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The Effects of Sampling from Finite Populations in a Mixed Effects Gage R&R Study

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Abstract

The goal of a gage repeatability and reproducibility (R&R) study is to determine how process variables and variability in the measurement process contribute to the observed variation in a measured product property. In an ANOVA for analyzing R&R data, it is assumed that the random effects (such as part-to-part variation) represent a random sample from a normal distribution. This, however, will not be the case in many applications, for example, when the random effects are sampled from a finite population of effects, such as sampling from a finite batch of available parts. In this paper, we examine the power of hypothesis tests in these R&R studies when sampling random effects from finite populations in a mixed effects two-factor factorial design. Populations of random effects are simulated from four finite distributions (two symmetric, two skewed), as well as from a normal distribution, for varying random effect population sizes, numbers of measurements per part, sampling fractions, interaction variances, and the error variances. The simulation results indicate that the power of F-tests is generally larger when random effects in an R&R study were sampled from any of the finite populations than under the traditional assumption that the random effects are normally distributed. However, exceptions may occur when the number of sampled effects is small, i.e., the sampling fraction (SF) is small. When the SF is large, the power is always larger when random effects are sampled from a finite distribution than under the normal distribution assumption. Thus, the SF can influence the power of hypothesis tests and ANOVA conclusions in a gage R&R study.

Keywords: Power, repeatability, reproducibility, sampling fraction, two-factor factorial design, variance components.

1. Introduction

One goal of manufacturing and process industries is continual improvement of a process and the resulting product quality. Through estimation and subsequent reduction of sources of process variation, this goal can be achieved. Using data from a designed experiment in a gage R&R study is a common method for performing tests of significance and providing estimates for sources of variation due to product or process variation as well as variation in the measurement process. That is, total variation is a combination of product or process variation and measurement error. A primary component of measurement error could be due to the capability of gages where gages are the measuring devices used to take measurements in a process. An improperly functioning gage will lead to less precise measurements of the true product properties which, in turn, may cloud assessment of the natural product variation.

For this research, a measurement system is defined as the gages and the procedure used to take measurements. A gage study can help identify areas of improvement in the measurement process and to ensure process signals are not obscured by noise introduced by the measurement gage. A gage repeatability and reproducibility (R&R) study can be designed to (i) estimate the total variability inherent in the measurement process; (ii) provide information on the magnitude and component sources of measurement error; (iii) estimate the closeness of the measurements to a target or true value; and (iv) determine the adequacy of the gage for use in the measurement process (Montgomery 2013). A good gage R&R study can be used to estimate the variability of the measurement process which can then be used to separate the actual process variability from the measured variability. Ultimately, the goal is to reduce the variability of the measurement process so that the difference between process variability and total variability is negligible.

The precision of a gage is denoted by the variance component σ^2_{gage} which can be separated into two components: repeatability and reproducibility. Reproducibility is the variability that results from using the measurement system (same gages) to make measurements under different operating conditions of normal use (e.g., different operators, different times or shifts, etc.). Reproducibility, denoted with variance component σ^2_{repro}, is the long-term variability that captures the changes in operating conditions. Repeatability, denoted by σ^2_{repeat}, is the variability in the measurement system. Repeatability is interpreted as the short-term variability that occurs under identical operating conditions. Both repeatability and reproducibility variance components can be estimated using a designed experiment where the potential sources of variation are changed systematically.

One of the most commonly-used gage R&R studies has a two-factor factorial structure. The protocol for conducting this study is the following: Take a sample of b items and a sample of a operating conditions. Then, for each of the $b \times a$ operating conditions, take n measurements on each item using the same gage. All other factors are held constant. In this research, we will adopt the commonly-used terminology that refers to the sampled items generically as ‘parts’ and the sampled operating conditions as ‘operators’.

The following is an example of a gage R&R study found in Montgomery (2013, 2017) from a R&R study conducted by Houf and Berman (1988). In this experiment, a random sample of $b = 10$ parts are taken. Each part was measured $n = 3$ times by $a = 3$ three operators where a ‘part’ is a power module for an induction motor starter, the ‘operators’ are inspectors, and the recorded measurements are on thermal impedance (in degrees C per watt×100). The study produced the following data

Part	Inspector 1			Inspector 2			Inspector 3		
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

For a review of gage R&R studies, see Montgomery et al. (1993a, b), Borror et al. (1997), Vardeman and VanValkenburg (1999), and Burdick et al. (2003, 2005a, b).

1.1. Mixed effect model

When designing an R&R study like this, the researcher must consider the suitability of the data in relation to a statistical analysis and the factors that are to be studied so that statistically valid conclusions can be drawn. Data from a factorial design (like in this R&R study) are typically analyzed using a multifactor ANOVA model associated with a random or a mixed effects model. ANOVA can be used to test for differences in fixed effects and to estimate variance components in models with random effects. In this research, the linear mixed effects model for response y can be written as:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \quad (1)$$

where μ is a baseline mean, τ_i is the i^{th} fixed effect of factor A, and β_j is the j^{th} random effect of factor B, respectively. The $(\tau\beta)_{ij}$ effect is the interaction effect for the combination of the i^{th} level of factor A and j^{th} level of factor B, and ε_{ijk} is a normal $N(0,\sigma^2)$ random error for the k^{th} observation from the $(i,j)^{\text{th}}$ levels of factors A and B. For example, in the inspector/part impedance gage R&R study, A is the inspector factor with fixed τ_i effects, B is the random part factor with variance component σ_P^2, A×B is the random interaction with variance component $\sigma_{I\times P}^2$, and ε represents a measurement error collected under identical operator/part conditions.

In the ANOVA, three F-tests are performed which correspond to the following null and alternative hypotheses for fixed operator (τ) effects and random part (β) and interaction $(\tau\beta)$ effects:

$$H_0 : \tau_i = 0 \text{ for all } i \qquad H_1 : \tau_i \neq 0 \text{ for some } i$$

$$H_0 : \sigma_P^2 = 0 \qquad H_1 : \sigma_P^2 > 0$$

$$H_0 : \sigma_{I\times P}^2 = 0 \qquad H_1 : \sigma_{I\times P}^2 > 0$$

For the operator and part impedance data, an ANOVA produced the following results:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F-statistic	p-value
Across Inspectors (I)	39.2667	2	19.6333	7.28	0.0048
Part-to-part (P)	3935.9556	9	437.3284	162.27	< 0.0001
Part/Operator Interaction ($I\times P$)	48.5111	18	2.6951	5.27	0.0001
Measurement Error (M)	30.6667	60	0.5111		
Total	4054.4	89			

with variance component estimates $\hat{\sigma}_P^2 = 48.29$, $\hat{\sigma}_{I\times P}^2 = 0.73$ and $\hat{\sigma}_M^2 = 0.51$, and estimates of the baseline mean $\hat{\mu} = 35.8$ and fixed inspector effects $\tau_1 = -0.9$, $\tau_1 = 0.67$, and $\tau_3 = 0.23$. Recall: for a mixed effects model, the mean square for the interaction is used in the denominator of the F-tests or operators and parts.

1.2. Model assumptions

Like any statistical procedure, there are assumption when using ANOVA to analyze data from a factorial experiment in a gage R&R study. One fundamental assumption involves normality of errors. In context, consider another example. Suppose a manufacturer suspects that the batches of raw material furnished by the supplier differ significantly in calcium content, and there are a large number of batches currently in the warehouse. Five of these are randomly selected for study from the population of available warehouse batches. It is typically assumed that the batch population size is large relative to the number of sampled batches to the extent that the population is treated as infinite, and, in particular, normally distributed, in the ANOVA (Graybill 1961, Searle 1971, Sahai and Ojeda 2005, Searle et al. 2006). In many cases, this will not be the truth. For example, if there are only 50 batches of raw material available and five batches were selected, then the population of batches in this study is finite and is not large relative to the number sampled. It is natural to ask what is the effect on an ANOVA if the researcher uses 5 of 50 batches (10%)? How does sampling from this finite population of batches affect the hypothesis tests for variance components. Specifically, what happens if we do not include a finite population correction in an ANOVA and just assume the sample comes from a normal distribution? These same questions can be asked of the data from the thermal impedance R&R study. Specifically, how large was the population of parts from which the sample of 10 parts was selected? What is the effect on the ANOVA F-tests and associated p-values if a finite population correction is ignored?

Many researchers have studied variance component estimation. Bennett and Franklin (1954), Cornfield and Tukey (1956), and Tukey (1956) studied variance component estimation for balanced designs under the assumptions of independence and normality, and obtained formulas for estimating variance components. Estimation for unbalanced designs was considered in Tukey (1957) and Searle (1961). Subsequently, Hartley (1967) developed a general procedure for determining the numerical values of the coefficients in formulas for expected mean squares (EMS) in random and mixed effects models for one-way and two-way classifications with unequal replicates when sampling from normal populations of random effects. It was useful to obtain closed-form EMS formulas with numerical coefficients which can then be used to give the variance and covariance formulas for mean squares, and associated ANOVA F-tests.

Various researchers have also studied the effect of finite population sampling in variance component models. When sampling effects from a finite population, a finite population correction (FPC) would be needed when estimating variance components. Gaylor and Hartwell (1969) considered a single unified procedure for obtaining the EMS for finite population sampling for multi-stage nested designs for both balanced and unbalanced random sampling from finite and infinite populations (e.g., normal) for each design factor classification. Searle and Fawcett (1970) studied the EMS in variance component models with random effects which are assumed to be sampled from finite populations. They developed rules for converting expectations under infinite population models for random effects to expectations under finite population models for random effects. In addition, the rules can be applied to balanced and unbalanced designs, and used for nested and crossed-factor classifications when it is assumed the population of levels of each factor are finite. What is curious is that they also assumed that the population of error terms was also finite which differs from the tradition linear models assumption of random normal errors. When sampling from a finite population, the variance components in EMS formulas assuming infinite population or continuous distribution sampling are simply replaced with variance components adjusted with a finite population correction (FPC). Simmachan et al. (2012) determined the EMS for a one-way ANOVA model assuming sampling from a finite population of effects and with normally distributed errors. An early and thorough review of sampling from finite populations can be found in the textbook by Bennett and Franklin (1954). Unfortunately, despite this early research on sampling from finite populations, no discussion that addresses this issue can be found in recent published statistical literature, as well as in current textbooks in experimental design and linear models. And, specifically, there does not appear that recent discussion exists that is related to gage R&R studies with finite population sampling in engineering and industrial applications.

In this research, Section 2 contains the sampling methodology for the simulation study, and the results of this research are presented in Section 3. A discussion appears in Section 4 and conclusions are summarized in Section 5.

2. Materials and Methods

In this section we will introduce methodology related to incorporating finite population corrections into analyses that are part of a gage R&R simulation study.

2.1. Finite population corrections

A population represents all objects or individuals of interest. For example, a population could the available batches of raw materials, machines in a factory, or patients in a hospital. In terms of size, populations can be classified as either finite or infinite. In this research, each part sampled from a population of interest has a corresponding model effect. If the model effects result from sampling a continuous random variable, then we say we have a sample from an "infinite" population, such as a normal distribution. However, if the model effects result from randomly sampling without replacement from a population of N effects, then we say we have a sample from a "finite" population.

