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The Alpha Power Shifted Exponential Distribution: Properties and Applications

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Abstract

This article proposes a new three parameter distribution in the family of the exponential distribution called the alpha power shifted exponential (APOSE) distribution. The structural mathematical properties of the APOSE model were derived and examined. The APOSE model parameters were established and obtained by maximum likelihood method. The flexibility, efficiency, and behavior of the APOSE model estimators were examined. The results show that the APOSE distribution is increasing, decreasing, unimodal, and right skewed property. The bivariate model of the APOSE distribution was also proposed. The empirical applicability and proficiency of the APOSE model was examined by a real-life dataset. The empirical results show that the proposed APOSE model provides a better goodness-of-fit when compared to existing models in statistical literature and can serve as an alternative model to those appearing in modeling Poisson processes.

Keywords: Alpha power distribution, bivariate model, exponential distribution, inverted exponential distribution, shifted distribution.

1. Introduction

Life processes decisions are made based on the probability distribution of the scenarios. Thus, researchers in distribution theory have focused on developing flexible models that are relevant and best suit in modeling these processes for better utilizations.

The exponential distribution is one of the distributions used to model memoryless processes that occur between events. This could be the case in mortality rate associated with some terminal endemic diseases that readily occurred in our societies. Hence, because of their pandemic in nature, the exponential distribution is used to model lifetime processes that are independently at a constant average rate. However, life-time processes are never constant. Hence, modeling with exponential distribution becomes difficult and complex because of its monotonicity. For example, the mortality rate associated with some pandemic diseases. The hazard rate will initially increase as the time increases and approach the peak after certain period of time and decline slowly Singh et al. (2013).

Thus, there is a need to extend the exponential model to accommodate the non-monotonicity property. Hence, the alpha power shifted exponential is proposed to accommodate the non-constant average rate. An exponential distribution is said to be shifted if the origin of the variable is shifted.

In a bid to solve the constant average rate, many researchers have work on diversifying the exponential distribution. Gupta and Kundu (1999) proposed a generalized exponential distribution using the exponentiation method proposed in Gupta et al. (1998). Keller et al. *(*1982*)* proposed the inverted exponential distribution with a bathtub hazard rate. Eghwerido et al. (2020a) proposed the Gompertz alpha power inverted exponential distribution. Efe-Eyefia et al. (2020) proposed the Weilbull alpha power inverted exponential distribution. Zelibe et al. (2019) proposed the Kumaraswamy alpha power inverted exponential distribution. Eghwerido et al. (2020b) Proposed the Gompertz extended generalized exponential distribution. Madi and Leonard (1996) proposed the Bayesian estimation for the shifted exponential. Chang et al. (2013) proposed the hypothesis testing for the parameter of shifted exponential distribution. Madi and Raqab (2003) used the shifted exponential distribution to analyze a doubly censored random sample of failure times. Sánchez et al. (2014) proposed a shifted exponential as likelihood function and conjugate inverted gamma prior for making Bayesian inference comparatively robust against a prior density poorly specified. Agu et al. (2020) proposed the exponentiated shifted exponential distribution.

The probability density function (pdf) of the shifted exponential distribution for a random variable M is defined as

$$g(m)=\lambda\exp\left(-\lambda\left(m-\theta\right)\right),\quad m\geq\theta,\,\lambda>0,\;\theta>0,\tag{1}$$

where λ and θ are the scale and location parameters respectively.

The cumulative distribution function (cdf) that corresponds to (1) is expressed as

$$G\left(m\right)=1-\exp\left(-\lambda\left(m-\theta\right)\right),\quad m\geq\theta,\,\lambda>0,\;\theta>0.\tag{2}$$

However, suppose $g\left(m\right)$ and $G\left(m\right)$ are the baseline density and distribution functions. Then, Mahdavi and Kundu (2017) proposed alpha power characterization for making a distribution more flexible. The pdf of the alpha power distribution is expressed as

$$u\left(m\right)=\begin{cases}\alpha^{G(m)}\dfrac{g(m)}{(\alpha-1)}\log\alpha, & \text{if } \alpha\in R^{+}-\{1\}\\ g(m), & \text{otherwise } \alpha=1.\end{cases}\tag{3}$$

