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Some Properties of the New Mixture of Nakagami Distribution

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Abstract

One new mixture Nakagami distribution has been introduced. Then some attributes of the proposed distribution have been explored. The comparisons of the effectiveness of estimators via the quasi-Newton and the simulated annealing were made by considering the bias, and the mean squared error (MSE). Lastly, to illustrate its usefulness, the model has been applied in describing real data sets.

Keywords: Survival distribution, right skewed distribution, quasi-Newton method, simulated annealing.

1. Introduction

Survival analysis is a field of statistics relating to death and failure. The research in survival analysis seeks the following four things:1. the survival rates of the population; 2. the force of mortality rate at a given time; 3. other factors causing death; 4. the environmental impact on studying.

The key data for survival analysis is the time to the occurrence of a given event, namely the death of patients, failed marriage ending in divorce, students resignation, or employees resignation. The time from the start to the given event occurs. It is survival time. The statistical analysis of the survival data focuses on survival function and hazard function.

In general, many physical causes for failure or death at a point in time. It is difficult to classify those causes and to explain them with a mathematical model. We select the parametric survival distributions and then bring them to estimate data characteristics. Some parametric distributions were used as models to describe the survival data, namely exponential distribution, Weibull Distribution, Log-normal distribution, Nakagami distribution, inverse Gaussian distribution, LogLogistic distribution, and so forth. In this article, we are interested in the Nakagami distribution for the development. In order to have the new alternative distribution which is applied to the survival analysis. The Nakagami mixture model is proposed by combining the Nakagami (Nak) distribution and the length-biased Nakagami (LBNak) distribution with the weighted parameter. The basic properties of the proposed distribution are established. The reliability of the proposed distribution, compared to an existing distribution is demonstrated.

2. The Nakagami distribution

The Nakagami distribution (Nak) was proposed by Nakagami (1960). This distribution is the intensity distribution due to rapid fading. The Nakagami distribution has two parameters; $\lambda \geq 0.5$ is the shape parameter and $\beta > 0$ is scale parameter. The corresponding cumulative distribution function (CDF) is given by

$$F(x) = \frac{1}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}x^2\right). \tag{1}$$

where Γ is the gamma function, and γ is the regularized (lower) incomplete gamma function. The probability density function (PDF) is as this formula:

$$f(x) = \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}x^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}x^2\right); \; x > 0, \tag{2}$$

and its mean is

$$E(X) = \int_0^\infty \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}x^{2\lambda}\exp\left(\frac{-\lambda}{\beta}x^2\right)\partial x = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{1/2}. \tag{3}$$

3. Theoretical Result

3.1. The Probability Density Function of the New Mixture of Nakagami Distribution

The length-biased probability density function (PDF) is defined as follows: let X be a non-negative random variable having a continuous PDF $f(\cdot)$ and a finite first moment $E[X]$ exist. We say that a non-negative random variable Y with the PDF $h(\cdot)$ has the length-biased random variable associated with X, if its PDF is given by this formula:

$$h(x) = \frac{xf(x)}{E[X]}, \; x > 0, \tag{4}$$

where $f(x)$ is the PDF of the original distribution. Suppose X_1 be an independent random variable of Nakagami (Nak) distribution, the corresponding probability density function of X_1 is given by

$$f(x_1) = \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}x_1^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}x_1^2\right); \; x_1 > 0. \tag{5}$$

Let X_2 be an independent random variable of the length-biased Nakagami (LBNak) distribution proposed by Abdullahi and Phaphan (2022), the probability density function of X_2 is

$$f(x_2) = \frac{2\lambda^{\lambda+1/2}x_2^{2\lambda}\mathrm{e}^{-\frac{\lambda}{\beta}x_2^2}}{\Gamma(\lambda+1/2)\beta^{\lambda+1/2}}; \; x_2 > 0. \tag{6}$$

The mixture approach is one of the prominent methods of obtaining new alternative probability distributions in the field of probability and statistics because of its simplicity and unambiguous interpretation for applications such as the research of Chananet and Phaphan (2021), Phaphan and Pongsart (2019), Gillariose and Tomy (2011), Bowonrattanaset and Budsaba (2011), Birnbaum and Saunders (1969a), and Birnbaum and Saunders (1969b). For the new mixture of Nakagami distribution, we consider the new random variable X such that

$$X = \begin{cases} X_1 \text{ with} & \text{probability } p \\ X_2 \text{ with} & \text{probability } 1-p, \end{cases}$$

where $0 \leq p \leq 1$. Obviously, X is a mixture of X_1 and X_2 and the PDF of X is given by

$$f_p(x) = pf(x_1) + (1-p)f(x_2), \tag{7}$$

insert (5) and (6) in (7) yield

$$f_p(x) = \left\{ \frac{2\lambda^\lambda}{\beta^\lambda} x^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta}x^2\right) \left[\frac{p}{\Gamma(\lambda)} + \frac{x(1-p)\sqrt{\lambda}}{\Gamma(\lambda+(1/2))\sqrt{\beta}} \right] \right\}, \tag{8}$$

where $\lambda \geq 0.5$ and $\beta > 0$. From Figure 1, it can be shown that the new mixture of Nakagami density function seems positively skewed, λ is a shape parameter, and β is a scale parameter. Besides, when $p = 0$ in the equation (8) then the equation yield the PDF of LBNak distribution, and also when $p = 1$ equation (8) then the equation yield the PDF of Nak distribution.