In an introductory statistics course, $Var(\overline{y}) = \sigma^2 / n$ is given as the variance of the sample mean where σ^2 is the population or distribution variance, n is the sample size, and it assumed that simple random sampling was used to collect the response (y) values. This formula is correct if a random sample is taken from an infinite population or a continuous distribution, and is appropriate to use when sampling from a very large population of size N for which the sampling fraction $\text{SF} = n / N$ is negligible. However, it often occurs that samples are taken from finite populations which are not large and n / N is not small enough to be ignored. In these cases, variance formulas have to be adjusted by a factor $(N-n)/N$ known as the finite population correction or FPC. For additional information on the use of FPCs when sampling from finite populations, see Kish (1965), Hedayat and Sinha (1991), Levy and Lemeshow (1991), and Lohr (2010). Thus, for a simple random sample (SRS) taken from a finite population without replacement, the variance of the sample mean $\bar{y}$ is

$$Var(\bar{y}_{SRS}) = \text{FPC} \times \frac{\sigma^2}{n} = \left(\frac{N-n}{N}\right)\frac{\sigma^2}{n} = \left(1-\frac{n}{N}\right)\frac{\sigma^2}{n} \tag{2}$$

with standard error $\text{s.e.}(\bar{y}_{SRS}) = \sqrt{Var(\bar{y}_{SRS})}$.

If the sampling fraction (SF) is small, then the FPC will be close to 1 ($\text{FPC} \approx 1$), in which case including a finite population size N has essentially little to no effect on the variance of $\bar{y}_{SRS}$ in (2). In practice, it has been suggested that the FPC can be ignored when the sampling fraction does not exceed 5% while others propose the FPC can be ignored if it is as high as 10% (Cochran 1977). Consequently, it will be useful to know what is the impact of ignoring the FPCs when using ANOVA methods in a gage R&R study with random effects.

In this research, we examine the impact on ANOVA due to random sampling β_i effects from a finite population for model (1) and the FPC is ignored. Population sizes and types of random effects are varied for different types of finite populations and for a varying sampling fraction. Moreover, we will compare the power of ANOVA F-tests when the random effects are sampled from a finite population to effects that are randomly sampling from a normal distribution.

2.2. Methodology and scope of the gage R&R simulation study

Distributions for four finite populations of random parts are considered. These distributions are a discrete uniform distribution (UNI) and three discrete triangular distributions which are defined as symmetric (SYM), extreme skewed right (ESR), and moderate skewed right (MSR). Simple random sampling without replacement will be used to sample model effects from these finite part populations. Finite populations of effects for part factor B will be denoted by G_β. This research will be restricted to two-factor factorial designs in the balanced (or equal replications) case as considered in Searle and Fawcett (1970).

Finite populations of random effects in the computer simulation will be generated for the following scenarios:

- For factor A, $a = 2$ or $a = 4$ are the number of fixed operator effects.
- For factor B, finite population sizes $N_B = 20$ and $N_B = 60$ parts are considered.
- The sampling fractions of parts $\left(\frac{b}{N_B}\right)$ from the finite population of parts were restricted to 10%, 20% and 50% of N_B. For $N_B = 60$, a sampling fraction of 5% was also considered.
- For each A×B operator/part combination is $n = 2$, 3, or 4 gage measurements.
- Let G_β be the (unscaled) finite population of N_B random β_j part effects.

1. For the UNI distribution, the N_B effects in G_β are uniformly spaced from $-\frac{N_B - 1}{2}$ to $\frac{N_B - 1}{2}$. Thus, the possible values of β_j are $\{-9.5, -8.5, ..., -0.5, 0.5, ..., 8.5, 9.5\}$ for $N_B = 20$ and $\{-29.5, -28.5, ..., -0.5, 0.5, ..., 28.5, 29.5\}$ for $N_B = 60$.

2. For the SYM, ESR, and MSR distributions, Table 1 contains a summary of the population of N_B (unscaled) β_j effect values and their frequencies.

3. Let $(\beta_1,\beta_2,...,\beta_b)$ be a SRS of size b selected from the N_B values in G_β and $(\tau_1,\tau_2,...,\tau_a)$ are a fixed effects. A SRS of interaction $(\tau\beta)_{ij}$ effects $(i=1,...,a,\ j=1,...,b)$ is selected from a discrete uniform distribution $G_{\tau\beta}$ with mean 0 and $\sigma^2_{\tau\beta}=k\sigma^2_\beta$ for $k=0$, 0.5, 1 and 2, $\sigma^2_\beta=$ 0, 0.5, 1 and 2. By definition, if $k=0$ then $(\tau\beta)_{ij}=0$ for all i, j.

These finite populations are centered so that each has mean 0. In the simulation, these effects will be scaled so that each population will have a variance that matches one considered for a normal distribution of effects. Let σ^2_β and $\sigma^2_{\tau\beta}$ be the variances of the scaled effects in finite G_β, and $G_{\tau\beta}$ populations, respectively. In the simulation study, MATLAB version 2017 was employed for the computation with 50,000 iterations taken for each simulated gage R&R data scenario.

We assume that the random error $\varepsilon_{ijk}\sim N(0,\sigma^2)$ in model (1). This differs from Searle and Fawcett (1970) who assumed the ε_{ijk} were from finite populations. We assume the random error is normal because the response is quantitative and not deterministic. That is, even if the same study design is replicated, the data that is collected will naturally vary due to uncontrollable random variation in experimental conditions (such as small changes in temperature, handling of materials, etc.). This is a critical distinction between our study and prior research which did not allow for variation due to the measurement process.

The power of ANOVA F-tests in a gage R&R study will depend on the magnitude of the measurement error variance. Thus, we allow σ^2 to vary in the simulation as a multiple of the part population variance σ^2_β. We consider $\sigma^2=R\sigma^2_\beta$ with $R=$ 0.25, 0.5, and 1. Because the power of a test is the probability of rejecting the null hypothesis when it is false, the power for each of the three F-tests can be estimated from the proportions of 50,000 simulation cases for which that null hypothesis was rejected.

In the normal distribution case, each F-statistic follows a non-central F-distribution with the same degrees of freedom as the central F but with a non-zero non-centrality parameter λ (Hocking 1984, Muller and Stewart 2006, Khuri 2010). For the finite populations, however, the distribution of the F-statistic is unknown. Thus, to assess the power of a hypothesis test assuming normality (even if that assumption is violated), a traditional ANOVA is still performed in the simulation. This can provide information regarding the robustness of ANOVA F-tests when sampling from finite distributions. Specifically, in the simulation power study, we compare the power when sampling β_j effects from a finite distribution with mean 0 and variance (σ^2_β) to the power when sampling $\beta_j\sim$ IID $N(0,\sigma^2_\beta)$. The steps of the simulation power study are:

Step 1: For each finite distribution G_β of N_β parts defined in the scope of the study, take SRSs of $b\beta_j$ effects from G_β and SRSs of $a\times b(\tau\beta)_{ij}$ effects from $G_{\tau\beta}$. Samples are taken without replacement.

Step 2: Sample n gage measurements $\varepsilon_{ijk}\in N(0,\sigma^2)$ for each $a\times b$ factor combination.

Step 3: Generate n responses for the mixed model $y_{ijk}=\mu+\tau_i+\beta_j+(\tau\beta)_{ij}+\varepsilon_{ijk}$ according to the conditions set for the τ_i fixed operator effects and the random β_j part and $(\tau\beta)_{ij}$ interaction effects.

Step 4: For each simulated data set of $a \times b \times n$ simulated responses, calculate the ANOVA F-statistics $(F_A = MS_A / MS_{AB},\ F_B = MS_B / MS_{AB}$, and $F_{AB} = MS_{AB} / MSE)$ for the three hypothesis tests, and determine if each null hypothesis would be rejected at nominal level $\alpha = 0.05$.

Step 5: Repeat Steps 1 to 4 for 50,000 iterations.

Step 6: Estimate the power for each F-test. Each power estimate is the proportion of the 50,000 tests for which the null hypothesis was correctly rejected.

Step 7: Repeat Steps 1 to 6, but instead take a SRS of effects $\beta_j \sim N(0, \sigma_\beta^2)$ and a SRS of effects $(\tau\beta)_{ij} \sim N(0, \sigma_{\tau\beta}^2)$.

Step 8: Compare the power of each test at Step 6 to the power of the same tests when sampling from normally distributed random effects at Step 7 having the same variance as those in G_β and $G_{\tau\beta}$. This provides an assessment of non-normal data sampled from finite populations on ANOVA conclusions. Specifically, assess if each null hypothesis is rejected more often, less often, or rejected at nearly the same rates when the normality assumption is violated.

3. Results

The simulation results of the power of ANOVA F-tests for each of the four finite distributions (UNI, SYM, ESR and MSR) are compared to each other and to the textbook case when the random effects are sampled from a normal population. We considered the relation between the power for different finite part population sizes (N_B), the number of gage measurements (n), the number of randomly selected parts b with σ_β^2, the interaction variance component $(\sigma_{\tau\beta}^2)$, and the coefficient multiplier R of the random error variance (σ^2).

The simulation study results are summarized in Tables 2 to 5. These tables list the estimated power of the F-tests for the a fixed operator effects, the b random part effects, and the ab random interaction A×B effects. The random part effects are sampled from a UNI, SYM, ESR or MSR finite population and the $a \times b$ random interaction effects are sampled from a UNI population. Random effects were also studied when sampling from a normal distribution.

The power results are shown in Tables 2 and 3 when b = 2, 4, and 10 part effects are sampled (i.e., for SF = 0.10, 0.20, and 0.50 for $N_B = 20$) with $a = 2$ and 4 fixed operator effects, four interaction variances $\sigma_{\tau\beta}^2 = k\sigma_\beta^2$ (k = 0, 0.5, 1 and 2), and three random error variance $\sigma^2 = R\sigma_\beta^2$ (R = 0.25, 0.5, and 1). Each table validates some results that we know should happen theoretically. That is, for each distribution from which the β_j are sampled, the power of F-tests for $H_0 : \sigma_\beta^2 = 0$ versus $H_1 : \sigma_\beta^2 > 0$:

(R1) increases as the number of gage measurements (n) increases from 2 to 4,

(R2) decreases when gage measurement variability σ^2 increases as the R multiplier for σ_β^2 increases from 0.25 to 1.

(R3) increases as the β_j sampling fraction SF of parts increases, and

(R4) decreases as interaction variability $\sigma_{\tau\beta}^2$ increases from 0 to $2\sigma_\beta^2$.

(R5) For the interaction $\sigma_{\tau\beta}^2$, results R1-R4 also apply to the power of F-tests for the $H_0 : \sigma_{\tau\beta}^2 = 0$ versus $H_1 : \sigma_{\tau\beta}^2 > .0$

Results (R1)-(R5) observed for a population of $N_B = 20$ parts also hold in Tables 4 and 5 for a population of $N_B = 60$ parts. For the fixed operator effects (Factor A), the power of the F-tests for fixed effects when $(\tau\beta)_{ij} \sim N(0,\sigma^2_{\tau\beta})$ are, in general, larger than or very close to the power of the F-tests when $(\tau\beta)_{ij} \sim G_{\tau\beta}(0,\sigma^2_{\tau\beta})$.

