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Analyzing the Prevalence of Overweight and Obesity of Pakistani Females in Bayesian Paradigm

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Abstract

This study analyzed the prevalence of over weight and obesity of Pakistani female in Bayesian paradigm. For the modeling of this data set Rayleigh-Rayleigh distribution (RRD) is used. The posterior distributions are evaluated using uniform, Jeffreys and exponential priors. These posterior distributions are not attained in closed form. Two approximation techniques, Lindley and Tierney-Kadane(T-K) are used to obtain the Bayes estimators under three different loss functions (squared error, Weighted and Precautionary loss functions). Monte Carlo simulation study and real life data about the prevalence of over weight and obesity of the Pakistani female is used to show the superiority of Bayes estimators over the maximum likelihood (ML) estimators. It is concluded that Bayes estimators under informative priors perform better than non informative priors due to minimum associated risks. It is also found that estimators obtained through Bayesian technique are better than most common ML method.

Keywords: Bayesian estimation, Lindley approximation, overweight, Pakistani female, Tierney-Kadane method

1. Introduction

Obesity is the excessive fat of a person that presents risk to health. Body mass index (BMI), mid upper arm circumference (MUAC), waist and hip circumference are commonly used screening tools of obesity and over weight. Chronic diseases like diabetes, cardiovascular, hypertensions and cancer are main reasons of over weight and obesity.

Friedrich (2002) stated that in developing countries, the epidemic of obesity is very rapid. Increase in urbanization, change in life style, economic growth, excessive food intake and lack of physical activities are major reason of this epidemic. Jafar et al. (2006) studied the prevalence of overweight and obesity in Pakistan and BMI cutoff values associated with hypertension and diabetes mellitus. Hajian-Tilaki and Heidari (2007) conducted cross sectional study to determine the prevalence rate of obesity and its associated factors in Iran. Hou et al. (2008) showed the prevalence of excessive weight in Shanghai metropolis population and its risk factors. Befort et al. (2012) analyzed the obesity prevalence from adults of urban and rural areas in the United States and its association with demographic, eating habits and physical activities are studied. Over the past three decades, a dramatic increase in excessive weight among Canadian is recorded. Twells et al. (2014) presented data about past and current excessive weight in Canada. It is predicted that obesity will continuously increase all over the country. Asif et al. (2020) examined current prevalence of overweight and obesity in Pakistani adults and sociodemographic factors associated with body weight are marital status, gender and residential areas. For this purpose, secondary data from 10,063 Pakistani adult is taken through population-based household survey. About women, it is concluded that 6.3% were obese and 23.9% were overweight.

Analysis and modeling of real life phenomena is an important aspect of statistics. Due to complexity, diversity and variation in real world, large number of statistical distributions and their generalizations are derived and studied. Parameters are the most important feature of a distribution, described the specific characteristic of the phenomena. That is why statistician are in effort of estimation of these parameters.

Rayleigh distribution was proposed by Lord Rayleigh. It is used to model the wave height of ocean, wind speed, random complex numbers which are composed of real and imaginary part and independently and identically normally distributed with zero mean and equal variances. The absolute value of the complex number follows Rayleigh distribution.

Kundu and Raqab (2005) estimated the parameters of generalized Rayleigh distribution using methods of maximum likelihood, modified moment, percentile, least square, weighted least square and modified L-moments. These estimates are compared through simulation study. Mousa and Al-Sagheer (2005) considered Bayesian prediction bounds from informative progressively type-II right censored sample from Rayleigh distribution. Merovci (2014) proposed the transmuted generalized Rayleigh distribution and discussed some of its properties like moments, order statistics, reliability analysis. The real life data is used to study the usefulness of the distribution. Dey et al. (2016) considered point and interval estimation of progressive Type-II censored two-parameter Rayleigh distribution. MLEs are estimated using profile log-likelihood function. Approximate Bayes estimators are obtained through importance sampling technique.

Ateeq et al. (2019) derived Rayleigh-Rayleigh distribution (RRD) and studied some properties of the distribution like moments, moment generating function, Shannon and Rényi entropies, order statistics, L-moments. Parameters are estimated through method of maximum likelihood technique. The Bayesian estimation of parameters of the distribution has not been addressed so far.

In this study, we proposed Bayesian estimation of the parameters of RRD using two approximation techniques, Lindley approximation by Lindley (1980) and Tirney-Kadane (T-K) method by Tierney and Kadane (1986). Square error loss function (SELF), weighted loss function (WLF) and precautionary loss function (PLF) are used to derive Bayes estimators and posterior risk functions. Uniform, Jeffreys and exponential priors are utilized for derivation of the posterior distributions. A real life data set about the prevalence of overweight and obesity of Pakistani female is taken from the (https://data.worldbank.org/country/pakistan). The Bayes estimators of the parameters of distribution are compared with MLEs. The purpose of study is to motivate practitioners toward Bayesian frame work.

Rest of the paper is organized as follows. Section 2 provides estimation techniques of the parameters of RRD in classical and Bayesian paradigm. In Section 3, the posterior distributions are derived under uniform, Jeffreys and exponential priors. In Section 4, the Bayes estimators under three symmetry and asymmetry loss functions are derived using Lindley and T-K approximation techniques. The article is concluded in Section 5.