3.2. The Cumulative Density Function of the New Mixture of Nakagami Distribution

Suppose $F(x_1)$ be a cumulative density function (CDF) of Nak distribution and $F(x_2)$ be a cumulative density function of LBNak distribution. Let X be a random variable of the new mixture of Nakagami distribution, then the CDF of X is given by

$$F_p(x) = p(F(x_1)) + (1-p)F(x_2). \tag{9}$$

From (9), the CDF can be expressed as

$$F_p(x) = \int_0^x \left\{ \frac{2\lambda^\lambda}{\beta^\lambda} t^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta}t^2\right) \left[\frac{p}{\Gamma(\lambda)} + \frac{t(1-p)\sqrt{\lambda}}{\Gamma(\lambda+(1/2))\sqrt{\beta}} \right] \right\} \partial t. \tag{10}$$

Let

$$k = \frac{\lambda}{\beta}t^2 \Rightarrow \partial t = \frac{\partial k \beta}{2\lambda t}. \tag{11}$$

Substituting equation (11) into (10) and integrate with respect to k, yields the cumulative density function of X in equation (13).

$$F_p(x) = \frac{p}{\Gamma(\lambda)} \int_0^x k^{\lambda-1} \mathrm{e}^{-k} \partial k + \frac{(1-p)}{\Gamma(\lambda+1/2)} \int_0^x k^{\lambda+(1/2)-1} \mathrm{e}^{-k} \partial k. \tag{12}$$

$$= p\left[\gamma_1\left(\lambda, \frac{\lambda}{\beta}x^2\right)\right] + (1-p)\left[\gamma_2\left((\lambda+(1/2)), \frac{\lambda}{\beta}x^2\right)\right] \tag{13}$$

where γ_1 and γ_2 are the lower incomplete gamma function.

3.3. The r^{th} Ordinary Moment of the New Mixture of Nakagami Distribution

The r^{th} ordinary moment of the new mixture of Nakagami distribution is defined by

$$E(X^r) = \int_0^\infty \left\{ \frac{2\lambda^\lambda}{\beta^\lambda} x^{2\lambda+r-1} \exp\left(\frac{-\lambda}{\beta}x^2\right) \left[\frac{p}{\Gamma(\lambda)} + \frac{x(1-p)\sqrt{\lambda}}{\Gamma(\lambda+(1/2))\sqrt{\beta}} \right] \right\} \partial x. \tag{14}$$

Substituting equation (11) into (14) and integrate with respect to k, yields the r^{th} ordinary moment of the new mixture of Nakagami distribution in equation (15) .

$$E(X^r) = \frac{p\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda)} \int_0^\infty k^{\lambda+(r/2)-1} \mathrm{e}^{-k} + \frac{(1-p)\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda+(1/2))} \int_0^\infty k^{\lambda+\frac{r-1}{2}} \mathrm{e}^{-k} \partial x,$$

$$= \frac{p\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda)} \Gamma\left(\lambda+\frac{r}{2}\right) + \frac{(1-p)\beta^{\frac{r+1}{2}}}{\lambda^{\frac{r+1}{2}}\Gamma(\lambda+(1/2))} \Gamma\left(\lambda+\left(\frac{r+1}{2}\right)\right) \quad r = 1, 2, \ldots. \tag{15}$$

The mean of the new mixture of Nakagami distribution takes the form:

$$E(X)=\frac{p\beta^{1/2}}{\lambda^{1/2}\Gamma(\lambda)}\Gamma\left(\lambda+\frac{1}{2}\right)+\frac{(1-p)\beta}{\lambda\Gamma(\lambda+(1/2))}\Gamma\left(\lambda+1\right). \tag{16}$$

$E(X^2)$ is the second moment obtained from equation (15) when $r=2$.

$$E(X^2)=\frac{p\beta}{\lambda\Gamma(\lambda)}\Gamma\left(\lambda+1\right)+\frac{(1-p)\beta^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}\Gamma(\lambda+(1/2))}\Gamma\left(\lambda+\frac{3}{2}\right). \tag{17}$$

The variance is obtained from equation (16) and (17)

$$\begin{aligned}Var(X)=&\frac{p\beta}{\lambda\Gamma(\lambda)}\Gamma\left(\lambda+1\right)+\frac{(1-p)\beta^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}\Gamma(\lambda+(1/2))}\Gamma\left(\lambda+\frac{3}{2}\right)\\&-\left[\frac{p\beta^{1/2}}{\lambda^{1/2}\Gamma(\lambda)}\Gamma\left(\lambda+\frac{1}{2}\right)+\frac{(1-p)\beta}{\lambda\Gamma(\lambda+(1/2))}\Gamma\left(\lambda+1\right)\right]^2.\end{aligned} \tag{18}$$

3.4. The Moment Generating Function of the New Mixture of Nakagami Distribution

The moment generating function of the new mixture of Nakagami distribution is given by

$$E(e^{Xt})=M_X(t)=\sum_{r=0}^{\infty}\frac{t^rE(X^r)}{r!}. \tag{19}$$

By inserting equation (15) into (19) yield the moment generating function of the new mixture of Nakagami distribution in equation (20).

