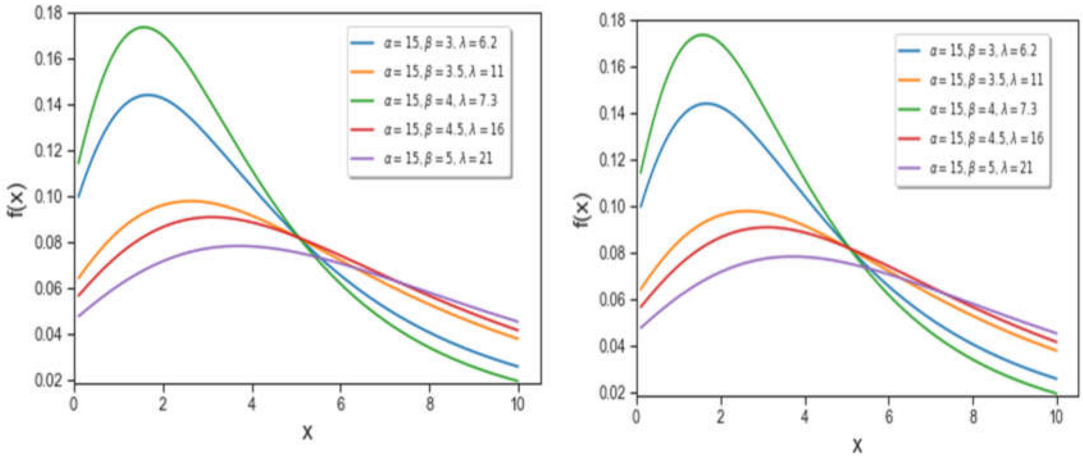

Figure 2 The pdf and hazard function plot of the APTLx distribution for $\alpha = 15$ and different values of λ and β

2. Statistical Properties

In this section, some statistical properties of the APTLx distribution are derived for the case when α, otherwise the distribution reduces to the Lomax model.

2.1. Quantile function

The quantile function plays an important role when simulating random variates from a statistical distribution. Using (8), the APTLx distribution can be simulated by

$$X = \lambda\left(\frac{\log\left(\alpha/\left(U(\alpha-1)+1\right)\right)}{\log\alpha}\right)^{-\frac{1}{\beta}} - \lambda, \tag{15}$$

where U follows the uniform distribution. The p^{th} quantile function of APTLx distribution is

$$x_p = \lambda\left(\frac{\log\left(\alpha/\left(p(\alpha-1)+1\right)\right)}{\log\alpha}\right)^{-\frac{1}{\beta}} - \lambda. \tag{16}$$

The median of APTLx distribution is

$$x_{0.5} = \lambda\left(\frac{\log\left(2\alpha/(\alpha-1)\right)}{\log\alpha}\right)^{-\frac{1}{\beta}} - \lambda. \tag{17}$$

2.2. Method of moments

The r^{th} moment and the moment generating function of the APTLx distribution are provided here. Using the series representation given as follows

$$\alpha^{-w} = \sum_{k=0}^{\infty}\frac{(-\log\alpha)^k w^k}{k!}. \tag{18}$$

The r^{th} moment of the random variable X having APTLx distribution is obtained as follows

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \left(\frac{\log\alpha}{\alpha-1} \left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)} \right) \alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)} \right) dx,$$

$$\mu_r' = \frac{\alpha\beta\lambda^r(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B\left[r+1,(\beta k+\beta)-r\right]. \tag{19}$$

The first moment and mean of APTLx distribution is obtained as follows

$$\mu_1' = \frac{\alpha\beta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B\left[2,(\beta k+\beta)-1\right]. \tag{20}$$

The r^{th} incomplete moment of the random variable X having APTLx distribution is obtained as follows

$$\varphi_r(t) = \int_0^t x^r f(x)dx = \int_0^t x^r \left(\frac{\log\alpha}{\alpha-1} \left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)} \right) \alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)} \right) dx,$$

$$\varphi_r(t) = \frac{\alpha\beta\lambda^r(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[r+1,(\beta k+\beta)-r\right]. \tag{21}$$