The cdf that corresponds to the alpha power characterization is given as

$$U\left(m\right)=\begin{cases}\dfrac{\alpha^{G(m)}-1}{(\alpha-1)} & \text{if } \alpha\in R^{+}-\{1\}\\ G(m), & \text{otherwise } \alpha=1\end{cases}.\tag{4}$$

The article is motivated by results obtained modeling with exponential and shifted exponential distributions. Thus, a flexible kurtosis shifted distribution called alpha power shifted exponential distribution is proposed to generate a distribution with a unimodal and right skewed shaped model with a better goodness-of-fit for life-time data.

The aim of this paper is to introduce a unimodal and right skewed model called alpha power shifted exponential distribution. Its major statistical properties were derived and investigated. A simulation and application to examined its flexibility and efficiency was also carried out.

2. The APOSE Distribution

This section introduces a class of the shifted exponential model and derived some of its reliability properties. Suppose M is a continuous random variable, then the APOSE pdf is expressed as

$$u(m)=\begin{cases}\alpha^{1-\exp(-\lambda(m-\theta))}\dfrac{\lambda\exp(-\lambda(m-\theta))}{(\alpha-1)}\log\alpha, & \text{if } \alpha\in R^{+}-\{1\}\\ \lambda\exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \quad (5)$$

The APOSE cdf is defined as

$$U(m)=\begin{cases}\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}, & \text{if } \alpha\in R^{+}-\{1\}\\ 1-\exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \quad (6)$$

In Equation (6), α is additional shape parameter that is responsible for the kurtosis and skewness. Figure 1 shows the density plot for different parameter cases for the APOSE density. In the plot, it was observed that the APOSE can be decreasing, when θ is decreasing, right skewed and unimodal.

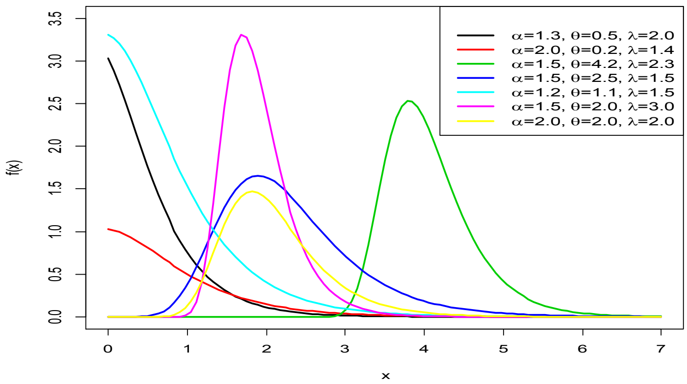

Figure 1 The APOSE density plot for different parameter cases

The APOSE survival and failure rate functions are expressed as

$$S(m)=\begin{cases}1-\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}, & \text{if } \alpha\in R^{+}-\{1\}\\ \exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \quad (7)$$

and

$$h(m)=\begin{cases}\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}\dfrac{\lambda\exp(-\lambda(m-\theta))}{(\alpha-1)}\log\alpha}{1-\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}}, & \text{if } \alpha\in R^{+}-\{1\}\\ \lambda, & \text{otherwise } \alpha=1.\end{cases} \quad (8)$$

Figure 2 shows the APOSE hazard rate function for different parameter cases. In the plot, we observed that the APOSE is increasing and uniform after a particular value.