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left\{\frac{p\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda)}\Gamma\left(\lambda+\frac{r}{2}\right)+\frac{(1-p)\beta^{\frac{r+1}{2}}}{\lambda^{\frac{r+1}{2}}\Gamma(\lambda+(1/2))}\Gamma\left(\lambda+\left(\frac{r+1}{2}\right)\right)\right\}. \tag{20}$$

3.5. The Incomplete Moments of the New Mixture of Nakagami Distribution

The incomplete moments of the new mixture of Nakagami distribution is given by

$$m_r'(t)=\int_0^t\left\{\frac{2\lambda^\lambda}{\beta^\lambda}x^{2\lambda+r-1}\exp\left(\frac{-\lambda}{\beta}x^2\right)\left[\frac{p}{\Gamma(\lambda)}+\frac{x(1-p)\sqrt{\lambda}}{\Gamma(\lambda+(1/2))\sqrt{\beta}}\right]\right\}\partial x. \tag{21}$$

Substituting equation (11) into (21) and integrate with respect to k yields

$$\begin{aligned}E(X^r)&=\frac{p\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda)}\int_0^t k^{\lambda+(r/2)-1}e^{-k}+\frac{(1-p)\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda+(1/2))}\int_0^t k^{\lambda+\frac{r-1}{2}}e^{-k}\partial x\\&=\frac{p\beta^{r/2}}{\lambda^{r/2}\Gamma(\lambda)}\gamma\left[\left(\lambda+\frac{r}{2}\right),t\right]+\frac{(1-p)\beta^{\frac{r+1}{2}}}{\lambda^{\frac{r+1}{2}}\Gamma(\lambda+(1/2))}\gamma\left[\left(\lambda+\left(\frac{r+1}{2}\right)\right),t\right]\ r=1,2,\ldots.\end{aligned}$$

If $r=1,2$ then we obtained the first and the second incomplete moment.

3.6. Survival Function and Hazard Rate Function of the New Mixture of Nakagami Distribution

Let X is a continuous random variable with a cumulative density function $F(x)$ on the interval $[0,\infty)$. The survival function of X, can be written in this form:

$$S(x)=1-F(x). \tag{22}$$

By inserting equation (13) into equation (22) yield the survival function of the new mixture of Nakagami distribution in equation (23).

$$S_p(x) = 1 - \left\{ p \left[\gamma_1 \left(\lambda, \frac{\lambda}{\beta} x^2 \right) \right] + (1-p) \left[\gamma_2 \left((\lambda + (1/2)), \frac{\lambda}{\beta} x^2 \right) \right] \right\}. \tag{23}$$

The hazard rate function of X can be defined as:

$$h_p(x) = \frac{f(x)}{S(x)}.$$

Hence the hazard rate function of the new mixture of Nakagami distribution is

$$h_p(x) = \frac{\left\{ \frac{2\lambda^\lambda}{\beta^\lambda} x^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta} x^2 \right) \left[\frac{p}{\Gamma(\lambda)} + \frac{x(1-p)\sqrt{\lambda}}{\Gamma(\lambda+(1/2))\sqrt{\beta}} \right] \right\}}{1 - \left\{ p \left[\gamma_1 \left(\lambda, \frac{\lambda}{\beta} x^2 \right) \right] + (1-p) \left[\gamma_2 \left((\lambda + (1/2)), \frac{\lambda}{\beta} x^2 \right) \right] \right\}}, \tag{24}$$

which increases monotonically from 0 to ∞ as time increases from 0 to ∞ (see Figure 2).

3.7. Parameter Estimation of the New Mixture of Nakagami Distribution

The maximum likelihood estimators of the parameters of the new mixture of Nakagami distribution will be estimated in this section. Let $x_1, \ldots, x_n$ be a random sample of size n drawn from the new mixture of Nakagami distribution, then the likelihood function of the new mixture of Nakagami distribution is given by:

$$\ell(\lambda, \beta) = \ln \prod_{i=1}^{n} \left\{ \frac{2\lambda^\lambda}{\beta^\lambda} x_i^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta} x_i^2 \right) \left[\frac{p}{\Gamma(\lambda)} + \frac{x_i(1-p)\sqrt{\lambda}}{\Gamma(\lambda + (1/2))\sqrt{\beta}} \right] \right\}. \tag{25}$$

The log likelihood function in equation (26) is obtained by taking the log of equation (25).

$$\begin{aligned} \ell(\lambda, \beta) = {} & n \ln(2) + n\lambda \ln(\lambda) + (2\lambda - 1) \ln \sum_{i=1}^{n} x_i - \left(\frac{\lambda}{\beta} \right) \sum_{i=1}^{n} x_i^2 \\ & + \sum_{i=1}^{n} \ln \left[\frac{p}{\Gamma(\lambda)} + \frac{x_i(1-p)\sqrt{\lambda}}{\Gamma(\lambda + (1/2))\sqrt{\beta}} \right]. \end{aligned} \tag{26}$$

The maximum likelihood estimators can be obtained by take derivative of equation (26) with respect to λ, β, p and solving for λ, β, p.