2. Estimation of Rayleigh-Rayleigh Distribution

Ateeq et al. (2019) derived the generalization of Rayleigh distribution named as Rayleigh-Rayleigh distribution (RRD) by using Transformed-Transformed technique proposed by Alzaatreh et al. (2013). By adding another scale parameter in the Rayleigh distribution, the generalized distribution is more flexible for complexed real-life phenomena.

Let the random variable X follow to the RRD having probability density function (PDF) and cumulative density function (CDF) as

$$g(x) = \frac{x^3}{2\beta^4\sigma^2} e^{-\frac{x^4}{8\beta^4\sigma^2}}, \qquad x, \beta, \sigma > 0$$

$$G(x) = 1 - e^{-\frac{x^4}{8\beta^4\sigma^2}}, \qquad x, \beta, \sigma > 0.$$

Both β and σ are the scale parameters.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from RRD. The likelihood and log likelihood function of the distribution are:

$$L(\beta, \sigma; x) = \frac{1}{(2\beta^4\sigma^2)^n} \prod_{i=1}^{n} x_i^3 e^{-\frac{1}{8\beta^4\sigma^2}\sum x_i^4} \tag{1}$$

$$l(\beta, \sigma; x) = \sum_{i=1}^{n} \log x^3 - n\log 2 - 4n\log\beta - 2n\log\sigma - \frac{\sum_{i=1}^{n} x_i^4}{8\beta^4\sigma^2}. \tag{2}$$

Equating score functions to zero, two normal equations are obtained.

$$\frac{\partial L(\beta, \sigma; x)}{\partial\beta} = -\frac{4n}{\beta} + \frac{\sum x_i^4}{2\beta^5\sigma^2} = 0$$

$$\frac{\partial L(\beta, \sigma; x)}{\partial\sigma} = -\frac{2n}{\sigma} + \frac{\sum x_i^4}{4\beta^4\sigma^3} = 0.$$

These normal equation cannot be solved simultaneously to obtain MLEs.

In this study, $maxLik$ package in R language is utilized for MLEs through Newton Raphson iterative process and Hessian matrix is also obtained. Information matrix is the negative expectation of Hessian matrix.

2.1. Bayesian estimation

In Bayesian paradigm, probability distribution is assigned not only to observed data but also to parameters which are considered random variables. Bayes rule is formal method to combine the prior knowledge about parameter with the sample information contained in the likelihood to provide posterior distributions, which gives full answer to any Bayesian problem.

2.2. Prior and posterior distributions

The distinctive feature of Bayesian approach is prior distribution, based on past studies or opinion of subject area experts. Calculation of posterior distribution is the main goal of Bayesian statistics. Bayesian approach makes use of expert's prior knowledge about the parameters. When this prior information is not available, it is possible to make use of the non-informative priors. Uniform and Jeffreys are two commonly used non-informative priors.

Laplace (1820) proposed the principal that uniform distribution can be used as prior when information about parameters is not available. This prior is not invariant under re-parameterization.

In this study, the prior distributions for the parameters β and σ are uniform, Jeffreys and exponential. Assuming the independence of prior,the joint uniform prior distribution for the parameters β and σ is;

$$\pi(\beta, \sigma) \propto 1. \tag{3}$$

The joint posterior distribution of the parameters β and σ using equations (1) and (3) is;

$$\pi(\beta, \sigma|\mathbf{x}) = \frac{\frac{1}{(2\beta^4\sigma^2)^n} \prod_{i=1}^{n} x_i^3 e^{-\frac{1}{8\beta^4\sigma^2}\sum x_i^4}}{\int_0^\infty \int_0^\infty \frac{1}{(2\beta^4\sigma^2)^n} \prod_{i=1}^{n} x_i^3 e^{-\frac{1}{8\beta^4\sigma^2}\sum x_i^4} d\beta d\sigma} \qquad \beta, \sigma > 0. \tag{4}$$

Table 1 Bayes Estimators and Posterior Risk under different Loss functions

Loss Function	Bayes Estimators	Posterior Risk
SELF	$E(\theta\|\mathbf{x})$	$E(\theta^2\|\mathbf{x})$ -$[E(\theta\|\mathbf{x})]^2$
WLF	$E(\theta^{-1}\|\mathbf{x})^{-1}$	$E(\theta\|\mathbf{x}) - [E(\theta^{-1}\|\mathbf{x})]^2$
PLF	$\sqrt{E(\theta^2\|\mathbf{x})}$	$2\left[\sqrt{E(\theta^2\|\mathbf{x})}\right.$ -$\left.E(\theta\|\mathbf{x})\right]$

Jeffreys (1946) proposed an invariant prior probability distribution, which capture all the information about parameters from sample. For parameter θ, the Jeffreys' prior is defined as;

$$\pi(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ is Fisher information of parameter θ, defined as;

$$I(\theta) = -nE\left[\frac{\partial^2 f(y;\theta)}{\partial\theta^2}\right].$$

Assuming the independence of priors, the joint prior distribution for the parameters of RRD is defined as;

$$\pi(\beta,\sigma) \propto \frac{1}{\beta}\frac{1}{\sigma} \qquad \beta,\sigma > 0. \tag{5}$$