The important results and new research knowledge, however, are uncovered in the comparisons of the powers across distributions. Within Tables 6 to 9, we summarize the power of F-tests for σ^2 for fixed τ_i -values with respect to the best and the worst distributions where U = Uniform distribution, N = Normal distribution, D = all other finite (discrete) distributions. The $\approx$ symbol is used when powers are very close across all distributions, and the "-" sign means the powers of all distributions are close to 1.

Tables (6a) and (6b) correspond to fixed operator effects $(\tau_1,\tau_2) = (-1,1)$, when $a = 2$ and $N_B = 20.$ It can be seen that

- the uniform (UNI) distribution has highest power when $k = 0$ while the normal (N) distribution has the lowest. Note when $k = 0,$ all $(\tau\beta)_{ij} = 0.$ Thus, the F-test is equivalent to using the mean squared for measurement error as the denominator of each F-test instead of the interaction mean square.
- when $b = 2$ parts and multiplier k =0.5, 1, 2 (where $\sigma^2_\beta = k\sigma^2_{\tau\beta}$), the power for all distributions are similar.
- for $k = 1$ and 2, and $b = 4$ and 10 parts, the powers associated with the four finite distributions are similar, but are all less than the power assuming a normal (N) distribution.

Tables (7a) and (7b) correspond to fixed operator effects $(\tau_1,\tau_2,\tau_3,\tau_4) = (1,-1,1,-1),$ when $a = 4$ and $N_B = 20.$ The results indicate

- the uniform (UNI) distribution has the highest power when $k = 0$ and 0.5, while the normal (N) distribution has the lowest.
- for $k = 1,$ powers across all five distributions are similar for $b = 2.$ However, for $b = 4$ or 10, the uniform (UNI) distribution has the highest power and the normal (N) distribution has the lowest power.
- for $k = 2,$ the normal (N) distribution has the highest power for $b = 4,$ but is worst for $b = 10.$

Tables (8a) and (8b) correspond to fixed operator effects $(\tau_1,\tau_2) = (-1,1),$ when $a = 2$ and $N_B = 60.$ The results indicate

- for $k = 0,$ the uniform (UNI) distribution is best and the normal (N) distribution is worst for $b = 3$, 6, and 12.
- for $k = 1$ or 2 and $b = 6$, 12, and 30, the normal (N) distribution has the highest power and all the finite distributions have lowest power for five of the six cases. The normal (N) distribution is the worst in the other case.
- for $k = 0.5,$ no clear pattern is observed for $b = 3$ or 6. However, for $b = 12$ and 30, the uniform (UNI) distribution has the highest power while the normal (N) distribution has the lowest power.

Tables (9a) and (9b) correspond to fixed operator effects $(\tau_1,\tau_2,\tau_3,\tau_4) = (1,-1,1,-1),$ when $a = 4$ and $N_B = 60.$ The results indicate

• the uniform (UNI) distribution is best and the normal (N) distribution is worst for the majority of cases.

• Powers are similar for all five distributions for three cases with k is small and b is large.

• the only one case when the normal (N) distribution has the highest power is $b = 3$ and $k = 2$.

4. Discussion

From Tables 6 to 9, it can be seen that the pattern of the results in these tables are similar when we change n or R. Thus, the patterns do not depend on n or R. These results show that, when $(\tau_1, \tau_2) = (-1,1)$, the powers when sampling parts from a finite population will be greater than for the normal distribution when $k = 0$, and the normal distribution has highest power when k and b are large. For k =0.5, 1, or 2 and b is small, all of distributions are similar. For $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,-1,1,-1)$, the powers in a gage R&R study with a finite population are the best when $k = 0$, 0.5 or k is small and b is large, while the normal distribution has highest power when k is large and b is small.

Next, we graphically summarize in Figures 1 and 2, the power of tests for σ_β^2 part variation for the case when the interaction variance, $\sigma_{\tau\beta}^2 = \sigma_\beta^2$. Figure 1 contains plots of the power of the F-test for σ_β^2 when $N_B = 20$ with various choices of a, b, n for operators, parts, and gage measurements. For R = 0.25, 0.5, and 1, plots in (1a), (1b), and (1c) are for the symmetric distributions and plots in (1d), (1e) and (1f) are for the finite skewed distributions. Subplots are separated vertically into three levels of part factor B and horizontally into two levels of operator factor A with 2, 3 and 4 gage measurements. Figure 2 (like Figure 1) contains plots of the power of the F-test for factor B but with $N_B = 60$. The patterns appearing in the plots in Figures 1 and 2 visually summarize the numerical results in Tables 2 to 5, but with $n = 3$ added. These figures visually support all tabular results for $N_B = 20$ and $N_B = 60$.

5. Conclusions

The purpose of this research was to examine the possible impact in ANOVA when the effects are sampled from finite populations in a mixed-effects two-factor factorial design in a gage R&R study. Specifically, the primary goal was to compare the powers of the ANOVA F-tests when the random part effects are sampled from finite and from normal populations. Although the results of Section 3 support the expectations that power will increase as the number of gage measurements increases, the simulation also quantifies the magnitude of the change in power under the different scenarios. It also showed that as the coefficient of the variance of random effects increases, the powers of the F-tests gradually decrease.

For $(\tau\beta)_{ij}$ that are randomly selected from a finite uniform distribution, the power of the F-tests for the interaction variance $\sigma_{\tau\beta}^2$ is greater when sampling from a finite population than from the normal distribution when $k = 0$ or k is small and b is large, while the power is highest when sampling from the normal distribution when k is large and b is small. For the examples in this research, we discovered that the sampling fraction SF and multiplier k can impact the power of the F-tests in a gage R&R study. When the SF is large and k is small, the power is greater when sampling from a finite distribution than from the normal distribution, while if k is large, the power when sampling from a normal distribution is greater than when sampling from a finite distribution.

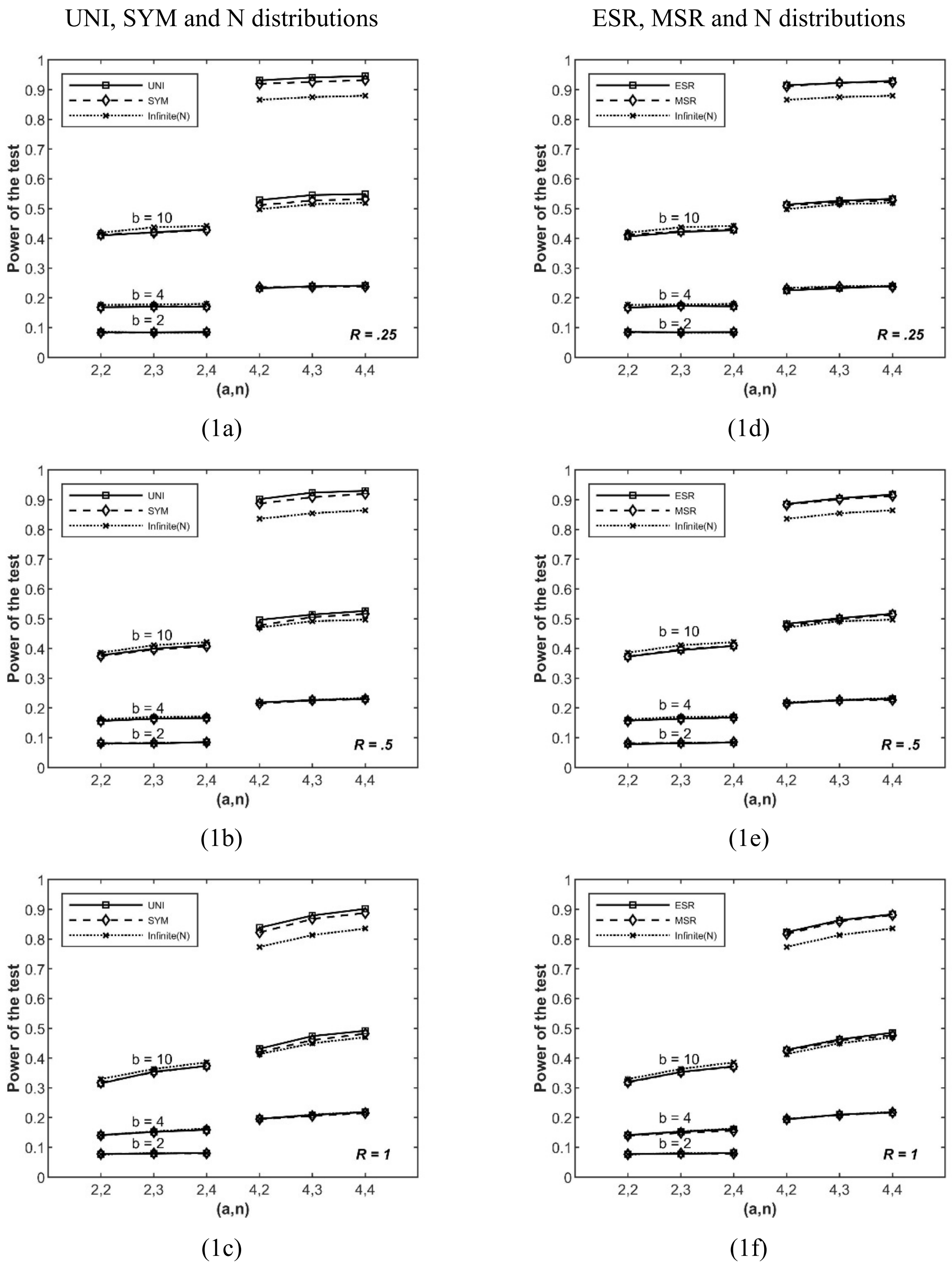

Figure 1 The comparison of the power of tests for factor B which contain the symmetric and skewed distributions at each level of SF (SF = 0.10, 0.20 and 0.50) with $N_B = 20$, $\sigma_{\tau\beta}^2 = \sigma_{\beta}^2$, and for $R = 0.25$, 0.5, and 1