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} &= n \ln \lambda + n + 2 \ln \sum_{i=1}^{n} x_i - \frac{1}{\beta} \sum_{i=1}^{n} x_i^2 \\ & \sum_{i=1}^{n} \left\{ \frac{2 \left[p\Psi(\lambda)\sqrt{\lambda}\Gamma(\lambda + \frac{1}{2})\sqrt{\beta} - x_i(p-1)\Gamma(\lambda) \left(\lambda\Psi\left(\lambda + \frac{1}{2}\right) - \frac{1}{2} \right) \right]}{\sqrt{\lambda} \left[2p\Gamma\left(\lambda + \frac{1}{2}\right)\sqrt{\beta} - 2x_i\sqrt{\lambda}\Gamma(\lambda)(p-1) \right]} \right\} = 0 \\ \frac{\partial \ell}{\partial \beta} &= \frac{\lambda}{\beta^2} \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \frac{(p-1)x_i\Gamma(\lambda)\sqrt{\lambda}}{2 \left[p\Gamma(\lambda + \frac{1}{2})\sqrt{\beta} - x_i\Gamma(\lambda)\sqrt{\lambda}(p-1) \right]} = 0 \\ \frac{\partial \ell}{\partial p} &= \frac{-x\sqrt{\lambda}\Gamma(\lambda) + \Gamma(\lambda + \frac{1}{2})\sqrt{\beta}}{p\Gamma(\lambda + \frac{1}{2})\sqrt{\beta} - x\sqrt{\lambda}\Gamma(\lambda)(p-1)} = 0 \end{aligned}$$

Because these equations are nonlinear equations, they can not be analytically solved but can be numerically solved through iterative methods. This article prefers the nlminb function in the R programming language for the maximum likelihood estimation (MLE) via the quasi-Newton method and the optim function for the MLE via the simulated annealing. For using the nlminb and the optim function, we mush to find the initial parameters λ, β, and p for a given sample that maximizes the likelihood function, likelihood function to be maximized, and parameter limits.

This paper generated random numbers of the new mixture of Nakagami distribution by using the accept-reject method through the Nakagami random number generation procedure from the nakagami package and the simulations were repeated 1,000 times for each model. Also, determining sample of size $n = 10, 30, 50, 100$, parameters $\lambda = 0.5, 1, 2$, $\beta = 0.5, 1, 3$, and give $p = 0.5$. All of the experiments were run on the R program version 4.0.5, set initial parameters for the nlminb function as c(0,0), and set the initial parameters for the optim function as c(0.2,0.1). Therefore, it shows the result from calculating 9 models of each method and each sample of size, see in Table 1-8.

Regarding the simulation results, it was observed that the maximum likelihood estimators via the quasi-Newton method worked well for parameter β, and the maximum likelihood estimators via simulated annealing worked well for both parameters λ and β and for all n. Hence, we can conclude that the maximum likelihood estimators via the optim function with simulated annealing are more efficient than the maximum likelihood estimators via the nlminb (quasi-Newton method) function in the R program.

Note that the bias will be less than Tables 1-8, if we select suitable initial parameters for each dataset but in this paper, we set the initial parameter to be the same for all datasets.

Table 1 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the quasi-Newton method $\hat{\lambda}_{MLE}$ and $\hat{\beta}_{MLE}$ for $n = 10$

λ	β	$\hat{\lambda}$	$\hat{\beta}$	Bias ($\hat{\lambda}$)	Bias($\hat{\beta}$)	MSE($\hat{\lambda}$)	MSE($\hat{\beta}$)
0.5	0.5	48.956	0.588	48.456	0.088	6312.395	0.064
	1	57.351	0.773	56.851	-0.227	6873.944	0.316
	3	62.922	1.530	62.422	-1.470	6441.232	4.902
1	0.5	74.515	0.582	73.515	0.082	10101.709	0.035
	1	98.450	0.895	97.450	-0.105	12614.077	0.272
	3	100.497	2.257	99.497	-0.743	10973.515	3.150
2	0.5	93.006	0.578	91.006	0.078	12833.254	0.025
	1	125.807	0.940	123.807	-0.060	16847.171	0.162
	3	117.110	2.682	115.110	-0.318	13428.995	1.716

4. Numerical Illustrations

To explore the potential of the proposed distribution, two actual data sets have been taken into consideration by using the AdequacyModel package in R, see R Core Team (2021a).