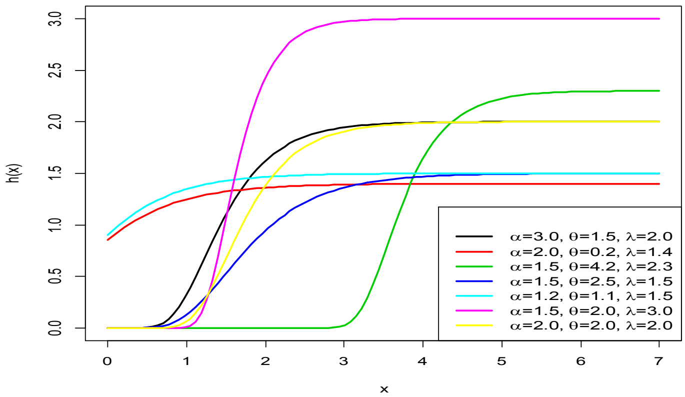

Figure 2 The APOSE hazard rate function plot for different parameter cases

The APOSE reversed hazard rate function is defined as

$$r(m)=\begin{cases}\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}\lambda\exp(-\lambda(m-\theta))\log\alpha}{\alpha^{1-\exp(-\lambda(m-\theta))}-1}, & \text{if } \alpha\in R^{+}-\{1\}\\[2ex] \dfrac{\lambda\exp(-\lambda(m-\theta))}{1-\exp(-\lambda(m-\theta))}, & \text{otherwise } \alpha=1.\end{cases} \tag{9}$$

The APOSE odds and cumulative hazard rate functions are given as

$$O(m)=\begin{cases}\dfrac{\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{\alpha-1}}{1-\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{\alpha-1}}, & \text{if } \alpha\in R^{+}-\{1\}\\[2ex] \dfrac{1-\exp(-\lambda(m-\theta))}{\exp(-\lambda(m-\theta))}, & \text{otherwise } \alpha=1.\end{cases} \tag{10}$$

and

$$H(m)=\begin{cases}-In\left[1-\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}\right], & \text{if } \alpha\in R^{+}-\{1\}\\[2ex] \lambda(m-\theta), & \text{otherwise } \alpha=1.\end{cases} \tag{11}$$

3. Linear Expansion

This section expresses the APOSE model as a linear function of the shifted exponential distribution. The linear representation will assist in simplifying the APOSE distribution in terms of the shifted exponential distribution and hence, enables a simplified statistical property of the proposed model. By Eghwerido et al. (2020c, 2021a, 2021b, 2021c), we have the following lemma as

Lemma 3.1 *For* $a>0,\ \alpha^{a}=\sum_{b=0}^{\infty}\dfrac{(\log\alpha)^{b}a^{b}}{b!}.$

Lemma 3.2 *For* $c-1>1,\ c\in Z^{+},\ (1-A)^{c-1}=\sum_{d=0}^{c-1}(-1)^{d}\binom{c-1}{d}A^{d}.$

Thus, by Lemma 3.1, $\alpha^{1-\exp(-\lambda(m-\theta))}=\sum_{b=0}^{\infty}\frac{(\log\alpha)^b\left(1-\exp(-\lambda(m-\theta))\right)^b}{b!}$. More so, by Lemma 3.2, $\left(1-\exp(-\lambda(m-\theta))\right)^b=\sum_{d=0}^{b}(-1)^d\frac{b!}{(b-d)!d!}\exp(-d\lambda(m-\theta))$. Thus,

$$\alpha^{1-\exp(-\lambda(m-\theta))}=\sum_{b=0}^{\infty}\sum_{d=0}^{b}\frac{(-1)^d(\log\alpha)^b\exp(-d\lambda(m-\theta))}{(b-d)!d!}.$$

However, in power series, the APOSE density is defined as

$$u(m)=\begin{cases}\sum_{b=0}^{\infty}\sum_{d}^{b}\frac{(-1)^d\lambda(\log\alpha)^{b+1}\exp(-\lambda(m-\theta)(d+1))}{(\alpha-1)(b-d)!d!}, & \text{if } \alpha\in R^+-\{1\}\\ \lambda\exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{12}$$

Hence, the APOSE distribution is a shifted exponential distribution with a shift parameter θ and scale parameter $\lambda(d+1)$. Also, the APOSE cdf in power series is defined as

$$U(m)=\begin{cases}(\alpha-1)^{-1}\left[\sum_{b=0}^{\infty}\sum_{d=0}^{b}\frac{(-1)^d\lambda(\log\alpha)^b\exp(-\lambda d(m-\theta))}{(b-d)!d!}-1\right], & \text{if } \alpha\in R^+-\{1\}\\ 1-\exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{13}$$

4. The APOSE Mathematical Properties

In this section, the mathematical and statistical properties of the APOSE distribution are derived and investigated. These properties include the quantile function, moments, moment generating and probability generating functions, probability weighted moments, entropy, order statistics, and moments of the residual life.