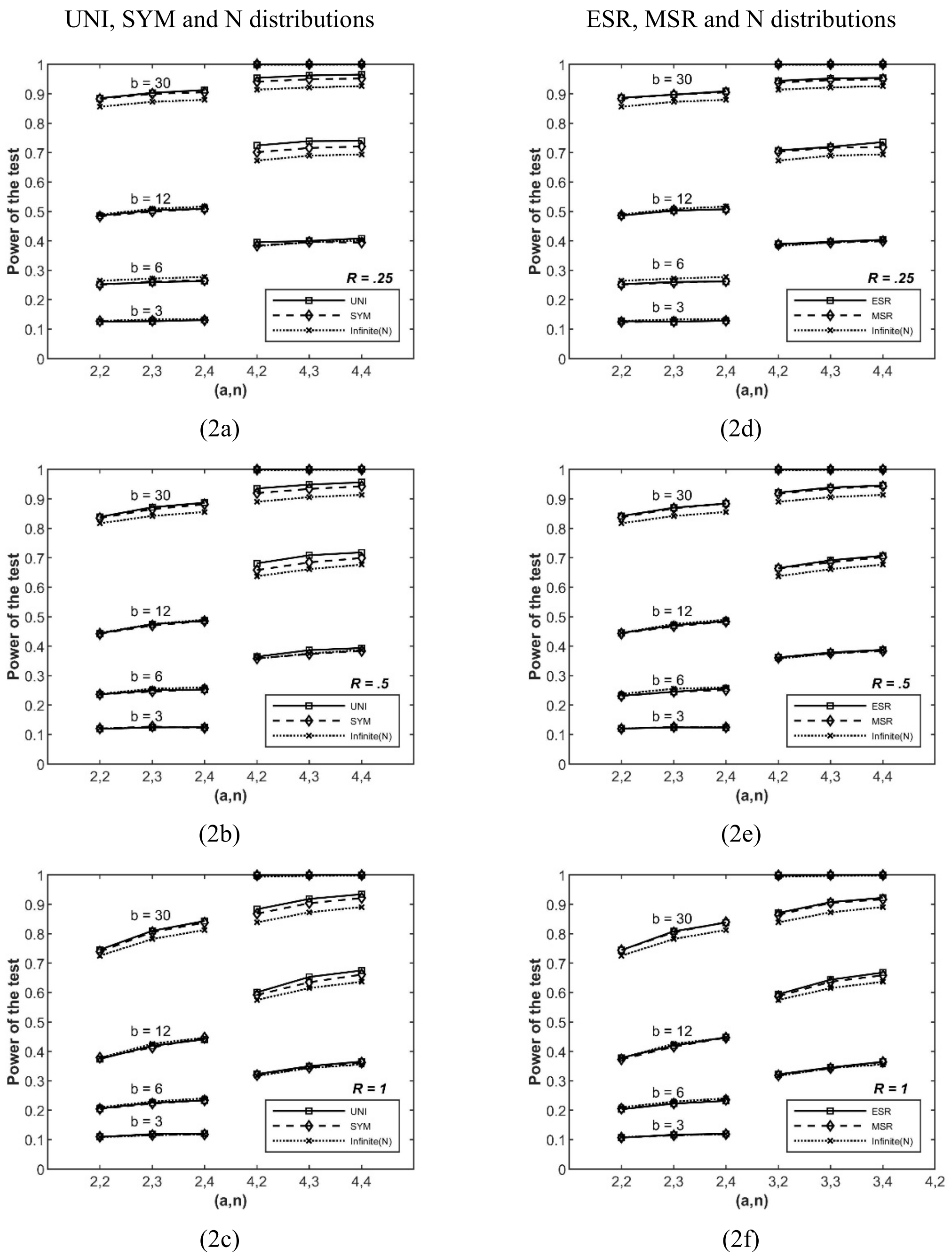

Figure 2 The comparison of the power of tests for factor B which contain the symmetric and skewed distributions at each level of SF (SF = 0.05, 0.10, 0.20 and 0.50) with $N_B = 60$, $\sigma^2_{\tau\beta} = \sigma^2_{\beta}$, and for $R = 0.25$, 0.5, and 1

Table 1 The frequencies of unscaled β_j values define the discrete triangular and the values of H make the means of the distributions equal zero

The population of β_j values	$N_B = 20$			$N_B = 60$		
	SYM	ESR	MSR	SYM	ESR	MSR
1-H	1	4	2	2	7	4
2-H	2	4	3	3	7	5
3-H	3	3	4	3	6	6
4-H	4	3	4	4	6	7
5-H	4	2	3	5	5	7
6-H	3	2	2	6	5	6
7-H	2	1	1	7	4	5
8-H	1	1	1	7	4	4
9-H				6	3	3
10-H				5	3	3
11-H				4	3	3
12-H				3	3	3
13-H				3	2	2
14-H				2	2	2
H	4.5	3.5	3.9	7.5	179/30	6.3

Consequently, under conditions related to the sampling fraction SF, the number of gage measurements n, the distribution of random part effects (discrete or normal), the size of the measurement error σ^2, the size of the interaction variance $\sigma^2_{\tau\beta}$, and the finite population variances σ^2_β, can influence conclusions regarding hypothesis testing when using ANOVA. In practical applications, we should consider whether or not it can be assumed that a normal population is appropriate when sampling random effects.

Finally, all of the results for the random effects in the mixed model R&R study can also be applied to a random effects study in which the operators are also considered random effects sampled from a finite population. That is, we can apply the results for testing $H_0 : \sigma^2_\beta = 0$ against $H_1 : \sigma^2_\beta > 0$ because the mean square for the interaction is also used for testing a random operator effects (i.e., $H_0 : \sigma^2_\tau = 0$ against $H_1 : \sigma^2_\tau > 0$).

Table 2 The powers of F-test for fixed $(\tau_1, \tau_2) = (-1,1)$, for $\beta_j \sim G_\beta(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim G_{\tau\beta}(0, \sigma_{\tau\beta}^2)$, and for $\beta_j \sim N(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0, \sigma_{\tau\beta}^2)$ with $\sigma^2 = R\sigma_\beta^2$, where $N_B = 20$

SF	Designs (a,b,n)	Factor	Dist	$\sigma_{\tau\beta}^2 = 0$			$\sigma_{\tau\beta}^2 = .5\sigma_\beta^2$			$\sigma_{\tau\beta}^2 = \sigma_\beta^2$			$\sigma_{\tau\beta}^2 = 2\sigma_\beta^2$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.10	(2,2,2)	B	N	.199	.149	.111	.101	.094	.087	.086	.079	.076	.071	.070	.067
			UNI	.207	.154	.112	.104	.096	.089	.084	.082	.077	.072	.070	.068
			SYM	.204	.151	.113	.100	.095	.086	.082	.082	.078	.069	.069	.068
			ESR	.207	.146	.112	.101	.095	.087	.085	.079	.077	.070	.069	.067
			MSR	.205	.151	.110	.102	.094	.086	.084	.082	.077	.070	.066	.067
		A	$\beta_j \sim$ N	.246	.177	.124	.113	.104	.090	.091	.085	.078	.072	.071	.070
			$\beta_j \sim G_\beta$	.246	.178	.124	.110	.101	.092	.089	.083	.079	.072	.071	.068
		AB	N	.049	.052	.051	.281	.185	.124	.407	.283	.184	.540	.407	.279
			$G_{\tau\beta}$	.052	.050	.051	.291	.187	.119	.419	.291	.187	.553	.416	.290
	(2,2,3)	B	N	.239	.175	.133	.103	.099	.090	.083	.083	.081	.072	.069	.067
			UNI	.251	.183	.138	.103	.101	.089	.084	.080	.079	.070	.071	.068
			SYM	.249	.181	.137	.103	.102	.092	.083	.082	.080	.069	.070	.070
			ESR	.245	.177	.133	.103	.098	.092	.084	.081	.078	.070	.071	.068
			MSR	.246	.177	.132	.105	.101	.090	.084	.082	.080	.071	.070	.070
		A	$\beta_j \sim$ N	.299	.212	.154	.115	.109	.099	.091	.088	.086	.073	.075	.070
			$\beta_j \sim G_\beta$	.305	.215	.155	.112	.109	.096	.088	.083	.083	.071	.071	.070
		AB	N	.051	.050	.051	.407	.284	.185	.540	.411	.282	.655	.539	.411
			$G_{\tau\beta}$	.049	.049	.051	.420	.289	.185	.555	.422	.293	.674	.556	.422
	(2,2,4)	B	N	.270	.202	.150	.105	.103	.095	.083	.084	.080	.071	.071	.071
			UNI	.286	.208	.154	.107	.103	.096	.086	.086	.081	.072	.071	.069
			SYM	.275	.207	.153	.106	.102	.093	.084	.084	.079	.071	.071	.068
			ESR	.274	.202	.150	.105	.101	.096	.085	.084	.081	.071	.069	.069
			MSR	.280	.205	.150	.106	.101	.097	.084	.085	.078	.071	.070	.070
		A	$\beta_j \sim$ N	.342	.249	.178	.121	.117	.105	.090	.088	.084	.074	.073	.072
			$\beta_j \sim G_\beta$	.344	.244	.176	.113	.110	.103	.089	.089	.085	.072	.072	.070
		AB	N	.051	.050	.050	.479	.347	.231	.608	.482	.349	.710	.608	.481
			$G_{\tau\beta}$	.049	.050	.050	.494	.361	.235	.620	.493	.358	.720	.620	.493
.20	(2,4,2)	B	N	.687	.489	.313	.266	.234	.188	.177	.161	.141	.115	.112	.103
			UNI	.745	.530	.326	.260	.230	.191	.168	.156	.141	.111	.110	.104
			SYM	.722	.509	.320	.256	.224	.186	.168	.157	.140	.112	.108	.107
			ESR	.718	.513	.319	.257	.227	.189	.167	.157	.141	.112	.109	.106
			MSR	.716	.511	.316	.257	.227	.183	.167	.157	.139	.115	.113	.105
		A	$\beta_j \sim$ N	.950	.754	.490	.413	.363	.289	.265	.241	.207	.164	.159	.148
			$\beta_j \sim G_\beta$	.950	.757	.490	.390	.344	.281	.245	.226	.205	.151	.149	.140
		AB	N	.051	.049	.051	.526	.325	.192	.724	.523	.325	.867	.723	.520
			$G_{\tau\beta}$	.050	.049	.050	.540	.333	.191	.748	.541	.330	.887	.749	.540
	(2,4,3)	B	N	.784	.605	.413	.280	.253	.218	.177	.170	.153	.117	.116	.109
			UNI	.843	.659	.439	.272	.248	.214	.170	.164	.152	.115	.114	.110
			SYM	.822	.640	.428	.268	.248	.216	.171	.164	.152	.115	.110	.109
			ESR	.813	.633	.429	.270	.247	.215	.173	.164	.153	.114	.110	.107
			MSR	.811	.633	.424	.265	.246	.214	.174	.164	.148	.117	.113	.108
		A	$\beta_j \sim$ N	.991	.889	.642	.437	.395	.336	.270	.256	.230	.167	.164	.154
			$\beta_j \sim G_\beta$	.990	.886	.649	.411	.373	.323	.249	.239	.220	.153	.155	.146
		AB	N	.049	.050	.051	.714	.509	.307	.860	.708	.509	.942	.858	.709
			$G_{\tau\beta}$	.051	.049	.049	.735	.524	.316	.880	.739	.521	.951	.877	.738
	(2,4,4)	B	N	.837	.685	.490	.288	.264	.235	.180	.171	.163	.118	.116	.112
			UNI	.892	.743	.525	.277	.259	.230	.170	.166	.159	.116	.113	.112
			SYM	.875	.724	.513	.276	.259	.228	.171	.166	.159	.115	.115	.110
			ESR	.868	.717	.508	.276	.255	.228	.171	.168	.161	.116	.115	.111
			MSR	.869	.714	.513	.275	.256	.226	.173	.167	.156	.119	.114	.111
		A	$\beta_j \sim$ N	.998	.951	.756	.448	.411	.364	.274	.262	.243	.170	.164	.160
			$\beta_j \sim G_\beta$	.998	.949	.753	.416	.392	.346	.250	.242	.228	.158	.153	.147
		AB	N	.050	.050	.050	.803	.624	.405	.910	.803	.619	.964	.911	.801
			$G_{\tau\beta}$	.051	.050	.050	.825	.645	.419	.926	.826	.646	.970	.926	.827