Illustration 1: The first data set depicts the fatigue fracture life of Kevlar 373/epoxy when subjected to steady pressure at 90 percent stress until all fail. Abdul-Moniem and Seham (2015) previously investigated this data set. It has 76 observations, which are as follows:
0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 2 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the simulated annealing $\tilde{\lambda}_{MLE}$ and $\tilde{\beta}_{MLE}$ for $n = 10$

λ	β	$\tilde{\lambda}$	$\tilde{\beta}$	Bias ($\tilde{\lambda}$)	Bias($\tilde{\beta}$)	MSE($\tilde{\lambda}$)	MSE($\tilde{\beta}$)
0.5	0.5	14.385	14.805	13.885	14.305	237.927	255.408
	1	12.749	13.641	12.249	12.641	212.151	250.768
	3	8.331	6.872	7.831	3.872	127.577	112.167
1	0.5	14.907	14.156	13.907	13.656	242.636	235.061
	1	12.141	13.017	11.141	12.017	176.612	228.706
	3	7.152	5.315	6.152	2.315	89.258	101.862
2	0.5	14.223	13.078	12.223	12.578	201.211	212.443
	1	12.055	11.682	10.055	10.682	165.664	209.125
	3	8.394	4.671	6.394	1.671	97.427	86.600

Table 3 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the quasi-Newton method $\hat{\lambda}_{MLE}$ and $\hat{\beta}_{MLE}$ for $n = 30$

λ	β	$\hat{\lambda}$	$\hat{\beta}$	Bias ($\hat{\lambda}$)	Bias($\hat{\beta}$)	MSE($\hat{\lambda}$)	MSE($\hat{\beta}$)
0.5	0.5	45.759	0.555	45.259	0.055	5227.054	0.024
	1	51.926	0.696	51.426	-0.304	5508.718	0.247
	3	59.112	1.369	58.612	-1.631	5591.255	4.341
1	0.5	66.334	0.556	65.334	0.056	8321.412	0.018
	1	101.067	0.866	100.067	-0.134	12165.150	0.181
	3	102.221	2.317	101.221	-0.683	10685.636	2.302
2	0.5	92.389	0.557	90.389	0.057	12285.127	0.014
	1	127.711	0.903	125.711	-0.097	16646.267	0.107
	3	115.994	2.737	113.994	-0.263	13146.579	1.027

Table 4 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the simulated annealing $\tilde{\lambda}_{MLE}$ and $\tilde{\beta}_{MLE}$ for $n = 30$.

λ	β	$\tilde{\lambda}$	$\tilde{\beta}$	Bias ($\tilde{\lambda}$)	Bias($\tilde{\beta}$)	MSE($\tilde{\lambda}$)	MSE($\tilde{\beta}$)
0.5	0.5	14.732	17.246	14.232	16.746	250.208	332.879
	1	13.229	17.635	12.729	16.635	217.652	356.283
	3	6.363	9.393	5.863	6.393	100.138	168.960
1	0.5	14.660	15.211	13.660	14.711	239.453	271.241
	1	12.452	14.007	11.452	13.007	187.928	258.724
	3	7.348	5.368	6.348	2.368	90.305	109.646
2	0.5	13.929	13.355	11.929	12.855	194.887	233.573
	1	11.353	10.741	9.353	9.741	145.153	201.359
	3	9.159	3.308	7.159	0.308	100.488	69.059

Table 5 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the quasi-Newton method $\hat{\lambda}_{MLE}$ and $\hat{\beta}_{MLE}$ for $n = 50$.

λ	β	$\hat{\lambda}$	$\hat{\beta}$	Bias ($\hat{\lambda}$)	Bias($\hat{\beta}$)	MSE($\hat{\lambda}$)	MSE($\hat{\beta}$)
0.5	0.5	47.844	0.553	47.344	0.053	5276.343	0.019
	1	53.412	0.659	52.912	-0.341	5423.619	0.221
	3	59.201	1.285	58.701	-1.715	5424.683	4.093
1	0.5	73.046	0.544	72.046	0.044	8959.075	0.011
	1	101.009	0.858	100.009	-0.142	11748.083	0.161
	3	101.758	2.287	100.758	-0.713	10420.076	2.015
2	0.5	95.992	0.541	93.992	0.041	12699.802	0.008
	1	128.565	0.883	126.565	-0.117	16534.429	0.097
	3	113.252	2.715	111.252	-0.285	12522.042	0.994

Table 6 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the simulated annealing $\tilde{\lambda}_{MLE}$ and $\tilde{\beta}_{MLE}$ for $n = 50$.

λ	β	$\tilde{\lambda}$	$\tilde{\beta}$	Bias ($\tilde{\lambda}$)	Bias($\tilde{\beta}$)	MSE($\tilde{\lambda}$)	MSE($\tilde{\beta}$)
0.5	0.5	12.801	15.771	12.301	15.271	221.083	304.841
	1	10.519	15.995	10.019	14.995	184.279	351.346
	3	5.066	8.809	4.566	5.809	77.166	158.901
1	0.5	10.951	12.549	9.951	12.049	179.015	220.565
	1	7.731	9.769	6.731	8.769	118.264	179.072
	3	4.882	3.618	3.882	0.618	46.574	62.679
2	0.5	10.312	10.274	8.312	9.774	146.608	175.622
	1	6.810	5.806	4.810	4.806	76.068	90.462
	3	7.559	1.760	5.559	-1.240	70.164	32.591

Table 7 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the quasi-Newton method $\hat{\lambda}_{MLE}$ and $\hat{\beta}_{MLE}$ for $n = 100$.