The first incomplete moment of APTLx distribution is obtained as follows

$$\varphi_1(t) = \frac{\alpha\beta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}.B_t\left[2,(\beta k+\beta)-1\right]. \tag{22}$$

The moment generating function of APTLx distribution is obtained as follows

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx} \left(\frac{\log\alpha}{\alpha-1} \left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)} \right) \alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)} \right) dx. \tag{23}$$

Using the (18) and the series representation, $e^{tx} = \sum_{j=1}^{\infty}\frac{t^j x^j}{j!}$, we get

$$M_x(t) = \frac{\alpha\beta\log\alpha}{\alpha-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\lambda^j(-\log\alpha)^k}{k!}B\left[j+1,(\beta k+\beta)-j\right]. \tag{24}$$

2.3. Lorenz curve and Bonferroni curve

The Bonferroni and Lorenz curves are important in economics, reliability, demography, insurance, and medicine. The Lorenz curve, $LO(x)$, and Bonferroni curve, $BO(x)$, are defined by

$$LO(x) = \frac{\varphi_1(x)}{E(X)} \text{ and } BO(x) = \frac{LO(x)}{F_{APTLx}(x)}.$$

By using (20) and (22), we get

$$LO(x) = \frac{\frac{\alpha\beta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]}{\frac{\alpha\beta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B\left[2,(\beta k+\beta)-1\right]},$$

$$LO(x) = \left(\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]\right)\times\left(\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B\left[2,(\beta k+\beta)-1\right]\right)^{-1}. \tag{25}$$

By using (10) and (25), we get

$$BO(x)=\frac{(\alpha-1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]\times\left(\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}B\left[2,(\beta k+\beta)-1\right]\right)^{-1}}{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}. \tag{26}$$

2.4. Probability weighted moments

The probability weighted moments (PWMs), denoted by $\pi_{r,s}$ for a random variable *X*, is defined as follows

$$\pi_{r,s}=E\left(x^r f(x)F(x)^s\right)=\int_0^\infty x^r f(x)F(x)^s\,dx$$

$$=\int_0^\infty x^r\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}\right)\left(\frac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}{\alpha-1}\right)^s dx,$$

$$\pi_{r,s}=\frac{\lambda^r\beta}{(\alpha-1)^{s+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{s}(-1)^{s+j-m}\binom{s}{m}\binom{i}{j}\frac{(m+1)^i(\log\alpha)^{i+1}}{i!}B\left[r+1,(\beta i+\beta)-r\right]. \tag{27}$$

2.5. Mean residual life time and mean inactivity time

The mean residual life time of APTLx distribution is given by

$$m_x(t)=E(X-t\,|\,X>t)=\frac{\mu-\varphi_1(x)}{S(t)}-t,\quad t>0\quad \text{where } \mu=\mu_1$$

By using (13) and (22),

$$m_x(t)=\frac{\mu-\frac{\alpha\beta\lambda\log\alpha}{\alpha-1}\sum_{k=1}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]}{\frac{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}{\alpha-1}}-t,$$

$$m_x(t)=\frac{\mu(\alpha-1)-\alpha\beta\lambda\log\alpha\sum_{k=1}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]}{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}}-t. \tag{28}$$

The mean inactivity time is defined by

$$\psi_x(t)=E(X-t\,|\,X<t)=t-\frac{\varphi_1(x)}{F(t)}.$$

By using (10) and (22),

$$\psi_x(t)=t-\frac{\dfrac{\alpha\theta\lambda\log\alpha}{\alpha-1}\displaystyle\sum_{k=1}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]}{\dfrac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}{\alpha-1}},$$

$$\psi_x(t)=t-\frac{\alpha\theta\lambda\log\alpha\displaystyle\sum_{k=1}^{\infty}\frac{(-\log\alpha)^k}{k!}B_t\left[2,(\beta k+\beta)-1\right]}{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}. \tag{29}$$

2.6. Stochastic ordering

Ordering of distributions, particularly among lifetime distributions plays an important role in the statistical literature. Specifically, we consider stochastic orders, the hazard rate, the mean residual life, and the likelihood ratio order for two independent APTLx random variables under a restricted parameter space. It may be recalled that if a family has a likelihood ratio ordering, it has the monotone likelihood ratio property. This implies that there exists a uniformly most powerful test for any one-sided hypothesis when the other parameters are known.