Table 2 (Continued)

SF	Designs (a,b,n)	Factor	Dist	$\sigma_{\tau\beta}^2=0$			$\sigma_{\tau\beta}^2=.5\sigma_{\beta}^2$			$\sigma_{\tau\beta}^2=\sigma_{\beta}^2$			$\sigma_{\tau\beta}^2=2\sigma_{\beta}^2$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.50	(2,10,2)	B	N	.990	.932	.746	.658	.583	.464	.419	.386	.329	.237	.221	.207
			UNI	.999	.975	.799	.694	.604	.476	.410	.377	.315	.222	.211	.192
			SYM	.998	.966	.790	.684	.600	.476	.413	.374	.318	.223	.212	.195
			ESR	.998	.965	.782	.683	.600	.475	.406	.373	.320	.224	.210	.198
			MSR	.998	.962	.786	.680	.600	.472	.412	.373	.318	.224	.212	.194
		A	$\beta_j\sim$ N	1	1	.977	.945	.902	.806	.756	.715	.632	.491	.468	.430
			$\beta_j\sim G_\beta$	1	1	.977	.952	.903	.803	.751	.703	.629	.468	.453	.413
		AB	N	.050	.051	.050	.872	.623	.347	.977	.872	.623	.997	.978	.873
			$G_{\tau\beta}$	.049	.052	.051	.898	.643	.351	.985	.895	.640	.999	.985	.896
	(2,10,3)	B	N	.998	.975	.872	.686	.629	.541	.437	.411	.363	.245	.234	.219
			UNI	1	.997	.930	.720	.661	.555	.420	.399	.354	.225	.219	.205
			SYM	1	.993	.916	.716	.649	.554	.419	.396	.352	.222	.215	.208
			ESR	1	.992	.914	.715	.653	.557	.422	.394	.353	.230	.220	.207
			MSR	1	.991	.913	.710	.656	.550	.424	.397	.353	.227	.219	.210
		A	$\beta_j\sim$ N	1	1	.998	.957	.929	.869	.771	.742	.681	.501	.484	.460
			$\beta_j\sim G_\beta$	1	1	.998	.965	.939	.870	.766	.732	.676	.474	.458	.436
		AB	N	.051	.051	.050	.969	.843	.574	.997	.969	.844	1	.997	.969
			$G_{\tau\beta}$	.053	.049	.051	.980	.868	.591	.999	.980	.870	1	.998	.980
	(2,10,4)	B	N	.999	.990	.934	.694	.656	.579	.442	.421	.386	.247	.236	.222
			UNI	1	.999	.976	.746	.690	.601	.430	.411	.373	.228	.221	.213
			SYM	1	.998	.966	.731	.684	.600	.428	.407	.374	.228	.223	.209
			ESR	1	.998	.965	.735	.682	.597	.427	.409	.372	.230	.220	.214
			MSR	1	.998	.962	.727	.677	.598	.432	.410	.371	.228	.221	.212
		A	$\beta_j\sim$ N	1	1	1	.963	.943	.898	.778	.758	.711	.505	.491	.470
			$\beta_j\sim G_\beta$	1	1	1	.971	.952	.903	.776	.753	.704	.478	.464	.448
		AB	N	.051	.049	.049	.989	.925	.723	.999	.989	.926	1	.999	.990
			$G_{\tau\beta}$	.049	.050	.051	.994	.944	.749	.999	.994	.947	1	1	.994

Table 3 The powers of F-test for fixed $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, -1, 1, -1)$, for $\beta_j \sim G_\beta(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim G_{\tau\beta}(0, \sigma_{\tau\beta}^2)$, and for $\beta_j \sim N(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0, \sigma_{\tau\beta}^2)$ with $\sigma^2 = R\sigma_\beta^2$, where $N_B = 20$

SF	Designs (a,b,n)	Factor	Dist	$\sigma_{\tau\beta}^2 = 0$			$\sigma_{\tau\beta}^2 = .5\sigma_\beta^2$			$\sigma_{\tau\beta}^2 = \sigma_\beta^2$			$\sigma_{\tau\beta}^2 = 2\sigma_\beta^2$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.10	(4,2,2)	B	N	.617	.500	.364	.327	.297	.251	.233	.218	.195	.159	.151	.144
			UNI	.660	.528	.393	.342	.306	.261	.232	.218	.196	.155	.148	.143
			SYM	.654	.522	.381	.331	.300	.256	.236	.215	.195	.156	.148	.142
			ESR	.636	.517	.380	.333	.300	.254	.225	.217	.194	.154	.149	.140
			MSR	.646	.513	.379	.328	.298	.255	.230	.215	.195	.157	.148	.140
		A	$\beta_j \sim$ N	.831	.565	.332	.279	.240	.191	.173	.161	.144	.116	.114	.105
			$\beta_j \sim G_\beta$	.835	.565	.330	.261	.232	.190	.167	.158	.140	.115	.111	.107
		AB	N	.050	.051	.050	.520	.322	.187	.726	.521	.322	.866	.724	.522
			$G_{\tau\beta}$	.050	.051	.051	.540	.326	.187	.747	.539	.333	.882	.745	.538
	(4,2,3)	B	N	.684	.569	.445	.336	.313	.280	.238	.227	.210	.158	.153	.146
			UNI	.720	.611	.474	.350	.331	.290	.239	.226	.209	.155	.151	.147
			SYM	.721	.603	.471	.343	.325	.288	.236	.225	.205	.156	.154	.147
			ESR	.702	.585	.459	.344	.321	.286	.233	.226	.210	.156	.150	.145
			MSR	.710	.594	.459	.342	.319	.283	.238	.226	.208	.153	.152	.142
		A	$\beta_j \sim$ N	.939	.731	.460	.292	.263	.219	.180	.168	.157	.115	.114	.109
			$\beta_j \sim G_\beta$	.941	.726	.461	.277	.252	.215	.174	.169	.153	.114	.112	.110
		AB	N	.051	.051	.049	.715	.508	.311	.860	.716	.506	.941	.863	.713
			$G_{\tau\beta}$	.049	.051	.049	.735	.524	.314	.875	.733	.525	.951	.877	.732
	(4,2,4)	B	N	.719	.618	.498	.341	.328	.292	.241	.234	.219	.162	.159	.148
			UNI	.759	.660	.534	.360	.340	.308	.241	.230	.218	.154	.155	.150
			SYM	.759	.652	.521	.347	.328	.300	.238	.229	.215	.153	.153	.149
			ESR	.740	.636	.513	.348	.330	.299	.239	.231	.217	.156	.151	.149
			MSR	.751	.645	.519	.348	.331	.299	.237	.227	.217	.154	.151	.145
		A	$\beta_j \sim$ N	.979	.833	.568	.299	.275	.235	.181	.174	.162	.117	.117	.109
			$\beta_j \sim G_\beta$	.979	.832	.567	.286	.268	.230	.176	.168	.158	.113	.117	.110
		AB	N	.050	.053	.050	.803	.621	.411	.913	.804	.622	.964	.910	.805
			$G_{\tau\beta}$	.051	.049	.051	.820	.639	.419	.923	.822	.641	.971	.922	.823
.20	(4,4,2)	B	N	.947	.874	.740	.681	.623	.540	.498	.470	.413	.320	.311	.282
			UNI	.979	.930	.803	.744	.683	.582	.529	.496	.431	.316	.302	.276
			SYM	.973	.911	.781	.710	.660	.562	.511	.477	.420	.318	.302	.277
			ESR	.965	.903	.777	.711	.661	.563	.513	.483	.428	.311	.303	.275
			MSR	.968	.904	.773	.700	.650	.559	.510	.476	.423	.308	.301	.275
		A	$\beta_j \sim$ N	1	.998	.921	.850	.776	.640	.585	.537	.461	.336	.319	.288
			$\beta_j \sim G_\beta$	1	.998	.921	.854	.773	.637	.567	.519	.448	.315	.295	.277
		AB	N	.050	.050	.051	.848	.589	.323	.970	.847	.584	.997	.970	.848
			$G_{\tau\beta}$	.050	.049	.051	.872	.601	.322	.981	.875	.603	.998	.981	.873
	(4,4,3)	B	N	.970	.925	.826	.695	.662	.592	.515	.492	.450	.325	.318	.300
			UNI	.990	.964	.891	.763	.721	.649	.546	.514	.474	.321	.305	.296
			SYM	.986	.954	.869	.735	.695	.628	.527	.506	.460	.316	.308	.290
			ESR	.982	.947	.861	.732	.693	.623	.527	.502	.463	.317	.310	.293
			MSR	.985	.948	.861	.726	.685	.620	.522	.499	.459	.318	.311	.293
		A	$\beta_j \sim$ N	1	1	.986	.874	.828	.726	.605	.569	.509	.341	.328	.310
			$\beta_j \sim G_\beta$	1	1	.987	.879	.823	.722	.577	.551	.490	.321	.305	.293
		AB	N	.049	.050	.052	.966	.832	.553	.996	.964	.829	1	.996	.967
			$G_{\tau\beta}$	.050	.050	.050	.977	.855	.572	.998	.978	.857	1	.998	.978
	(4,4,4)	B	N	.980	.949	.874	.708	.678	.625	.519	.497	.470	.334	.317	.307
			UNI	.994	.979	.927	.772	.741	.683	.549	.526	.492	.322	.312	.298
			SYM	.992	.972	.915	.746	.714	.661	.532	.516	.482	.317	.311	.301
			ESR	.989	.968	.904	.740	.710	.658	.533	.517	.485	.318	.313	.298
			MSR	.991	.968	.904	.735	.704	.653	.528	.513	.476	.317	.313	.301
		A	$\beta_j \sim$ N	1	1	.998	.887	.851	.773	.615	.585	.537	.349	.333	.317
			$\beta_j \sim G_\beta$	1	1	.998	.896	.851	.768	.589	.564	.519	.320	.312	.298
		AB	N	.051	.051	.049	.988	.920	.714	.999	.989	.921	1	.999	.988
			$G_{\tau\beta}$	.050	.048	.052	.993	.939	.734	.999	.995	.943	1	1	.993

Table 3 (Continued)