λ	β	$\hat{\lambda}$	$\hat{\beta}$	Bias ($\hat{\lambda}$)	Bias($\hat{\beta}$)	MSE($\hat{\lambda}$)	MSE($\hat{\beta}$)
0.5	0.5	47.171	0.538	46.671	0.038	4852.809	0.012
	1	55.248	0.647	54.748	-0.353	5436.870	0.207
	3	63.634	1.306	63.134	-1.694	5658.259	3.832
1	0.5	72.891	0.539	71.891	0.039	8367.096	0.008
	1	102.453	0.823	101.453	-0.177	11471.591	0.130
	3	100.155	2.322	99.155	-0.678	9938.825	1.490
2	0.5	95.567	0.534	93.567	0.034	12259.945	0.005
	1	127.249	0.898	125.249	-0.102	16004.806	0.070
	3	113.105	2.677	111.105	-0.323	12479.566	0.642

Table 8 The average estimates, the bias, the mean squared errors, and the simulated variance of the maximum likelihood estimators via the simulated annealing $\tilde{\lambda}_{MLE}$ and $\tilde{\beta}_{MLE}$ for $n = 100$.

λ	β	$\tilde{\lambda}$	$\tilde{\beta}$	Bias $(\tilde{\lambda})$	Bias$(\tilde{\beta})$	MSE$(\tilde{\lambda})$	MSE$(\tilde{\beta})$
0.5	0.5	12.049	15.249	11.549	14.749	218.982	312.654
	1	8.384	13.209	7.884	12.209	153.024	301.827
	3	4.210	7.561	3.710	4.561	63.616	171.043
1	0.5	8.491	10.046	7.491	9.546	147.107	178.373
	1	3.380	4.931	2.380	3.931	49.570	81.970
	3	3.113	1.240	2.113	-1.760	18.072	18.058
2	0.5	7.839	8.034	5.839	7.534	118.693	131.934
	1	3.792	2.389	1.792	1.389	32.183	31.393
	3	6.656	0.453	4.656	-2.547	50.977	10.839

From Figure 3, it clear that the data is right-skewed data. Hence, the Nakagami distribution (Nak), the length-biased Nakagami distribution (LBNak), the new mixture of Nakagami distribution (MNak), and the Nakagami exponential distribution (NakExp) which proposed by Abdullahi and Obalowu (2020), have been fitted to data by MLE via the simulated annealing from the Adequacy-Model package. The parameter estimates, Akaike's information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS) test, and Anderson-Darling (AD) test are exhibited in Table 9. Based on the results, it clearly suggests that the new mixture of Nakagami distribution outperforms the other three competing models.

Table 9 The MLE of the model parameters for the prostate cancer data, AIC measure, BIC measure, KS test, and AD test

Fitting Dist.	Estimate parameters			AIC	BIC	KS test		AD
	λ	β	p			Statistic	p-value	Statistic
Nak	0.3197	36.4452	-	312.1152	316.7767	0.3770	0.0000	0.8414
LBNak	0.2615	17.5919	-	402.1553	406.8168	0.6425	0.0000	0.8420
MNak	0.5529	7.4346	1.1669	255.1742	262.1664	0.1429	0.0812	1.1252
NakExp	0.2182	32.8661	0.3904	317.0142	324.0064	0.3576	0.0000	4.3393

Illustration 2: The second data set is about survival times in months after mastectomy of women with breast cancer without censored observation from Everitt and Rabe-Hesketh (2001), and R Core Team (2021b), which are as follows:
23, 47, 69, 148, 181, 5, 8, 10, 13, 18, 24, 26, 26, 31, 35, 40, 41, 48, 50, 59, 61, 68, 71, 113, 118, 143.
From Figure 4, it clear that this data is right-skewed data. Again, models like the Nakagami distribution (Nak), the length-biased Nakagami distribution (LBNak), the new mixture of Nakagami distribution (MNak), and the Nakagami exponential distribution (NakExp) are being fitted to the given data set and the parameters of each model were estimated by MLE via the simulated annealing. In order to test the goodness of fit, AIC statistic, BIC statistic, Kolmogorov-Smirnov test, and Anderson-Darling test have been employed. Based on the results of AIC statistics and Kolmogorov-Smirnov test that there exists enough statistical evidence that the length-biased Nakagami distribution, the new mixture of Nakagami distribution, and the Nakagami exponential distribution fits the breast cancer data very well (Table 10).

Table 10 The MLE of the model parameters for the survival times after mastectomy for cancer, AIC measure, BIC measure, KS test, and AD test

Fitting Dist.	Estimate parameters			AIC	BIC	KS test		AD
	λ	β	p			Statistic	p-value	Statistic
Nak	0.0054	30.6568	-	480.3764	482.8925	0.9741	0.0000	1.1603
LBNak	0.0026	27.9594	-	264.9343	267.4505	0.1436	0.6575	0.4303
MNak	0.0028	29.4391	0.3*	283.0474	285.5636	0.3331	0.0062	1.0654
NakExp	0.2364	48.8641	0.0179	278.6505	282.4248	0.3228	0.0089	1.4514

*fitted value

5. Conclusions

This article proposes a new distribution base upon Nakagami distribution, establishes some attributes of the proposed distribution. A simulation study for performing the maximum likelihood estimators via the quasi-Newton method and the simulated annealing were compared by Bias, and mean squared error (MSE). To illustrate its usefulness, the model has been applied in describing real data sets and compared to an existing distribution. The proposed model can be applied in a wide variety of real situations, including reliability and survival data, especially the “small” values data.