If X and Y are independent random variables with cdfs F_X and F_Y, respectively, then X is said to be smaller than Y in the

- stochastic order $(X\leq_{st}Y)$ if $F_X(x)\geq F_Y(x)$ for all x,
- hazard rate order $(X\leq_{hr}Y)$ if $h_X(x)\geq h_Y(x)$ for all x,
- mean residual life order $(X\leq_{mrl}Y)$ if $m_X(x)\geq m_Y(x)$ for all x,
- likelihood ratio order $(X\leq_{lr}Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following method is well known for establishing stochastic ordering of distributions. Let $Y\sim APTLx(\alpha_1,\beta_1,\lambda_1)$, and $Y\sim APTLx(\alpha_2,\beta_2,\lambda_2)$ then the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)}=\frac{\left(\dfrac{\log\alpha_1}{\alpha_1-1}\left(\dfrac{\beta_1}{\lambda_1}\left(1+\dfrac{x}{\lambda_1}\right)^{-(\beta_1+1)}\right)\alpha_1^{\left(1-\left(1+\frac{x}{\lambda_1}\right)^{-\beta_1}\right)}\right)}{\left(\dfrac{\log\alpha_2}{\alpha_2-1}\left(\dfrac{\beta_2}{\lambda_2}\left(1+\dfrac{x}{\lambda_2}\right)^{-(\beta_2+1)}\right)\alpha_2^{\left(1-\left(1+\frac{x}{\lambda_2}\right)^{-\beta_2}\right)}\right)}$$

$$=\frac{\left((\alpha_2-1)\log\alpha_1\left(\dfrac{\beta_1}{\lambda_1}\left(1+\dfrac{x}{\lambda_1}\right)^{-(\beta_1+1)}\right)\alpha_1^{\left(1-\left(1+\frac{x}{\lambda_1}\right)^{-\beta_1}\right)}\right)}{\left((\alpha_1-1)\log\alpha_2\left(\dfrac{\beta_2}{\lambda_2}\left(1+\dfrac{x}{\lambda_2}\right)^{-(\beta_2+1)}\right)\alpha_2^{\left(1-\left(1+\frac{x}{\lambda_2}\right)^{-\beta_2}\right)}\right)}.$$

Taking the log of both sides, we get

$$\log\frac{f_X(x)}{f_Y(x)}=\log\left\{\frac{\left((\alpha_2-1)\log\alpha_1\left(\frac{\beta_1}{\lambda_1}\left(1+\frac{x}{\lambda_1}\right)^{-(\beta_1+1)}\right)\alpha_1^{\left(1-\left(1+\frac{x}{\lambda_1}\right)^{-\beta_1}\right)}\right)}{\left((\alpha_1-1)\log\alpha_2\left(\frac{\beta_2}{\lambda_2}\left(1+\frac{x}{\lambda_2}\right)^{-(\beta_2+1)}\right)\alpha_2^{\left(1-\left(1+\frac{x}{\lambda_2}\right)^{-\beta_2}\right)}\right)}\right\} \quad (30)$$

$$=\log(\alpha_2-1)+\log(\log\alpha_1)+\log\beta_1+\log\lambda_2-(\beta_1+1)\log\left(1+\frac{x}{\lambda_1}\right)+$$

$$\left(1-\left(1+\frac{x}{\lambda_1}\right)^{-\beta_1}\right)\log\alpha_1-\log(\alpha_1-1)-\log(\log\alpha_2)-\log\beta_2+\log\lambda_1+$$

$$(\beta_2+1)\log\left(1+\frac{x}{\lambda_2}\right)-\left(1-\left(1+\frac{x}{\lambda_2}\right)^{-\beta_2}\right)\log\alpha_2,$$

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}=\frac{\beta_2+1}{\left(1+\frac{x}{\lambda_2}\right)\lambda_2}-\frac{\beta_1+1}{\left(1+\frac{x}{\lambda_1}\right)\lambda_1}+\frac{\beta_1\log\alpha_1}{\lambda_1\left(1+\frac{x}{\lambda_1}\right)^{(\beta_1+1)}}-\frac{\beta_2\log\alpha_2}{\lambda_2\left(1+\frac{x}{\lambda_2}\right)^{(\beta_2+1)}}. \quad (31)$$