SF	Designs (a,b,n)	Factor	Dist	$\sigma^2_{\tau\beta}=0$			$\sigma^2_{\tau\beta}=.5\sigma^2_\beta$			$\sigma^2_{\tau\beta}=\sigma^2_\beta$			$\sigma^2_{\tau\beta}=2\sigma^2_\beta$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.50	(4,10,2)	B	N	1	.998	.982	.968	.946	.896	.866	.836	.773	.637	.611	.567
			UNI	1	1	.998	.995	.988	.955	.931	.902	.838	.671	.638	.584
			SYM	1	1	.996	.991	.979	.940	.919	.887	.822	.659	.632	.582
			ESR	1	1	.995	.990	.978	.937	.915	.886	.823	.664	.632	.583
			MSR	1	1	.995	.990	.976	.937	.911	.883	.817	.659	.633	.580
		A	$\beta_j\sim$ N	1	1	1	1	1	.994	.988	.980	.951	.846	.824	.777
			$\beta_j\sim G_\beta$	1	1	1	1	1	.995	.992	.984	.954	.851	.825	.775
		AB	N	.050	.050	.050	.997	.926	.625	1	.997	.925	1	1	.997
			$G_{\tau\beta}$	.049	.052	.049	.999	.944	.636	1	.999	.943	1	1	.999
	(4,10,3)	B	N	1	1	.996	.975	.963	.932	.875	.855	.814	.643	.630	.595
			UNI	1	1	1	.997	.993	.978	.941	.924	.879	.679	.659	.620
			SYM	1	1	.999	.994	.988	.968	.926	.909	.868	.674	.652	.612
			ESR	1	1	.999	.993	.986	.967	.922	.905	.863	.671	.655	.615
			MSR	1	1	.999	.992	.986	.964	.923	.901	.859	.668	.649	.617
		A	$\beta_j\sim$ N	1	1	1	1	1	.999	.991	.985	.971	.854	.838	.807
			$\beta_j\sim G_\beta$	1	1	1	1	1	.999	.995	.989	.976	.857	.841	.804
		AB	N	.050	.050	.049	1	.995	.898	1	1	.995	1	1	1
			$G_{\tau\beta}$	.049	.050	.050	1	.998	.916	1	1	.998	1	1	1
	(4,10,4)	B	N	1	1	.998	.975	.968	.947	.879	.864	.835	.649	.636	.612
			UNI	1	1	1	.998	.996	.987	.945	.930	.902	.686	.669	.640
			SYM	1	1	1	.996	.991	.979	.932	.919	.888	.674	.658	.630
			ESR	1	1	1	.993	.989	.978	.930	.917	.883	.673	.662	.632
			MSR	1	1	1	.993	.989	.976	.925	.913	.881	.675	.662	.634
		A	$\beta_j\sim$ N	1	1	1	1	1	.999	.992	.989	.978	.856	.845	.825
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	.995	.992	.983	.861	.849	.826
		AB	N	.050	.050	.051	1	.999	.973	1	1	1	1	1	1
			$G_{\tau\beta}$	.053	.049	.051	1	1	.985	1	1	1	1	1	1

Table 4 The powers of F-test for fixed $(\tau_1,\tau_2)=(-1,1)$, for $\beta_j \sim G_\beta(0,\sigma_\beta^2)$, $(\tau\beta)_{ij} \sim G_{\tau\beta}(0,\sigma_{\tau\beta}^2)$, and for $\beta_j \sim N(0,\sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0,\sigma_{\tau\beta}^2)$ with $\sigma^2 = R\sigma_\beta^2$, where $N_B = 60$

SF	Designs (a,b,n)	Factor	Dist	$\sigma_{\tau\beta}^2=0$			$\sigma_{\tau\beta}^2=.5\sigma_\beta^2$			$\sigma_{\tau\beta}^2=\sigma_\beta^2$			$\sigma_{\tau\beta}^2=2\sigma_\beta^2$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.05	(2,3,2)	B	N	.470	.323	.208	.179	.160	.136	.128	.121	.108	.091	.091	.086
			UNI	.503	.338	.211	.179	.161	.135	.125	.119	.109	.092	.088	.087
			SYM	.489	.328	.211	.178	.161	.134	.127	.120	.110	.095	.088	.088
			ESR	.488	.335	.210	.179	.162	.136	.127	.120	.107	.093	.092	.086
			MSR	.486	.329	.212	.177	.160	.137	.124	.120	.107	.094	.090	.086
		A	$\beta_j \sim$ N	.704	.471	.288	.245	.216	.179	.168	.156	.139	.112	.109	.106
			$\beta_j \sim G_\beta$	.708	.472	.290	.236	.210	.174	.156	.149	.134	.111	.106	.103
		AB	N	.049	.053	.050	.413	.261	.157	.593	.414	.259	.752	.593	.413
			$G_{\tau\beta}$	.050	.048	.051	.426	.261	.157	.614	.424	.263	.767	.613	.424
	(2,3,3)	B	N	.567	.409	.269	.185	.174	.151	.133	.124	.115	.096	.090	.090
			UNI	.609	.429	.279	.184	.173	.149	.127	.124	.119	.097	.094	.091
			SYM	.588	.422	.275	.185	.171	.150	.128	.127	.115	.096	.092	.088
			ESR	.594	.419	.275	.184	.171	.151	.125	.124	.117	.097	.094	.093
			MSR	.588	.420	.274	.183	.173	.151	.126	.126	.114	.093	.095	.089
		A	$\beta_j \sim$ N	.838	.607	.390	.258	.239	.205	.170	.161	.150	.118	.111	.106
			$\beta_j \sim G_\beta$	.839	.605	.388	.248	.227	.194	.159	.155	.147	.113	.110	.105
		AB	N	.050	.049	.050	.590	.409	.249	.747	.591	.405	.856	.749	.589
			$G_{\tau\beta}$	.048	.050	.051	.610	.421	.258	.768	.604	.418	.871	.764	.604
	(2,3,4)	B	N	.637	.472	.323	.194	.179	.163	.133	.125	.118	.094	.093	.090
			UNI	.675	.504	.338	.190	.179	.159	.131	.125	.120	.094	.091	.091
			SYM	.658	.489	.330	.188	.178	.160	.130	.122	.117	.096	.092	.092
			ESR	.656	.490	.329	.190	.174	.161	.129	.124	.120	.093	.094	.091
			MSR	.652	.488	.330	.187	.178	.160	.128	.125	.119	.096	.094	.092
		A	$\beta_j \sim$ N	.907	.706	.471	.268	.246	.219	.171	.163	.155	.116	.116	.112
			$\beta_j \sim G_\beta$	.909	.710	.472	.254	.237	.209	.164	.158	.149	.111	.109	.106
		AB	N	.051	.051	.051	.678	.509	.328	.814	.683	.503	.899	.812	.679
			$G_{\tau\beta}$	.050	.049	.049	.696	.521	.339	.827	.699	.522	.909	.832	.700
.10	(2,6,2)	B	N	.893	.729	.497	.423	.366	.288	.264	.238	.210	.162	.152	.142
			UNI	.935	.780	.520	.420	.365	.290	.252	.236	.206	.153	.148	.139
			SYM	.922	.757	.511	.419	.368	.288	.250	.236	.204	.151	.150	.137
			ESR	.920	.760	.517	.418	.366	.290	.252	.232	.203	.153	.147	.139
			MSR	.915	.756	.511	.421	.362	.289	.251	.231	.205	.154	.148	.140
		A	$\beta_j \sim$ N	1	.971	.791	.695	.618	.507	.460	.427	.366	.279	.267	.243
			$\beta_j \sim G_\beta$	1	.972	.788	.687	.611	.498	.441	.410	.355	.259	.250	.234
		AB	N	.050	.051	.052	.687	.442	.249	.876	.694	.440	.963	.876	.687
			$G_{\tau\beta}$	.050	.048	.051	.709	.452	.245	.894	.705	.448	.972	.895	.703
	(2,6,3)	B	N	.947	.843	.636	.442	.404	.337	.272	.255	.229	.161	.155	.148
			UNI	.975	.888	.677	.448	.402	.338	.259	.250	.226	.157	.151	.146
			SYM	.963	.869	.662	.439	.397	.334	.262	.246	.223	.155	.151	.147
			ESR	.959	.870	.669	.441	.400	.335	.260	.245	.222	.154	.151	.145
			MSR	.956	.860	.660	.441	.402	.335	.258	.245	.223	.154	.151	.147
		A	$\beta_j \sim$ N	1	.996	.921	.728	.678	.583	.474	.446	.405	.285	.274	.259
			$\beta_j \sim G_\beta$	1	.996	.919	.714	.660	.569	.451	.428	.391	.266	.258	.244
		AB	N	.050	.048	.049	.862	.658	.412	.958	.863	.662	.991	.959	.862
			$G_{\tau\beta}$	.050	.050	.050	.881	.682	.418	.968	.879	.678	.993	.968	.881
	(2,6,4)	B	N	.970	.893	.729	.456	.424	.366	.277	.260	.240	.161	.156	.153
			UNI	.987	.937	.779	.455	.427	.365	.264	.252	.233	.157	.154	.148
			SYM	.981	.921	.756	.456	.417	.364	.266	.254	.234	.154	.155	.149
			ESR	.978	.920	.762	.452	.419	.366	.262	.257	.232	.157	.154	.149
			MSR	.975	.914	.759	.450	.418	.364	.263	.251	.235	.156	.152	.149
		A	$\beta_j \sim$ N	1	1	.969	.745	.695	.624	.484	.459	.427	.286	.275	.267
			$\beta_j \sim G_\beta$	1	1	.969	.733	.690	.608	.456	.443	.407	.265	.262	.249
		AB	N	.051	.049	.051	.924	.777	.538	.981	.925	.779	.996	.980	.922
			$G_{\tau\beta}$	.050	.049	.050	.936	.798	.557	.984	.939	.801	.997	.985	.936

Table 4 (Continued)