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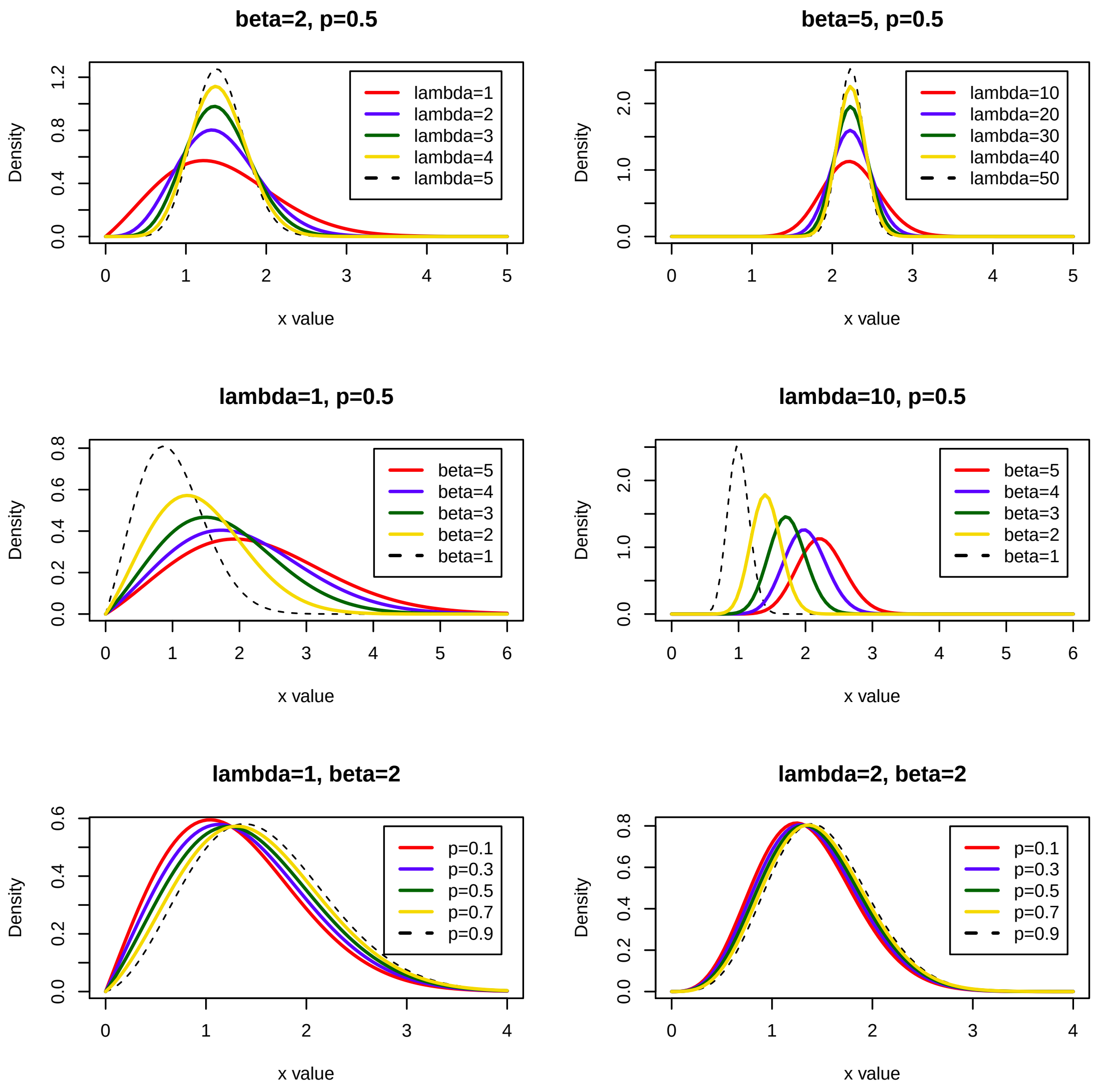