Now if $\alpha_1=\alpha_2=\alpha,\ \beta_1=\beta_2=\beta,\ \theta_1=\theta_2=\theta$ and $\lambda_1>\lambda_2$ then $\frac{d}{dx}\log\left[\frac{f_X(x)}{f_Y(x)}\right]\le 0 \Rightarrow X\le_{lr} Y$ and hence $X\le_{lr} Y, X\le_{hr} Y, X\le_{mrl} Y,$ and $X\le_{st} Y.$

2.7. Rényi entropy

The entropy of the random variable X measures the variation of uncertainty. The Rényi entropy is defined as

$$RE_x(v)=\frac{1}{1-v}\log\left\{\int_{-\infty}^{\infty}f(x)^v\,dx\right\},\ v>0, v\neq 1$$

Rényi entropy of the APTLx distribution is given below:

$$RE_x(v)=\frac{1}{1-v}\log\left\{\int_{-\infty}^{\infty}\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}\right)^v dx\right\}$$

$$RE_x(v)=\frac{v}{1-v}\left\{\log\left(\frac{\alpha\log\alpha}{\alpha-1}\right)+\log\beta-\log\lambda\right\}+\frac{1}{1-v}\log\left\{\lambda\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}\left(\frac{1}{v(\beta k+\beta+1)-1}\right)\right\} \quad (32)$$

2.8. Stress strength parameter

The stress strength parameter, $R=P\left(X_1<X_2\right)$ in the lifetime model, describes the lifetime component which has a random stress X_1 that is subjected to a random strength X_2. Let X_1 and X_2 be two independent random variables, $X_1 \sim APTLx\left(\alpha_1,\beta_1,\lambda\right)$ and $X_2 \sim APTLx\left(\alpha_2,\beta_2,\lambda\right)$ then the stress strength parameter, say R, is defined as follows:

$$R=\int_0^\infty f_1(x)F_2(x)dx$$

By using (10) and (11), we get

$$R=\frac{\beta_1\log\alpha_1}{\lambda\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\int_0^\infty\left(\left(1+\frac{x}{\lambda}\right)^{-\left(\beta_1+1\right)}\alpha_1^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta_1}\right)}\left(\alpha_2^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta_2}\right)}-1\right)\right)dx.$$

By using (18), R can be simplified to

$$R=\frac{\alpha_1\beta_1\log\alpha_1}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\sum_{k=0}^{\infty}\frac{\left(-\log\alpha_1\right)^k}{k!}\left\{\alpha_2\sum_{m=0}^{\infty}\frac{\left(-\log\alpha_2\right)^m}{m!\ \left(\beta_1+\beta_1k+\beta_2m\right)}-\left(\frac{1}{\beta_1+\beta_1k}\right)\right\}. \qquad (33)$$

2.9. Order statistics

Let $X_1,X_2,...,X_n$ be random sample, and let $X_{k:n}$ denotes that i^{th} order statistic, then the pdf of $X_{k:n}$ is given by

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}F(x)^{k-1}f(x)\left(1-F(x)\right)^{n-k}.$$

We can rewrite the above equation as follows

$$f_{k:n}(x)=\frac{1}{B\left(k,n-k+1\right)}F(x)^{k-1}f(x)\left(1-F(x)\right)^{n-k}.$$

Substituting (10) and (11), we can write

$$f_{k:n}(x)=\frac{1}{B\left(k,n-k+1\right)}\left(\frac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}{\alpha-1}\right)^{k-1}\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}\right)\left(1-\left(\frac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)}-1}{\alpha-1}\right)\right)^{n-k}.$$

By using the binomial expansion, we obtain

$$f_{k:n}(x)=\frac{\beta\left(\log\alpha\right)}{B\left(k,n-k+1\right)\lambda\left(\alpha-1\right)^n}\sum_{j=0}^{n-k}\sum_{r=0}^{k-1}\frac{\left(-1\right)^{-r+k+j-1}}{\alpha^{k-r-n-1}}\binom{k-1}{r}\binom{n-k}{j}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\left(\alpha^{-\left(1+\frac{x}{\lambda}\right)^{-\beta}}\right)^{r+j+1}. \qquad (34)$$

3. Parameter Estimation

The population parameters of the APTPL distribution can be estimated using maximum likelihood, least squares, and weighted least squares methods of estimation.