By using Equations (5) and (1) the posterior distribution is given as;

$$\pi(\beta,\sigma|\mathbf{x}) = \frac{\frac{1}{2^n\beta^{4n+1}\sigma^{2n+1}}\prod_{i=1}^n x_i^3 \exp(-\frac{1}{8\beta^4\sigma^2}\sum x_i^4)}{\int_0^\infty\int_0^\infty \frac{1}{2^n\beta^{4n+1}\sigma^{2n+1}}\prod_{i=1}^n x_i^3 \exp(-\frac{1}{8\beta^4\sigma^2}\sum x_i^4)d\beta d\sigma} \qquad \beta,\sigma > 0. \tag{6}$$

An informative prior leads to reduction of posterior risk of the Bayes estimators because the use of prior information is equal to adding a number of observations to the given sample size. Sindhu et al. (2013) used inverse levy and gamma prior for the Bayesian estimation of Kumaraswamy distribution under failure censoring sampling scheme. The joint prior distribution for the parameters of RRD using exponential distribution is:

$$\pi(\beta,\sigma) \propto e^{-\beta a_1 - \sigma a_2}, \qquad \beta,\sigma,a_1,a_2 > 0 \tag{7}$$

where a_1, a_2 are hyper parameters By using (7) and (1) the posterior distribution under exponential prior is:

$$\pi(\beta,\sigma|\mathbf{x}) = \frac{\frac{1}{(2\beta^4\sigma^2)^n}\prod_{i=1}^n x_i^3 \exp(-\frac{1}{8\beta^4\sigma^2}\sum x_i^4 - a_1\beta - a_2\sigma)}{\int_0^\infty\int_0^\infty \frac{1}{(2\beta^4\sigma^2)^n}\prod_{i=1}^n x_i^3 \exp(-\frac{1}{8\beta^4\sigma^2}\sum x_i^4 - a_1\beta - a_2\sigma)} \qquad \beta,\sigma > 0. \tag{8}$$

Equations (4), (6) and (8) are not in closed form. So we cannot determine marginal posterior distributions, Bayes estimators and posterior risk of β and σ.

3. Bayes Estimators and Posterior Risk

Loss functions are the basic tool in decision theory, used to check goodness of an estimators. Bayes estimators are the values obtained by minimizes the average of loss function.

In this study, Bayes estimators of the parameters of RRD are evaluated under SELF, WLF and PLF. The Bayes estimators and posterior risk of parameter θ under defined loss function is shown in Table 1.

Bayes estimators and risk function are evaluated by taking posterior expectations of function of parameters $U(\beta,\sigma)$.

$$\tilde{U}(\beta,\sigma) = E_{(\beta,\sigma|\mathbf{x})}[U(\beta,\sigma)] = \frac{\int_0^\infty \int_0^\infty U(\beta,\sigma)L(\beta,\sigma;x)\pi(\beta,\sigma)d\beta d\sigma}{\int_0^\infty \int_0^\infty L(\beta,\sigma;x)\pi(\beta,\sigma)d\beta d\sigma}, \quad \beta,\sigma > 0. \quad (9)$$

It is tedious to solve due to ratio of two integrals. We used Lindley and T-K approximation techniques for this purpose and estimators obtained are compared through minimum values of risk.

3.1. Lindley's approximation technique

In this section, three loss function are used for the derivation of Bayes estimators and posterior risk of unknown parameters β and σ under the exponential prior. Equation (9) can be written as

$$\begin{aligned} E_{(\beta,\sigma|\mathbf{x})}[U(\beta,\sigma)] &= \frac{\int_0^\infty \int_0^\infty U(\beta,\sigma)e^{l(\beta,\sigma;x)+\rho(\beta,\sigma)}d\beta d\sigma}{\int_0^\infty \int_0^\infty e^{l(\beta,\sigma;x)+\rho(\beta,\sigma)}d\beta d\sigma} \\ &= \frac{\int_0^\infty \int_0^\infty U(\beta,\sigma)e^{Q(\beta,\sigma)}d\beta d\sigma}{\int_0^\infty \int_0^\infty e^{Q(\beta,\sigma)}d\beta d\sigma} \end{aligned} \quad (10)$$

where Q(β,σ)=$l(\beta,\sigma;x)$+$\rho(\beta,\sigma)$ and $\rho(\beta,\sigma) = ln\pi(\beta,\sigma)$. Using Lindley approximation technique, for sufficient large sample size Equation (10) can be approximated as

$$\begin{aligned} \tilde{U}(\beta,\sigma) \approx & U(\hat{\beta},\hat{\sigma}) + \frac{1}{2}\sum_i\sum_j u_{ij}\sigma_{ij} + \sum_j U_j\rho_j + \frac{1}{2}\Big[L_{30}\sigma_{11}U_1 + L_{12}(2\sigma_{12}U_1 + \sigma_{11}U_2) \\ & + L_{12}(\sigma_{22}U_1 + 2\sigma_{12}U_2) + L_{03}\sigma_{22}U_2\Big], \qquad i,j = 1,2 \end{aligned} \quad (11)$$

where $\hat{\beta}$ and $\hat{\sigma}$ are MLEs of β and σ. The elements of variance covariance matrix σ_{ij} can be obtained by taking the inverse of Fisher information matrix. Also