Figure 1 PDF plot for different values of the parameter λ, β, and p

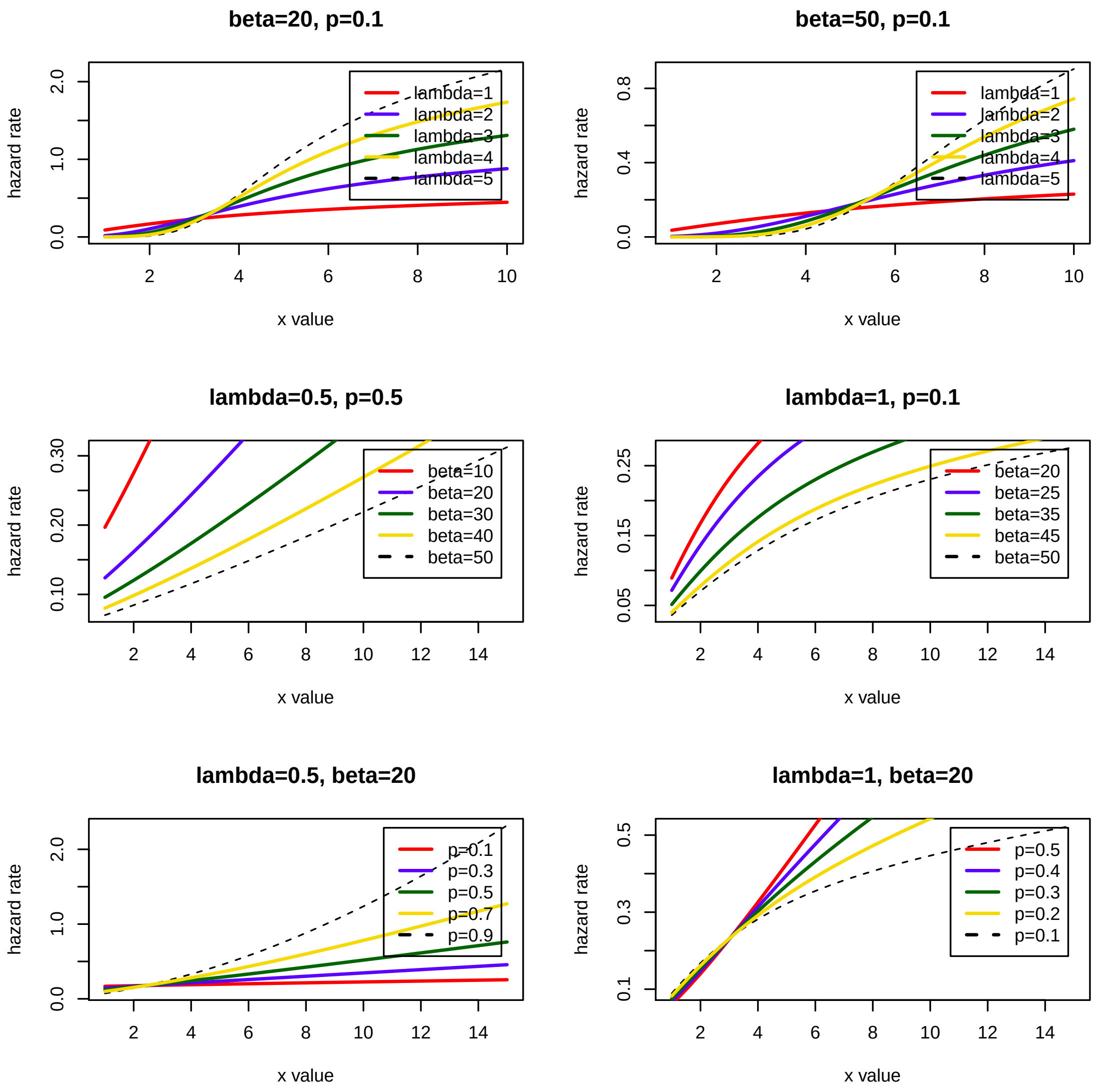

Figure 2 The hazard rate function plot for different values of the parameter λ, β, and p

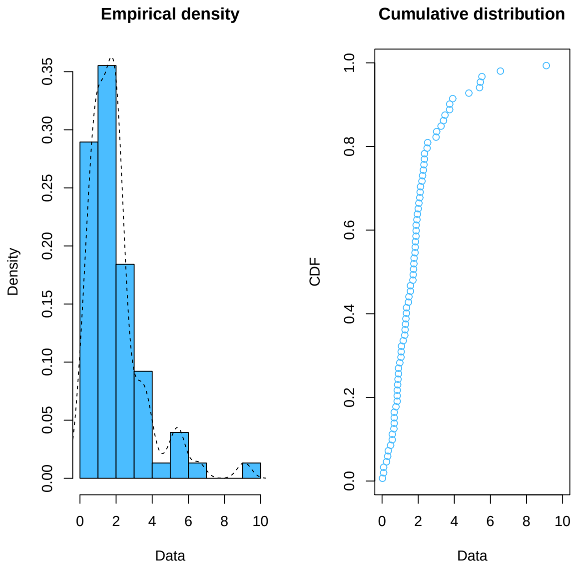

Figure 3 Histogram and distribution function plots of the empirical distribution of the fatigue fracture life of Kevlar 373/epoxy

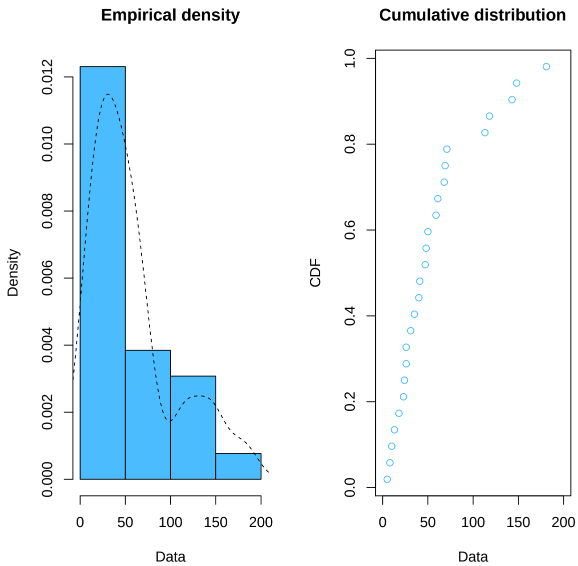

Figure 4 Histogram and distribution function plots of the empirical distribution of the survival times after mastectomy