3.1. Maximum likelihood estimation (MLE)

Let $X_1, X_2, ..., X_n$ be random samples from the APTLx distribution with unknown parameters α, β, λ then the likelihood function is given by

$$L(\alpha,\beta,\lambda) = \left(\frac{\log\alpha}{\alpha-1}\right)^n \left(\frac{\beta}{\lambda}\right)^n \alpha^{n-\sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} \prod_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-(\beta+1)}.$$

Taking log on both sides we get the log likelihood function,

$$l(\alpha,\beta,\lambda) = \log\left(L(x,\alpha,\beta,\lambda)\right)$$

$$= n\log\left(\frac{\log\alpha}{\alpha-1}\right) + n\log\beta - n\log\lambda - (\beta+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) + n\log\alpha - \sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\log\alpha. \tag{35}$$

Therefore, to obtain the MLEs of α, β and $\lambda,$ we find the first order partial derivatives of (35) with respect to the parameters and equate them to zero;

$$\frac{\partial l}{\partial\alpha} = \frac{n(\alpha-1-\alpha\log\alpha)}{\alpha(\alpha-1)\log\alpha} + \frac{n}{\alpha} - \frac{\sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}{\alpha} = 0 \tag{36}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) + \sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)\left(1+\frac{x_i}{\lambda}\right)^{\beta}\log\alpha = 0 \tag{37}$$

$$\frac{\partial l}{\partial\lambda} = -\frac{n}{\lambda} + (\beta+1)\sum_{i=1}^{n}\frac{x_i}{\lambda(\lambda+x_i)} - \sum_{i=1}^{n}\frac{\beta x_i(\lambda+x_i)^{-(\beta+1)}}{\lambda^{-\beta+1}}\log\alpha = 0 \tag{38}$$

Then, the maximum likelihood estimations of the parameters α, λ and β can be obtained by solving system of (36)-(38). Fisher information I_{ij} matrix for APTLx distribution given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix},$$

$$I_{11} = E\left[-\frac{\partial^2\log L}{\partial\alpha^2}\right], I_{22} = E\left[-\frac{\partial^2\log L}{\partial\beta^2}\right], I_{33} = E\left[-\frac{\partial^2\log L}{\partial\lambda^2}\right],$$

$$I_{12} = I_{21} = E\left[-\frac{\partial^2\log L}{\partial\alpha\,\partial\beta}\right], I_{13} = I_{31} = E\left[-\frac{\partial^2\log L}{\partial\alpha\,\partial\lambda}\right], I_{23} = I_{32} = E\left[-\frac{\partial^2\log L}{\partial\beta\,\partial\lambda}\right],$$

$$\frac{\partial^2\log L}{\partial\alpha^2} = \frac{-n\alpha(\alpha-1)(\log\alpha)^2 - n(\alpha-1-\alpha\log\alpha)(\alpha+2\alpha\log\alpha-1-\log\alpha)}{\left(\alpha(\alpha-1)\log\alpha\right)^2} - \frac{n}{\alpha^2} + \frac{\sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}{\alpha^2},$$

$$\frac{\partial^2\log L}{\partial\beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\left[\log\left(1+\frac{x_i}{\lambda}\right)\right]^2 .\log\alpha,$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{n}{\lambda^2} - (\beta+1)\sum_{i=1}^{n} \frac{x_i\left[\lambda(1+x_i)+(\lambda+x_i)\right]}{\left[\lambda(\lambda+x_i)\right]^2}$$
$$-\sum_{i=1}^{n} \frac{\left[\lambda^{-\beta+1}\beta x_i(-\beta-1)(\lambda+x_i)^{-\beta-2} - \beta x_i(\lambda+x_i)^{-\beta-1}(-\beta+1)\lambda^{-\beta}\right]}{\left[\lambda(\lambda+x_i)\right]^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha\, \partial \beta} = -\frac{\sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta} \log\left(1+\frac{x_i}{\lambda}\right)}{\alpha},$$