$$\begin{aligned} u_1 &= \frac{\partial U(\beta,\sigma)}{\partial\beta}, \qquad u_2 = \frac{\partial U(\beta,\sigma)}{\partial\sigma} \\ u_{11} &= \frac{\partial^2 U(\beta,\sigma)}{\partial\beta^2}, \qquad u_{22} = \frac{\partial^2 U(\beta,\sigma)}{\partial\sigma^2}, \qquad u_{12} = u_{21} = \frac{\partial^2 U(\beta,\sigma)}{\partial\beta\partial\sigma} \\ U_1 &= u_1\sigma_{11} + u_2\sigma_{12}, \qquad U_2 = u_1\sigma_{21} + u_2\sigma_{22} \\ \rho_1 &= \frac{\partial\rho(\beta,\sigma)}{\partial\beta} = -\frac{1}{\beta}, \qquad \rho_2 = \frac{\partial\rho(\beta,\sigma)}{\partial\sigma} = -\frac{1}{\sigma} \qquad \text{(for exponential prior)} \\ L_{12} &= \frac{\partial^3 l(\beta,\sigma)}{\partial\beta\partial\sigma^2} = \frac{3\sum x_i^4}{\sigma^4\beta^5} \\ L_{21} &= \frac{\partial^3 l(\beta,\sigma)}{\partial\sigma\partial\beta^2} = \frac{5\sum x_i^4}{\beta^6\sigma^3} \\ L_{30} &= \frac{\partial^3 l(\beta,\sigma)}{\partial\beta^3} = \frac{-8n}{\beta^3} + \frac{15\sum x_i^4}{\beta^7\sigma^2} \\ L_{03} &= \frac{\partial^3 l(\beta,\sigma)}{\partial\sigma^3} = \frac{-4n}{\sigma^3} + \frac{3\sum x_i^4}{\beta^4\sigma^5}. \end{aligned}$$

The parameters of RRD are estimated using Equation (11) by taking different value of $U(\beta,\sigma)$ for different loss functions as defined in Table 1.

3.1.1 Under square error loss function (SELF)

Bayes estimator and posterior risk for the parameter β under SELF using $U(\beta,\sigma)=\beta$, are;

$$\tilde{\beta}_{LS} = \hat{\beta}+\sigma_{11}\rho_1+\sigma_{21}\rho_2+\frac{1}{2}\Big[L_{30}\sigma_{11}^2+L_{21}(3\sigma_{11}\sigma_{12})+L_{12}(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)+L_{03}(\sigma_{22}\sigma_{21})\Big]$$

$$\rho_{\beta(LS)} = \Big[\hat{\beta}^2+\sigma_{11}+2\hat{\beta}(\sigma_{11}\rho_1+\sigma_{21}\rho_2)+\frac{\hat{\beta}}{2}\big[L_{30}(2\sigma_{11}^2)+L_{21}(6\sigma_{11}\sigma_{21})+L_{12}(2\sigma_{11}\sigma_{22}+4\sigma_{12}^2) + L_{03}(2\sigma_{12}\sigma_{22})\big]\Big]-\Big[\hat{\beta}+\sigma_{11}\rho_1+\sigma_{21}\rho_2+\frac{1}{2}\big[L_{30}\sigma_{11}^2+L_{21}(3\sigma_{11}\sigma_{12})+L_{12}(\sigma_{11}\sigma_{22}+2\sigma_{12}^2) + L_{03}(\sigma_{22}\sigma_{21})\big]\Big]^2.$$

3.1.2 Under weighted loss function (WLF)

Bayes estimator and posterior risk for the parameter β under WLF using $U(\beta,\sigma)=\beta^{-1}$, are;

$$\tilde{\beta}_{LW} = \Big[\hat{\beta^{-1}}+\hat{\beta^{-3}}\sigma_{11}-\hat{\beta^{-2}}(\sigma_{11}\rho_1+\sigma_{21}\rho_2)-\frac{\hat{\beta^{-2}}}{2}\big[L_{30}\sigma_{11}^2+3L_{21}\sigma_{11}\sigma_{12}+L_{12}(\sigma_{22}\sigma_{11}+2\sigma_{12}^2) + L_{03}\sigma_{22}\sigma_{21}\big]\Big]^{-1}$$

$$\rho_{\beta(LW)} = \tilde{\beta}_{LS}-\tilde{\beta}_{LW}.$$

3.1.3 Under precautionary loss function (PLF)

Bayes estimator and posterior risk for the parameter β under PLF using $U(\beta,\sigma)=\beta^2$, are;

$$\tilde{\beta}_{LP} = \Big[\hat{\beta}^2+\sigma_{11}+2\hat{\beta}(\sigma_{11}\rho_1+\sigma_{21}\rho_2)+\frac{\hat{\beta}}{2}\big[2L_{30}\sigma_{11}^2+6L_{21}\sigma_{11}\sigma_{21} + L_{12}(2\sigma_{11}\sigma_{22}+4\sigma_{12}^2)+2L_{03}\sigma_{12}\sigma_{22}\big]\Big]^{1/2}$$

$$\rho_{\beta(LP)} = 2[\tilde{\beta}_{LP}-\tilde{\beta}_{LS}].$$

Similarly Bayes estimators and function of posterior risk for other parameter σ can be derive. For different priors, values of ρ_1 and ρ_2 are used as:

$$\begin{aligned} &\rho_1=\rho_2=0, && \text{for uniform prior}\\ &\rho_1=-\frac{1}{\beta},\rho_2=-\frac{1}{\sigma}, && \text{for Jeffreys prior}\\ &\rho_1=-a_1,\rho_2=-a_2, && \text{for exponential prior}\end{aligned} \tag{12}$$