SF	Designs (a,b,n)	Factor	Dist	$\sigma^2_{\tau\beta}=0$			$\sigma^2_{\tau\beta}=.5\sigma^2_{\beta}$			$\sigma^2_{\tau\beta}=\sigma^2_{\beta}$			$\sigma^2_{\tau\beta}=2\sigma^2_{\beta}$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.20	(2,12,2)	B	N	.996	.966	.827	.740	.664	.543	.489	.445	.377	.273	.263	.236
			UNI	1	.987	.864	.774	.692	.555	.486	.443	.374	.260	.246	.224
			SYM	.999	.981	.849	.764	.682	.546	.483	.442	.378	.258	.251	.226
			ESR	.999	.981	.854	.768	.687	.548	.486	.445	.378	.259	.251	.230
			MSR	.998	.978	.850	.762	.686	.551	.487	.443	.373	.262	.245	.227
		A	$\beta_j\sim$ N	1	1	.993	.978	.953	.885	.844	.804	.730	.578	.555	.519
			$\beta_j\sim G_\beta$	1	1	.994	.982	.956	.884	.845	.804	.730	.564	.546	.498
		AB	N	.050	.049	.050	.920	.694	.395	.990	.918	.691	.999	.990	.919
			$G_{\tau\beta}$	.048	.050	.051	.934	.709	.397	.994	.934	.708	1	.993	.936
	(2,12,3)	B	N	1	.991	.927	.763	.713	.621	.509	.475	.426	.279	.266	.249
			UNI	1	.998	.961	.805	.745	.641	.504	.474	.419	.263	.252	.242
			SYM	1	.996	.949	.792	.735	.633	.500	.470	.413	.264	.260	.245
			ESR	1	.996	.950	.796	.739	.636	.503	.471	.419	.262	.257	.240
			MSR	1	.995	.948	.794	.737	.635	.504	.467	.416	.266	.257	.241
		A	$\beta_j\sim$ N	1	1	1	.985	.970	.934	.855	.831	.783	.590	.572	.546
			$\beta_j\sim G_\beta$	1	1	1	.989	.975	.934	.859	.829	.779	.576	.561	.531
		AB	N	.050	.052	.049	.986	.894	.642	.999	.986	.896	1	.999	.986
			$G_{\tau\beta}$	.046	.050	.050	.991	.911	.653	1	.990	.910	1	.999	.990
	(2,12,4)	B	N	1	.997	.967	.781	.739	.666	.517	.489	.447	.278	.271	.263
			UNI	1	1	.987	.823	.776	.694	.509	.487	.441	.267	.261	.251
			SYM	1	.999	.981	.806	.763	.683	.509	.485	.447	.271	.262	.245
			ESR	1	.999	.981	.812	.767	.685	.508	.485	.448	.270	.262	.251
			MSR	1	.999	.978	.808	.765	.683	.507	.483	.447	.271	.262	.251
		A	$\beta_j\sim$ N	1	1	1	.987	.978	.951	.865	.846	.804	.594	.580	.553
			$\beta_j\sim G_\beta$	1	1	1	.991	.983	.956	.866	.844	.803	.584	.568	.543
		AB	N	.052	.052	.049	.996	.957	.785	1	.996	.958	1	1	.996
			$G_{\tau\beta}$	.053	.050	.050	.997	.969	.805	1	.998	.970	1	1	.998
.50	(2,30,2)	B	N	1	1	.996	.985	.963	.899	.856	.817	.725	.544	.519	.462
			UNI	1	1	.999	.997	.984	.925	.885	.839	.746	.546	.510	.458
			SYM	1	1	.999	.995	.982	.922	.882	.834	.739	.543	.517	.461
			ESR	1	1	.999	.995	.981	.922	.886	.842	.744	.540	.514	.461
			MSR	1	1	.999	.995	.981	.920	.883	.837	.744	.539	.513	.460
		A	$\beta_j\sim$ N	1	1	1	1	1	1	.999	.997	.990	.952	.942	.916
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	.999	.998	.992	.956	.945	.918
		AB	N	.050	.050	.050	.999	.958	.706	1	.999	.960	1	1	.999
			$G_{\tau\beta}$	.050	.049	.049	1	.968	.716	1	1	.968	1	1	1
	(2,30,3)	B	N	1	1	1	.989	.978	.943	.873	.842	.782	.554	.532	.498
			UNI	1	1	1	.998	.993	.969	.903	.872	.810	.555	.533	.493
			SYM	1	1	1	.997	.991	.967	.900	.865	.806	.555	.531	.496
			ESR	1	1	1	.997	.992	.967	.897	.870	.808	.551	.529	.497
			MSR	1	1	1	.997	.990	.964	.897	.868	.805	.551	.533	.498
		A	$\beta_j\sim$ N	1	1	1	1	1	1	.999	.998	.995	.955	.949	.933
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	1	.999	.996	.958	.952	.936
		AB	N	.049	.048	.048	1	.997	.930	1	1	.998	1	1	1
			$G_{\tau\beta}$	.051	.051	.051	1	.999	.940	1	1	.999	1	1	1
	(2,30,4)	B	N	1	1	1	.990	.984	.963	.880	.855	.813	.561	.544	.520
			UNI	1	1	1	.999	.997	.984	.913	.887	.843	.560	.541	.512
			SYM	1	1	1	.998	.995	.981	.905	.881	.838	.556	.539	.513
			ESR	1	1	1	.998	.995	.981	.909	.884	.838	.557	.541	.512
			MSR	1	1	1	.997	.994	.982	.905	.884	.839	.555	.539	.514
		A	$\beta_j\sim$ N	1	1	1	1	1	1	.999	.999	.997	.958	.953	.941
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	1	.999	.998	.962	.957	.946
		AB	N	.048	.049	.049	1	1	.982	1	1	1	1	1	1
			$G_{\tau\beta}$	.050	.049	.050	1	1	.988	1	1	1	1	1	1

Table 5 The powers of F-test for fixed $(\tau_1,\tau_2,\tau_3,\tau_4)=(1,-1,1,-1)$, for $\beta_j \sim G_\beta(0,\sigma_\beta^2)$, $(\tau\beta)_{ij} \sim G_{\tau\beta}(0,\sigma_{\tau\beta}^2)$, and for $\beta_j \sim N(0,\sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0,\sigma_{\tau\beta}^2)$ with $\sigma^2 = R\sigma_\beta^2$, where $N_B = 60$

SF	Designs (a,b,n)	Factor	Dist	$\sigma_{\tau\beta}^2=0$			$\sigma_{\tau\beta}^2=.5\sigma_\beta^2$			$\sigma_{\tau\beta}^2=\sigma_\beta^2$			$\sigma_{\tau\beta}^2=2\sigma_\beta^2$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.05	(4,3,2)	B	N	.860	.749	.598	.530	.484	.411	.382	.357	.316	.246	.240	.218
			UNI	.887	.793	.635	.567	.516	.437	.395	.364	.323	.244	.228	.212
			SYM	.878	.775	.618	.554	.502	.423	.383	.359	.319	.238	.230	.216
			ESR	.868	.772	.618	.556	.508	.427	.389	.362	.322	.238	.231	.215
			MSR	.867	.766	.614	.553	.498	.423	.387	.360	.318	.237	.231	.213
		A	$\beta_j \sim$ N	.999	.956	.729	.623	.542	.426	.385	.349	.301	.217	.211	.190
			$\beta_j \sim G_\beta$	.999	.956	.728	.610	.526	.419	.361	.332	.289	.208	.199	.187
		AB	N	.052	.051	.049	.731	.471	.258	.907	.726	.467	.979	.908	.725
			$G_{\tau\beta}$	.051	.050	.050	.748	.483	.260	.925	.751	.477	.985	.925	.747
	(4,3,3)	B	N	.901	.821	.689	.548	.520	.461	.395	.375	.344	.252	.243	.226
			UNI	.922	.857	.735	.587	.547	.485	.400	.386	.350	.244	.239	.222
			SYM	.921	.843	.716	.568	.533	.469	.396	.373	.345	.238	.232	.221
			ESR	.912	.835	.714	.569	.540	.480	.397	.379	.345	.241	.235	.224
			MSR	.908	.826	.711	.567	.535	.475	.394	.375	.342	.241	.236	.223
		A	$\beta_j \sim$ N	1	.994	.886	.660	.595	.501	.401	.372	.332	.224	.217	.202
			$\beta_j \sim G_\beta$	1	.995	.888	.638	.579	.484	.372	.350	.318	.213	.206	.195
		AB	N	.049	.052	.049	.900	.707	.446	.975	.896	.707	.995	.976	.898
			$G_{\tau\beta}$	.051	.049	.049	.917	.730	.456	.983	.916	.728	.997	.983	.916
	(4,3,4)	B	N	.924	.859	.750	.558	.539	.487	.402	.387	.356	.249	.244	.238
			UNI	.944	.889	.791	.601	.573	.515	.408	.394	.365	.243	.239	.232
			SYM	.939	.880	.778	.581	.547	.503	.395	.384	.362	.242	.240	.226
			ESR	.931	.869	.772	.581	.553	.511	.404	.388	.365	.240	.242	.229
			MSR	.930	.867	.762	.579	.549	.509	.399	.383	.364	.245	.237	.231
		A	$\beta_j \sim$ N	1	.999	.956	.671	.625	.544	.404	.386	.350	.221	.221	.210
			$\beta_j \sim G_\beta$	1	.999	.956	.657	.606	.529	.377	.361	.328	.214	.207	.200
		AB	N	.051	.050	.050	.950	.826	.585	.990	.950	.823	.999	.990	.949
			$G_{\tau\beta}$	.049	.051	.052	.962	.845	.606	.994	.963	.849	.999	.993	.963
.10	(4,6,2)	B	N	.993	.970	.891	.846	.800	.715	.673	.637	.575	.447	.425	.393
			UNI	.998	.987	.936	.900	.857	.766	.724	.681	.601	.453	.429	.390
			SYM	.996	.981	.919	.878	.832	.744	.701	.658	.592	.446	.430	.392
			ESR	.995	.978	.919	.879	.838	.744	.708	.665	.594	.451	.429	.398
			MSR	.995	.974	.909	.871	.829	.743	.703	.664	.587	.447	.430	.394
		A	$\beta_j \sim$ N	1	1	.997	.984	.962	.891	.850	.806	.723	.557	.535	.491
			$\beta_j \sim G_\beta$	1	1	.996	.989	.968	.892	.852	.808	.717	.543	.518	.468
		AB	N	.051	.049	.050	.959	.758	.442	.997	.956	.756	1	.997	.957
			$G_{\tau\beta}$	.049	.051	.050	.971	.779	.447	.999	.970	.777	1	.999	.970
	(4,6,3)	B	N	.997	.986	.946	.860	.833	.773	.690	.662	.615	.456	.444	.415
			UNI	.999	.996	.974	.914	.888	.828	.739	.709	.653	.461	.449	.415
			SYM	.999	.992	.964	.889	.860	.800	.716	.684	.635	.446	.439	.412
			ESR	.998	.991	.962	.896	.864	.807	.720	.692	.644	.457	.444	.420
			MSR	.998	.990	.958	.886	.856	.799	.717	.684	.637	.454	.443	.418
		A	$\beta_j \sim$ N	1	1	1	.990	.979	.942	.864	.834	.776	.569	.548	.516
			$\beta_j \sim G_\beta$	1	1	1	.994	.982	.945	.868	.837	.775	.549	.530	.496
		AB	N	.050	.048	.050	.996	.945	.721	1	.997	.944	1	1	.996
			$G_{\tau\beta}$	.051	.049	.049	.998	.961	.736	1	.998	.960	1	1	.998
	(4,6,4)	B	N	.999	.993	.970	.868	.846	.801	.694	.677	.637	.458	.445	.432
			UNI	1	.998	.987	.921	.902	.856	.740	.718	.675	.463	.450	.433
			SYM	.999	.996	.981	.897	.877	.832	.721	.699	.661	.455	.444	.423
			ESR	.999	.995	.978	.898	.878	.837	.736	.707	.668	.461	.453	.426
			MSR	.999	.994	.975	.892	.873	.828	.719	.702	.659	.460	.448	.429
		A	$\beta_j \sim$ N	1	1	1	.992	.984	.962	.873	.851	.804	.570	.556	.531
			$\beta_j \sim G_\beta$	1	1	1	.996	.989	.965	.876	.851	.804	.556	.543	.514
		AB	N	.051	.050	.049	.999	.984	.861	1	.999	.984	1	1	.999
			$G_{\tau\beta}$	.048	.051	.050	1	.992	.884	1	1	.991	1	1	1

Table 5 (Continued)