$$\frac{\partial^2 \log L}{\partial \alpha\, \partial \lambda} = -\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^{-\beta-1}\left(\frac{x_i(\lambda-1)}{\lambda^2}\right),$$

$$\frac{\partial^2 \log L}{\partial \beta\, \partial \lambda} = \sum_{i=1}^{n}\left(\frac{x_i}{\lambda(\lambda+x_i)}\right)$$
$$-\frac{\sum_{i=1}^{n}\lambda^{-\beta+1}\left[\beta x_i(\lambda+x_i)^{-\beta-1}\log(\lambda+x_i)+(\lambda+x_i)^{-\beta-1}.x_i\right]-\beta x_i(\lambda+x_i)^{-\beta-1}\lambda^{-\beta+1}\log\lambda}{\left(\lambda^{-\beta+1}\right)^2}.$$

So, we obtain the asymptotic $100(1-\alpha)\%$ confidence intervals of the unknown parameters can be easily obtained for α, β and λ as given in equation below

$$\alpha \in \left[\hat{\alpha} - z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}, \hat{\alpha} + z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}\right],$$

$$\beta \in \left[\hat{\beta} - z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}, \hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}\right],$$

$$\lambda \in \left[\hat{\lambda} - z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}, \hat{\lambda} + z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}\right],$$

where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ quantile of the standard normal distribution.

3.2. Least squares and weighted least squares estimator

Let $X_1, X_2, ..., X_n$ be a random sample of size n from APTLx distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denotes the order statistics. Then,

$$E\left(F(X_{(i)})\right) = \frac{i}{n+1} \text{ and } V\left(F(X_{(i)})\right) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

1) Least squares estimator

We obtain the estimators by minimizing $\sum_{i=1}^{n}\left[F\left(X_{(i)}\right) - E\left(F\left(X_{(i)}\right)\right)\right]^2$, with respect to the unknown parameters. The least squares estimators of the APTLx distribution are obtained by minimizing following quantity with respect to α, β and λ,

$$\sum_{i=1}^{n}\left[\frac{\alpha^{\left(1-\left(1+\frac{x_i}{\lambda}\right)\right)}-1}{\alpha-1}-\frac{i}{n+1}\right]^2. \tag{39}$$

2) Weighted least squares estimator

The weighted least squares estimator of APTLx distribution can be obtained by minimizing the following $\sum_{i=1}^{n} w_i\left[F\left(X_{(i)}\right)-E\left(F\left(X_{(i)}\right)\right)\right]^2$, with respect to unknown parameters, where

$$w_i=\frac{1}{V\left(F\left(X_{(i)}\right)\right)}=\frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Which implies that,

$$\sum_{i=1}^{n}\frac{1}{V\left(F\left(X_{(i)}\right)\right)}\left[\frac{\alpha^{\left(1-\left(1+\frac{x_i}{\lambda}\right)\right)}-1}{\alpha-1}-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{\alpha^{\left(1-\left(1+\frac{x_i}{\lambda}\right)\right)}-1}{\alpha-1}-\frac{i}{n+1}\right]^2. \tag{40}$$

3.3. Monte Carlo simulation for APTLx distribution

In this section, we perform a simulation study to assess the performance and examine the mean estimate, average bias, root mean square error of the maximum likelihood estimators and obtained the confidence interval for each parameter. We study the performance of MLE of the APTLx distribution by conducting various simulations for different sample sizes and different parameter values. Quantile function is used to generate random data from the APTLx distribution. The simulation study is repeated for $N=1000$ times each with sample size $n=20,50,75,100$ and parameter values, case I: $(\alpha=0.3,\beta=0.2,\lambda=0.8)$ and case II: $(\alpha=1.5,\beta=1.6,\lambda=0.4)$. Five quantities are computed in this simulation study,

a) Mean estimate of the MLE $\hat{\vartheta}$ of the parameter $\vartheta=\alpha,\beta,\lambda$ which is $\frac{1}{N}\sum_{i=1}^{N}\hat{\vartheta}_i$,

b) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta=\alpha,\beta,\lambda$ which is $\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta}_i-\vartheta)$,

c) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta=\alpha,\beta,\lambda$ which is $\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta}_i-\vartheta)^2}$,

d) Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta=\alpha,\beta,\lambda$, i.e., the percentage of intervals that contain the true value of parameter ϑ.

e) Average width (AW) of 95% confidence intervals of the parameter $\vartheta=\alpha,\beta,\lambda$.