3.2. Tierney-Kadane's approximation (T-K) technique

Tierney and Kadane (1986) proposed an approximate procedure for the evaluation of ratio of two integrals. Although Lindley's approximation is also the solution of this problem, but it involve third derivative of log likelihood function, which is tedious to solve in some situations. Now the posterior expectation given in Equation (9) can be written as;

$$\tilde{U}(\beta,\sigma)=\frac{\int_0^\infty\int_0^\infty e^{nl_2(\beta,\sigma)}d\beta d\sigma}{\int_0^\infty\int_0^\infty e^{nl_1(\beta,\sigma)}d\beta d\sigma} \tag{13}$$

where

$$l_1(\beta,\sigma)=\frac{1}{n}[l(\beta,\sigma;x)+\rho(\beta,\sigma)]$$
$$l_2(\beta,\sigma)=\frac{1}{n}[lnU(\beta,\sigma)+l(\beta,\sigma;x)+\rho(\beta,\sigma)].$$

By solving Equation (13) using T-K approximation technique, we get;

$$\tilde{U}(\beta,\sigma)\approx\left[\frac{\det\sum^*}{\det\sum}\right]^{\frac{1}{2}}exp\big[nl_2(\hat{\beta}_{l_2},\hat{\sigma}_{l_2})-nl_1(\hat{\beta}_{l_1},\hat{\sigma}_{l_1})\big] \tag{14}$$

where $\sum^*$ and $\sum$ are negative inverse Hessian matrix of $l_2(\beta,\sigma)$ and $l_1(\beta,\sigma)$ respectively . Also $\hat{\beta}_{l_2}$ and $\hat{\sigma}_{l_2}$ are point of maxima of $l_2(\beta,\sigma)$, similarly $\hat{\beta}_{l_1}$ and $\hat{\sigma}_{l_1}$ for $l_1(\beta,\sigma)$.

Now the Bayes estimates of β,σ are derived under defined loss functions using T-K approximation.

3.2.1 Under square error loss function (SELF)

For Bayes estimator of β under SELF, U(β,σ)=β, then

$$\tilde{\beta}_{TKS}=\left[\frac{|\sum^*|}{|\sum|}\right]^{\frac{1}{2}}exp\left[ln\hat{\beta}_{l_2}+nl_1(\hat{\beta}_{l_2},\hat{\sigma}_{l_2})-nl_1(\hat{\beta}_{l_1},\hat{\sigma}_{l_1})\right]$$
$$\rho_{\beta(TKS)}=\left[\frac{|\sum^*|}{|\sum|}\right]^{\frac{1}{2}}exp\left[2ln\hat{\beta}_{l_2}+nl_1(\hat{\beta}_{l_2},\hat{\sigma}_{l_2})-nl_1(\hat{\beta}_{l_1},\hat{\sigma}_{l_1})\right]-\left[\tilde{\beta}_{TKS}\right]^2.$$

3.2.2 Under weighted loss function (WLF)

For Bayes estimator of β under WLF, U(β,σ)=β^{-1}, then

$$\tilde{\beta}_{TKW}=\left[\left[\frac{|\sum^*|}{|\sum|}\right]^{\frac{1}{2}}exp\left[-ln\hat{\beta}_{l_2}+nl_1(\hat{\beta}_{l_2},\hat{\sigma}_{l_2})-nl_1(\hat{\beta}_{l_1},\hat{\sigma}_{l_1})\right]\right]^{-1}$$
$$\rho_{\beta(TKW)}=\tilde{\beta}_{TKS}-\tilde{\beta}_{TKW}.$$

3.2.3 Under precautionary loss function (PLF)

For Bayes estimator of β under PLF, U(β,σ)=β^2, then

$$\tilde{\beta}_{TKP}=\sqrt{\left[\frac{|\sum^*|}{|\sum|}\right]^{\frac{1}{2}}exp\left[2ln\hat{\beta}_{l_2}+nl_1(\hat{\beta}_{l_2},\hat{\sigma}_{l_2})-nl_1(\hat{\beta}_{l_1},\hat{\sigma}_{l_1})\right]}$$
$$\rho_{\beta(TKP)}=2[\tilde{\beta}_{TKP}-\tilde{\beta}_{TKS}].$$

Similarly, Bayes estimators of other parameters σ and corresponding function of posterior risk are derived. Taking different values of ρ_1 and ρ_2 defined in Equation (12), Bayes estimators under uniform, Jeffreys and exponential prior are derived.

4. Simulation Study

A Monte Carlo simulation study is carried out for the evaluation of MLEs and Bayes estimators. These estimates are compared on the basis of values of corresponding risk functions. For this purpose sample of different sizes n = 10, 30, 50, 70, 100, 120, 170, 200, 250, 300, 400 and 500 are generated from RRD by using the random number generator $X=\left(8\beta^4\sigma^2ln(\frac{1}{1-U})\right)^{\frac{1}{4}}$, proposed by Ateeq et al. (2019). MLEs are estimated numerically using R-software $maxLik$. Bayes estimators and values of risk function of β and σ are evaluated through results derived above under Lindley and T-K methods using SELF,WLF and PLF. Three priors uniform, Jeffreys and exponential are used. Values of parameters are selected as $\beta = 2.67$, and $\sigma = 12.5$. Elicited values of hyper parameters for exponential priors are $a_1 = 2$ and $a_2 = 5$. This process is repeated 5000 times and R-program is formulated for this purpose.