SF	Designs (a,b,n)	Factor	Dist	$\sigma^2_{\tau\beta}=0$			$\sigma^2_{\tau\beta}=.5\sigma^2_\beta$			$\sigma^2_{\tau\beta}=\sigma^2_\beta$			$\sigma^2_{\tau\beta}=2\sigma^2_\beta$		
				R			R			R			R		
				.25	.5	1	.25	.5	1	.25	.5	1	.25	.5	1
.20	(4,12,2)	B	N	1	1	.994	.986	.972	.937	.914	.890	.838	.707	.682	.631
			UNI	1	1	.999	.997	.992	.971	.954	.935	.883	.741	.711	.658
			SYM	1	1	.997	.993	.987	.959	.941	.919	.868	.725	.698	.652
			ESR	1	1	.997	.994	.987	.961	.944	.921	.871	.737	.705	.651
			MSR	1	1	.996	.992	.984	.955	.938	.917	.864	.727	.701	.652
		A	$\beta_j\sim$ N	1	1	1	1	1	.999	.998	.994	.981	.918	.901	.862
			$\beta_j\sim G_\beta$	1	1	1	1	1	.999	.999	.995	.984	.924	.905	.868
		AB	N	.050	.050	.050	.999	.959	.702	1	.999	.959	1	1	.999
			$G_{\tau\beta}$	.050	.049	.051	1	.969	.705	1	1	.971	1	1	1
	(4,12,3)	B	N	1	1	.999	.989	.981	.963	.921	.906	.873	.714	.700	.665
			UNI	1	1	1	.998	.995	.987	.962	.949	.918	.750	.730	.694
			SYM	1	1	1	.996	.992	.978	.949	.934	.904	.736	.719	.682
			ESR	1	1	1	.996	.992	.980	.952	.939	.907	.741	.723	.689
			MSR	1	1	.999	.995	.990	.976	.947	.935	.905	.737	.720	.686
		A	$\beta_j\sim$ N	1	1	1	1	1	1	.998	.996	.991	.922	.913	.888
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	.999	.998	.993	.929	.916	.891
		AB	N	.051	.049	.050	1	.998	.938	1	1	.999	1	1	1
			$G_{\tau\beta}$	.049	.050	.050	1	.999	.952	1	1	.999	1	1	1
	(4,12,4)	B	N	1	1	1	.990	.985	.973	.926	.914	.891	.717	.705	.684
			UNI	1	1	1	.999	.997	.992	.965	.956	.934	.756	.739	.710
			SYM	1	1	1	.996	.994	.986	.952	.943	.921	.740	.725	.699
			ESR	1	1	1	.996	.993	.986	.954	.945	.922	.745	.731	.707
			MSR	1	1	1	.995	.992	.984	.949	.943	.917	.742	.725	.702
		A	$\beta_j\sim$ N	1	1	1	1	1	1	.999	.997	.994	.926	.920	.901
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	.999	.998	.996	.931	.921	.904
		AB	N	.051	.050	.048	1	1	.988	1	1	1	1	1	1
			$G_{\tau\beta}$	.047	.049	.050	1	1	.993	1	1	1	1	1	1
.50	(4,30,2)	B	N	1	1	1	1	1	1	.999	.998	.993	.962	.953	.931
			UNI	1	1	1	1	1	1	1	1	.999	.985	.978	.959
			SYM	1	1	1	1	1	1	1	1	.999	.981	.973	.954
			ESR	1	1	1	1	1	1	1	1	.998	.981	.974	.954
			MSR	1	1	1	1	1	1	1	1	.998	.982	.973	.953
		A	$\beta_j\sim$ N	1	1	1	1	1	1	1	1	1	1	1	1
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	1	1	1	1	1	1
		AB	N	.049	.051	.051	1	1	.964	1	1	1	1	1	1
			$G_{\tau\beta}$	.049	.048	.051	1	1	.972	1	1	1	1	1	1
	(4,30,3)	B	N	1	1	1	1	1	1	.999	.999	.997	.965	.959	.945
			UNI	1	1	1	1	1	1	1	1	1	.987	.982	.973
			SYM	1	1	1	1	1	1	1	1	.999	.983	.979	.969
			ESR	1	1	1	1	1	1	1	1	1	.985	.981	.968
			MSR	1	1	1	1	1	1	1	1	.999	.982	.979	.967
		A	$\beta_j\sim$ N	1	1	1	1	1	1	1	1	1	1	1	1
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	1	1	1	1	1	1
		AB	N	.050	.049	.051	1	1	1	1	1	1	1	1	1
			$G_{\tau\beta}$	.050	.050	.051	1	1	1	1	1	1	1	1	1
	(4,30,4)	B	N	1	1	1	1	1	1	.999	.999	.998	.967	.962	.953
			UNI	1	1	1	1	1	1	1	1	1	.987	.984	.978
			SYM	1	1	1	1	1	1	1	1	1	.985	.981	.974
			ESR	1	1	1	1	1	1	1	1	1	.985	.983	.976
			MSR	1	1	1	1	1	1	1	1	1	.984	.980	.973
		A	$\beta_j\sim$ N	1	1	1	1	1	1	1	1	1	1	1	1
			$\beta_j\sim G_\beta$	1	1	1	1	1	1	1	1	1	1	1	1
		AB	N	.049	.052	.051	1	1	1	1	1	1	1	1	1
			$G_{\tau\beta}$	.049	.051	.050	1	1	1	1	1	1	1	1	1

Table 6 The best and worst distribution of powers of F-test for $(\tau_1, \tau_2) = (-1,1)$ is fixed, $a = 2,\ N_B = 20$

(6a) The best distribution

b	k = 0	.5	1	2
2	U	≈	≈	≈
4	U	N	N	N
10	U	U	N	N

(6b) The worst distribution

b	k = 0	.5	1	2
2	N	≈	≈	≈
4	N	D	D	D
10	N	N	D	D

Table 7 The best and worst distribution of powers of F-test for $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,-1,1,-1)$ is fixed, $a = 4,\ N_B = 20$

(7a) The best distribution

b	k = 0	.5	1	2
2	U	U	≈	≈
4	U	U	U	N
10	U	U	U	U

(7b) The worst distribution

b	k = 0	.5	1	2
2	N	N	≈	≈
4	N	N	N	D
10	N	N	N	N

Table 8 The best and worst distribution of powers of F-test for $(\tau_1, \tau_2) = (-1,1)$ is fixed, $a = 2,\ N_B = 60,\ (S_1, S_2, S_3)$ represents the best designs for $n = 2,\ 3$ and 4

(8a) The best distribution

b	k = 0	.5	1	2
3	U	(≈, ≈, N)	≈	≈
6	U	(≈, N or U, N or U)	N	N
12	U	U	N	N
30	≈	U	U	N

(8b) The worst distribution

b	k = 0	.5	1	2
3	N	(≈, ≈, D)	≈	≈
6	N	≈	D	D
12	N	N	D	D
30	≈	N	N	D

Table 9 The best and worst distribution of powers of F-test for $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,-1,1,-1)$ is fixed, $a = 4,\ N_B = 60$

(9a) The best distribution

b	k = 0	.5	1	2
3	U	U	U	N
6	U	U	U	U
12	-	U	U	U
30	-	-	U	U

(9b) The worst distribution

b	k = 0	.5	1	2
3	N	N	N	D
6	N	N	N	-
12	-	N	N	N
30	-	-	N	N

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References

Bennett CA, Franklin NL. Statistical analysis in chemistry and the chemical industry. New York: John Wiley & Sons; 1954.

Borror CM, Montgomery DC, Runger GC. Confidence intervals for variance components from gauge capability studies. Qual Reliab Eng Int. 1997; 13(6): 361-369.

Burdick RK, Borror CM, Montgomery DC. A review of methods for measurement systems capability analysis. J Qual Technol. 2003; 35(4): 342-354.

Burdick RK, Borror CM, Montgomery DC. Design and analysis of gauge R&R studies: making decisions with confidence intervals in random and mixed ANOVA models. ASA-SIAM Ser Stat Appl Probab: SIAM, Philadelphia, ASA, Alexandria, VA; 2005a.

Burdick RK, Park Y-J, Montgomery DC, Borror CM. Confidence intervals for misclassification rates in a gauge R&R study. J Qual Technol. 2005b; 37(4): 294-303.

Cochran WG. Sampling techniques. New York: John Wiley & Sons; 1977.

Cornfield J, Tukey JW. Average values of mean squares in factorials. Ann Math Stat. 1956; 27(4): 907-949.

Gaylor DW, Hartwell TD. Expected mean squares for nested classifications. Biometrics. 1969; 25(2): 427-430.

Graybill FA. An introduction to linear statistical models: volume 1. New York: McGraw-Hill; 1961.

Hartley HO. Expectations, variances and covariances of ANOVA mean squares by ‘synthesis’. Biometrics. 1967; 23(1): 105-114.

Hedayat AS, Sinha BK. Design and inference in finite population sampling. New York: John Wiley & Sons; 1991.

Hocking RR. The analysis of linear models. Monterey: Brooks/Cole; 1984.

Houf RF, Berman DB. Statistical analysis of power module thermal test equipment performance. IEEE Trans Components, Hybrids Manuf Technol. 1988; 11(4): 516-520.

Khuri AI. Linear model methodology. Boca Raton: CRC Press; 2010.

Kish L. Survey sampling. New York: John Wiley & Sons; 1965.

Levy PS, Lemeshow S. Sampling of populations: methods and applications. New York: John Wiley & Sons; 1991.

Lohr SL. Sampling: Design and analysis. 2nd edition. Boston: Brooks/Cole; 2010.

Montgomery DC. Statistical quality control. New York: John Wiley & Sons; 2013.

Montgomery DC. Design and analysis of experiments. New York: John Wiley & Sons; 2017.

Montgomery DC, Runger GC. Gauge capability and designed experiments part I: basic methods. Qual Eng. 1993a; 6(1): 115-135.

Montgomery DC, Runger GC. Gauge capability analysis and designed experiments part II: experimental design models and variance component estimation. Qual Eng. 1993b; 6(2): 289-305.

Muller KE, Stewart PW. Linear model theory: univariate, multivariate, and mixed models. New York: John Wiley & Sons; 2006.

Sahai H, Ojeda MM. Analysis of variance for random model volume II: unbalanced data. Boston: Birkhaäuser; 2005.

Searle SR. Variance components in the unbalanced 2-way nested classification. Ann Math Stat. 1961; 32(4): 1161-1166.

Searle SR. Linear models. New York: John Wiley & Sons; 1971.

Searle SR, Casella G, Mcculloch CE. Variance components. Hoboken: John Wiley & Sons; 2006.

Searle SR, Fawcett RF. Expected mean squares in variance component models having finite population. Biometrics. 1970; 26(2): 243-254.

Simmachan T, Borkowski JJ, Budsaba K. Expected mean squares for the random effects one-way ANOVA model when sampling from a finite population. Thail Stat. 2012; 10(1): 121-128.

Tukey JW. Variances of variance components I: balanced designs. Ann Math Stat. 1956; 27(3): 722-736.

Tukey JW. Variances of variance components II: the unbalanced single classification. Ann Math Stat. 1957; 28(1): 43-56.

Vardeman SB, VanValkenburg ES. Two-way random-effects analyses and gauge R&R studies. Technometrics. 1999; 41(3): 202-211.