Table 1 presents the average bias, RMSE, CP and AW values of the parameters α,β,λ for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay toward zero. The average biases for the parameter α are all positive and slightly larger for small to moderate sample sizes but tend to get smaller as the sample size n increases. We also observe that for all the parametric values, the biases decrease as the sample size n increases. Also, the table shows that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases. Hence the ML estimates of APTLx distribution are consistent and efficient.

Table 1 Monte Carlo simulation results: average bias, RMSE, CP and AW

Parameter	n	Case I: $(\alpha=0.3,\beta=0.2,\lambda=0.8)$					Case II: $(\alpha=1.5,\beta=1.6,\lambda=0.4)$				
		Mean	AB	RMSE	CP	AW	Mean	AB	RMSE	CP	AW
α	25	2.2904	1.9904	5.8283	0.879	37.9046	3.9189	2.4189	10.2275	0.753	64.8764
	50	1.7892	1.4892	5.6869	0.836	24.4469	3.7302	2.2302	8.0649	0.748	49.7192
	75	1.0736	0.7736	2.8841	0.857	12.0661	3.4364	1.9364	7.3856	0.732	42.0305
	100	0.9528	0.6528	2.2517	0.890	9.4323	3.0993	1.5993	6.0635	0.704	37.1034
β	25	0.2333	0.0333	0.1077	0.929	0.4909	0.6683	0.0683	0.5988	0.982	1.9935
	50	0.2122	0.0122	0.0817	0.921	0.3569	0.5948	−0.0052	0.2812	0.970	0.9152
	75	0.2108	0.0108	0.0739	0.946	0.3078	0.5649	−0.0351	0.1666	0.969	0.6809
	100	0.2112	0.0112	0.0671	0.944	0.2703	0.5545	-0.0455	0.1482	0.962	0.6130
λ	25	1.0180	0.2180	1.0263	0.873	4.1924	0.9357	0.5357	2.1877	0.968	8.3538
	50	0.8707	0.0707	0.6613	0.902	2.5161	0.6541	0.2541	0.8533	0.956	3.3671
	75	0.8003	0.0003	0.4675	0.922	1.9218	0.5964	0.1964	0.4906	0.963	2.4774
	100	0.7852	−0.0148	0.3974	0.928	1.6512	0.5911	0.1911	0.4649	0.949	2.1337

4. Applications

In this section, we consider a data used by Lee and Wang (2003) in their paper corresponding to the remission times of a random sample of the 128 patients who are affected by bladder cancer. The data are as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23,3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76,26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

We have fitted the alpha power transformed Lomax (APTLx) distribution to the data using MLE, and APTLx distribution is compared with Lomax, KW Lomax, exponential Lomax, G-Lomax, transmuted exponentiated Lomax, WLomax, extended Poisson Lomax. The model selection is carried out by using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan Quinn information criterion):

$$AIC=-2\log L(\hat{\theta})+2q,\quad BIC=-2\log L(\hat{\theta})+q\log(n),$$

$$CAIC=-2\log L(\hat{\theta})+\frac{2qn}{(n-q-1)},\quad HQIC=-2L(\hat{\theta})+2q\log(\log(n)),$$

where $L(\hat{\theta})$ denotes the log-likelihood function evaluated at the MLEs, q is the number of parameters, and n is the sample size. Here, θ denotes the parameters $\theta=\alpha,\beta,\lambda.$ An iterative procedure is applied to solve the equations (36), (37) and (38) and we obtain,

$\hat{\theta} = (\hat{\alpha} = 28.5412,\ \hat{\beta} = 2.873798,\ \hat{\lambda} = 8.271523)$. The model with minimum AIC (or BIC, CAIC) values is chosen as the best model to fit the data.