Table 2 MLEs, Bayes estimators and posterior risk using uniform prior for the parameter β

			Loss functions				
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	3.779533	3.779763	3.779531	3.779879	3.071579	3.071579	3.07171
	0.000878	0.000878	0.0002324	0.000232	0.000804	0.000730	0.000262
50	3.779851	3.779669	3.779532	3.779737	3.07156	3.07156	3.071699
	0.000516	0.000516	0.000136	0.000136	0.000855	1e-07	0.000278
70	3.779672	3.779629	3.779532	3.779678	3.071569	3.071569	3.071658
	0.000366	0.000366	0.000097	0.000097	0.000543	0.0000410	0.000176
100	3.779650	3.779601	3.779533	3.779635	3.071465	3.071465	3.071639
	0.000257	0.000257	0.000068	0.000068	0.001068	0.000210	0.000348
120	3.779592	3.779589	3.779533	3.779617	3.07168	3.07168	3.071725
	0.000212	0.000212	0.000056	0.000056	0.000274	0.000710	8.9e-05
170	3.779598	3.779573	3.779533	3.779592	3.071585	3.071585	3.07171
	0.000149	0.000149	0.000039	0.000039	0.000770	0.000051	0.000250
250	3.779815	3.779560	3.779533	3.779573	3.071577	3.071577	3.07171
	0.000101	0.000101	0.000026	0.000026	0.000818	0.000071	0.000266
300	3.779681	3.779555	3.779533	3.779567	3.071737	3.071736	3.071643
	0.000084	0.000084	0.000022	0.000022	0.000571	0.000710	0.000186
400	3.779686	3.779550	3.779533	3.779558	3.071821	3.071821	3.071717
	0.000063	0.000063	0.000016	0.000016	0.000639	2e-07	0.000208
500	3.779876	3.779546	3.779533	3.779553	3.071476	3.071476	3.071649
	0.000050	0.000050	0.000013	0.000013	0.001064	0.000041	0.000346

Table 3 MLEs, Bayes estimators and Posterior risk using Jeffreys' prior for the parameter β

		Loss functions					
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	3.07171	3.071897	3.071757	3.071968	3.07171	3.07171	3.071678
	0.000432	0.000432	0.000140	0.000140	0.000194	0.000761	6.34e-05
50	3.07178	3.071822	3.071738	3.071864	3.779533	3.779532	3.779486
	0.000259	0.000025	0.000084	0.000084	0.000352	1.2e-06	9.33e-05
70	3.07188	3.071790	3.071730	3.071820	3.071453	3.071453	3.071704
	0.000184	0.000184	0.000059	0.000059	0.001539	4e-07	0.000501
100	3.071790	3.071766	3.071724	3.071786	3.779064	3.779062	3.779533
	0.000128	0.000128	0.000041	0.000041	0.003549	1.9e-06	0.000939
120	3.07187	3.071756	3.071721	3.071773	3.071441	3.07144	3.07171
	0.000106	0.000106	0.000034	0.000034	0.001654	4e-07	0.000538
170	3.07179	3.071743	3.071718	3.071755	3.779561	3.77956	3.77977
	0.000075	0.000075	0.000024	0.000024	0.001578	1.5e-06	0.000417
250	3.07191	3.071732	3.071715	3.071740	3.071723	3.071711	3.071782
	0.000050	0.000050	0.000016	0.000016	2e-06	0.000006	6e-07
300	3.07170	3.071728	3.071715	3.071735	3.071721	3.071876	3.071761
	0.000042	0.0000042	0.000003	0.000001	0.000002	0.000002	0.000005
400	3.07171	3.071724	3.071713	3.071729	3.071434	3.071434	3.071572
	0.000031	0.000031	0.000010	0.000010	0.000848	0.000912	0.000276
500	3.0717	3.071721	3.071713	3.071725	3.071715	3.071706	3.071711
	0.000025	0.0000025	0.000008	0.000008	0.000005	0.000008	0.000007

Table 4 MLEs, Bayes estimators and Posterior risk using exponential prior for the parameter β

		Loss functions					
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	3.079885	3.078951	3.080348	3.482737	3.071236	3.071235	3.071347
	0.002923	0.002854	0.000934	0.000926	0.000678	1.5e-06	0.000220
50	3.076466	3.075918	3.076738	3.296010	3.071247	3.071248	3.071473
	0.001700	0.001477	0.000547	0.000445	0.001389	6e-07	0.000452
70	3.075092	3.074702	3.075287	3.227620	3.071479	3.071477	3.071593
	0.001209	0.000997	0.000390	0.000289	0.000703	1.2e-06	0.000229
100	3.074078	3.073804	3.074215	3.179113	3.07171	3.071819	3.071927
	0.000846	0.000740	0.000244	0.000273	0.000668	0.000017	0.000217
120	3.073685	3.073456	3.073799	3.160718	3.07171	3.07148	3.07125
	0.000705	0.000662	0.000218	0.000226	0.00141	0.000341	0.000459
170	3.073103	3.072941	3.073183	3.133925	3.071479	3.07171	3.071941
	0.000497	0.000396	0.000161	0.000141	0.001418	0.000431	0.000461
250	3.072650	3.072540	3.072704	3.113374	3.778954	3.779533	3.780113
	0.000335	0.000275	0.000109	0.000089	0.004379	2e-06	0.001158
300	3.072495	3.072404	3.072541	3.106453	3.779533	3.778953	3.778372
	0.000280	0.000220	0.000091	0.000081	0.004386	0.000432	0.000160
400	3.072297	3.072229	3.072331	3.097591	3.071816	3.071724	3.071632
	0.000209	0.000189	0.000068	0.000058	0.000567	0.000014	0.000184
500	3.072179	3.072124	3.072206	3.092347	3.071475	3.071649	3.071822
	0.000167	0.000147	0.000054	0.000044	0.001065	0.0000178	0.000346

Table 5 MLEs, Bayes estimators and Posterior risk using uniform prior for the parameter σ

Loss functions							
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	0.589911	0.589939	0.589911	0.589954	0.589896	0.589897	0.589897
	1.7e-05	1.7e-06	2.88e-06	2.8e-06	3e-07	0.000002	4e-07
50	0.589912	0.58992	0.58991	0.58993	0.58989	0.589901	0.589903
	9.9e-06	9.9e-06	1.6e-05	1.6e-05	2e-06	0.000007	3.4e-06
70	0.588219	0.589923	0.589911	0.589929	0.589906	0.589904	0.589902
	7.1e-06	3.1e-07	1.2e-05	1.2e-05	2.5e-06	0.000008	4.3e-06
100	0.589919	0.589919	0.589911	0.589923	0.589906	0.589905	0.589904
	4.9e-05	3.9e-06	8.4e-06	8.4e-06	6e-07	0.000005	1.1e-06
120	0.589917	0.589918	0.589911	0.589921	0.589907	0.589908	0.589909
	4.1e-06	3.1e-06	6.1e-06	6.9e-06	8e-07	0.000008	1.4e-06
170	0.589219	0.589916	0.589912	0.589918	0.589905	0.589908	0.589911
	2.9e-05	2.9e-06	4.9e-07	4.8e-07	3.3e-06	0.000007	5.6e-06
250	0.589442	0.589914	0.589911	0.589916	0.589905	0.589909	0.589912
	2.0e-05	2.0e-06	3.9e-06	3.3e-06	3.1e-06	0.000004	5.7e-06
300	0.589520	0.589913	0.589912	0.589915	0.589911	0.589908	0.589904
	1.6e-05	1.6e-06	2.2e-06	2.8e-06	4.1e-06	0.000002	6.9e-06
400	0.589618	0.589913	0.589913	0.589914	0.589910	0.589912	0.589914
	1.2e-05	1.2e-06	2.2e-06	2.1e-06	2.1e-06	0.000002	3.5e-06
500	0.589677	0.589912	0.589919	0.589913	0.589905	0.589908	0.589919
	1.0e-05	1.0e-06	1.7e-06	1.5e-06	2.3e-06	0.000005	6.5e-06

Table 6 MLEs,Bayes estimators and Posterior risk using Jeffreys' prior for the parameter σ

Loss functions							
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	0.589947	0.589856	0.589992	0.577269	0.589864	0.589864	0.589865
	5.3e-05	2.3e-05	9.0e-05	9.0e-05	9e-06	0.0000006	1.5e-06
50	0.589932	0.589878	0.589959	0.582269	0.596360	0.596371	0.596381
	3.1e-05	3.1e-05	5.4e-05	5.4e-05	1.6e-05	0.000007	2.1e-05
70	0.589926	0.589888	0.589945	0.584460	0.589891	0.589894	0.589897
	2.2e-05	2.2e-05	3.8e-05	3.8e-05	3.3e-06	0.0000082	5.5e-06
100	0.589921	0.589895	0.589935	0.586103	0.596463	0.596449	0.796435
	1.5e-05	1.5e-06	2.6e-05	2.6e-05	2.2e-05	0.000007	2.8e-05
120	0.589920	0.589897	0.589931	0.586753	0.589901	0.589904	0.589904
	1.3e-05	1.3e-06	2.2e-06	2.2e-06	3.7e-06	0.0000034	6.2e-06
170	0.589917	0.589901	0.589925	0.587677	0.596500	0.596484	0.596469
	9.2e-05	9.2e-06	1.5-05	1.6e-05	2.4e-05	1e-06	3.0e-05
250	0.589915	0.589904	0.589920	0.588396	0.589904	0.589907	0.589911
	6.3e-06	6.1e-07	1.0e-07	1.0e-07	3.9e-06	0.0000031	6.5e-06
300	0.589914	0.589905	0.589919	0.588654	0.589904	0.589907	0.589911
	5.2e-06	5.2e-07	8.8e-07	8.2e-07	3.9e-06	0.0000051	6.5e-06
400	0.589913	0.589907	0.589917	0.588965	0.589911	0.589904	0.589898
	3.9e-05	3.9e-06	6.1e-06	6.6e-06	7.7e-06	0.0000061	1.3e-05
500	0.589913	0.589908	0.589915	0.589155	0.589904	0.589907	0.589911
	3.1e-05	3.1e-06	5.3e-06	5.1e-06	3.9e-06	0.0000041	6.6e-06

Table 7 MLEs, Bayes estimators and posterior risk using exponential prior for the parameter σ