Table 2 MLEs and the measures of AIC, BIC, HQIC, CAIC

Distribution	Estimates	$-\log L$	AIC	BIC	HQIC	CAIC
Lomax	$\hat{\alpha}$ =13.9384 $\hat{\lambda}$ =121.023	413.84	831.68	837.38	833.99	831.78
KW Lomax	$\hat{\alpha}$ =0.3911 $\hat{\lambda}$ =12.2973 $\hat{a}$ =1.5162 $\hat{\eta}$ =11.0323	409.94	827.88	839.29	832.52	828.21
Exp.Lomax	$\hat{\alpha}$ =1.0644 $\hat{\beta}$ =0.0800 $\hat{\lambda}$ =0.0060	414.97	835.94	844.49	839.42	836.13
G-Lomax	$\hat{\alpha}$ =4.7540 $\hat{\beta}$ =20.581 $\hat{a}$ =1.5858	410.08	826.16	834.72	829.64	826.35
TE-Lomax	$\hat{\alpha}$ =1.71418 $\hat{\gamma}$ =0.05456 $\hat{\lambda}$ =0.24401 $\hat{\theta}$ =3.33911	410.43	828.86	840.27	833.51	829.19
WLomax	$\hat{\alpha}$ =0.25661 $\hat{\beta}$ =1.57945 $\hat{a}$ =2.42151 $\hat{b}$ =1.86389	410.81	829.62	841.03	834.26	829.95
Power Lomax	$\hat{\alpha}$ =2.07012 $\hat{\beta}$ =1.4276 $\hat{\lambda}$ =34.8626	409.74	825.48	834.04	828.96	825.67
APTLx	$\hat{\alpha}$ =28.5412 $\hat{\beta}$ =2.87379 $\hat{\lambda}$ =8.27152	409.39	824.78	833.34	828.26	824.97

From the Table 2, we conclude that the alpha power transformation of Lomax (APTLx) distribution is best when compared to Lomax, KW Lomax, exponential Lomax (Exp.Lomax), G-Lomax, transmuted exponential Lomax (TE- Lomax), WLomax and power Lomax distributions.

For an ordered sample, from $APTL(\alpha,\beta,\lambda)$, where the parameters α,β and λ are unknown, the Kolmogorov-Smirnov D_n, Cramér-von Mises W_n^2, Anderson and Darling A_n^2 tests statistics are given as follows:

$$D_n = \max_i \left(\frac{i}{n} - F\left(x_i,\hat{\alpha},\hat{\beta},\hat{\lambda}\right), F\left(x_i,\hat{\alpha},\hat{\beta},\hat{\lambda}\right) - \frac{i-1}{n} \right),$$

$$A_n^2 = -n - \sum_{i=1}^{n} \frac{2i-1}{n}\left(\ln F\left(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right) - \ln F\left(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)\right), \tag{41}$$

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{2i-1}{2n} - F\left(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)\right)^2.$$

Table 3 Test statistics for the goodness-of-fit tests

Distribution	D_n	W_n^2	A_n^2
Lomax	0.096669	0.21258940	1.374568
KW Lomax	0.038908	0.02295290	0.159531
Exp.Lomax	0.076702	0.17967690	1.090800
G-Lomax	0.040639	0.02619050	0.180890
TE-Lomax	0.039910	0.03143840	0.227535
WLomax	0.041403	0.03829510	0.262735
Power Lomax	0.035055	0.01754725	0.120466
APTLx	0.028119	0.01339500	0.083468

Table 3 indicates that the test statistics D_n, W_n^2 and A_n^2 have the smallest values for the data set under alpha power transformation of Lomax distribution model with regard to the other models. The APTLx distribution approximately provides an adequate fit for the data.

5. Conclusion

In this paper, a new three parameter distribution is proposed called APTLx distribution based on alpha power transformation. The aim of this study is to bring more flexibility to the distribution. Various mathematical properties such as moments, moment generating function, quantile function etc. are discussed. The maximum likelihood estimation is used to estimate the model parameters. The usefulness of the proposed model is illustrated by means of real-life data set consists 128 bladder cancer patients, whereby it is shown that APTLx distribution gives a better fit than other competitive models. We hope that the new model will be useful for wider application in several areas.

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