	Loss functions						
n	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
30	0.588610	0.588001	0.588918	0.508422	0.589602	0.589602	0.589610
	0.000363	0.000261	0.000609	0.000584	8.7e-06	3e-07	1.4e-05
50	0.589156	0.588800	0.589335	0.539519	0.58973	0.589731	0.589729
	0.000211	0.000180	0.000356	0.000257	3.1e-06	1e-07	5.3e-06
70	0.589374	0.589121	0.589501	0.553138	0.589778	0.589778	0.589786
	0.000150	0.000120	0.000253	0.000224	9.7e-06	1e-07	1.6e-05
100	0.589535	0.589357	0.589624	0.563660	0.589906	0.589913	0.589921
	0.000105	0.000085	0.000177	0.000168	9e-06	0.000001	1.5e-05
120	0.589598	0.589449	0.589672	0.567855	0.589834	0.589839	0.589845
	0.000087	0.000067	0.000148	0.000138	6.5e-06	0.000004	1.1e-05
170	0.589690	0.589585	0.589742	0.574178	0.589861	0.589858	0.589856
	0.000061	0.000051	0.000104	0.000093	3.2e-06	0.0000042	5.5e-06
250	0.589762	0.589691	0.589797	0.579209	0.596252	0.596252	0.596262
	0.000041	0.000031	0.000070	0.000062	1.5e-05	2e-07	1.9e-05
300	0.589786	0.589727	0.589816	0.580938	0.796328	0.796328	0.796309
	0.000034	0.000023	0.000059	0.000039	3.0e-05	1e-07	3.7e-05
400	0.589818	0.589774	0.589840	0.583181	0.589910	0.589910	0.589912
	0.000026	0.000024	0.000044	0.000024	2.6e-06	0.000004	4.5e-06
500	0.589836	0.589801	0.589854	0.584523	0.589905	0.589905	0.589908
	0.000020	0.000010	0.000035	0.000027	3.3e-06	0.000006	5.5e-06

The results of Monte Carlo simulation study, obtained from Table 2 to Table 7 showed that all the Bayes estimators performed better than MLEs due to minimum values of associated risk. All the Bayes estimators and MLEs approaches to true values of parameters by increasing the sample size. For both parameters β and σ, estimators under informative prior are more efficient than non-informative priors. So it is preferable to use informative prior more than the non-informative priors when the information about the parameters are available. For both parameters, estimators obtained through T-K method perform better than Lindley approximation method. The Bayes estimators under WLF using all the priors show better results as the values of risks are minimum.

5. Illustrative Real Life Example

A real life data set about the prevalence of overweight and obesity of Pakistani females for the year 1976-2018 is taken from (https://data.worldbank.org/country/pakistan). This data set is used to check the performance of Bayes and MLEs of the parameters of RRD.

From this data set, the MLEs of the parameters of RRD are evaluated using $maxLik$ package in R language. The Bayes estimators and associated posterior risk of the parameters are obtained through the datasets. Three loss functions SELF, WLF and PLF are used for the evaluation of Bayes estimators and associated posterior risk. The values of hyper parameters used in informative prior is elicited as $a_1 = 2$ and $a_2 = 5$.

The results obtained from real life data set are shown in Table 8. It is assessed that all the Bayes estimators using informative and non-informative priors are better than MLEs due to minimum values of risks. The estimators obtained through $T - K$ approximation are superior than Lindley approximation under each loss function and all the priors. Results obtained from real life data sets are nearly same as findings of simulation study.

Table 8 MLEs, Bayes estimators and posterior risk of β and σ using uniform, Jeffreys and exponential priors

	MLE	SELF	WLF	PLF	SELF	WLF	PLF
		Lindley Method				**T-K Method**	
				Uniform Prior			
β	3.779533	3.779577	3.779533	3.779598	3.779533	3.779533	3.779347
	0.0001648	0.0001647	4.360e-05	4.359e-05	0.0000140	6.874e-12	0.0003717
σ	0.796537	0.796567	0.796537	0.796583	0.796525	0.796525	0.796525
	2.441e-05	2.441e-05	3.065e-05	3.064e-05	2.694e-10	3.397e-10	3.382e-10
				Jeffrey's Prior			
β	3.779533	3.779591	3.779548	3.779613	3.779533	3.779533	3.779347
	0.0001648	0.0001621	4.359e-05	4.339e-05	0.0000135	6.874e-12	0.0003517
σ	0.796537	0.7965494	0.796518	0.796564	0.796525	0.796525	0.796525
	2.441e-05	2.371e-05	3.065e-05	3.065e-05	2.694e-10	3.397e-10	3.382e-10
				exponential Prior			
β	3.779533	3.780018	3.779974	3.791415	3.779533	3.779533	3.779347
	0.0001648	0.0001445	4.354e-05	4.253e-05	0.0000120	6.875e-12	0.0002715
σ	0.796537	0.796460	0.793193	0.796475	0.796525	0.796525	0.796525
	2.441e-05	2.240e-05	2.043e-05	3.064e-05	2.717e-10	3.368e-10	3.411e-10

6. Concluding Remarks

In this paper, classical and Bayesian estimation of the parameters of the RRD is performed. Further the issue of the modeling of prevalence of over weight and obesity is resolved. The practitioner can use this distribution for the evaluation of probabilities and prediction about the prevalence and obesity of female. The posterior distribution of the parameters of the RRD are not in closed form, so the Bayes estimators and corresponding posterior risk cannot be obtained. We used two approximation techniques to resolved this issue. Results of simulation study and real life data are evident that the Bayes estimators perform better than the most commonly used MLEs. In the comparison of two approximation techniques $T - K$ method is significantly better than Lindley methods.

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