4.1. The quantile function

Suppose M is an APOSE random variable. Then, the quantile function of M for $v\in(0,1)$ is defined as

$$m_v=\theta-\lambda^{-1}\log\left(1-(\log\alpha)^{-1}\log\left[v(\alpha-1)+1\right]\right). \tag{14}$$

For $v=0.5$ in (14), we obtain the median (Med) of M as

$$Q_2=\begin{cases}\theta-\lambda^{-1}\log\left(1-(\log\alpha)^{-1}\log\left[0.5(\alpha-1)+1\right]\right), & \text{if } \alpha\in R^+-\{1\}\\ \theta-\lambda^{-1}\log(0.5), & \text{otherwise } \alpha=1.\end{cases} \tag{15}$$

However, the third and first quantiles are obtained as

$$Q_3=\begin{cases}\theta-\lambda^{-1}\log\left(1-(\log\alpha)^{-1}\log\left[0.75(\alpha-1)+1\right]\right), & \text{if } \alpha\in R^+-\{1\}\\ \theta-\lambda^{-1}\log(0.75), & \text{otherwise } \alpha=1.\end{cases} \tag{16}$$

and

$$Q_1=\begin{cases}\theta-\lambda^{-1}\log\left(1-(\log\alpha)^{-1}\log\left[0.25(\alpha-1)+1\right]\right), & \text{if } \alpha\in R^+-\{1\}\\ \theta-\lambda^{-1}\log(0.25), & \text{otherwise } \alpha=1.\end{cases} \tag{17}$$

The APOSE coefficient of kurtosis is obtained in terms of the Moor's formula as

$$K = \frac{m_{0.125} - m_{0.375} - m_{0.625} + m_{0.875}}{m_{0.75} - m_{0.25}}.$$

Also, the APOSE coefficient of skewness is obtained in terms of the Bowley's formula as

$$S = \frac{m_{0.25} - 2m_{0.5} + m_{0.75}}{m_{0.75} - m_{0.25}}.$$

The APOSE kurtosis (K), median (Q_2), skewness (S), first quantile (Q_1) and third quantile (Q_3) are evaluated generating sequence of random numbers with the aid of the quantile function obtained in (14). The model parameters are set as follows: $\lambda = 1.0, 1.5, 2.0, 5.0$ and 10, $\alpha = 0.5, 1.5, 2.5, 3.0$ and 5.0 and $\theta = 0.2, 0.5, 1.2, 1.5$ and 2.0.

Table 1 shows the results. However, the results in Table 1, the λ, α and θ values increases with decreases in skewness and kurtosis. It was observed that as θ increases, the skewness and kurtosis are constant.

Table 1 APOSE skewness, kurtosis and quantile results for different values of parameter cases

α	θ	λ	S	K	Q_1	Q_2	Q_3
		1.0	0.2991	0.7109	0.4139	0.7362	1.3334
		1.5	0.2991	0.7109	0.3426	0.5574	0.9556
0.5	0.2	2.0	0.2991	0.7109	0.3069	0.4681	0.7667
		5.0	0.2991	0.7109	0.2427	0.3072	0.4266
		10.0	0.2991	0.7109	0.2213	0.2536	0.31335
		1.0	0.2372	0.5475	0.8431	1.2992	2.0389
		1.5	0.2372	0.5475	0.7287	1.0328	1.5259
1.5	0.5	2.0	0.2372	0.5475	0.6715	0.8996	1.2694
		5.0	0.2372	0.5475	0.5686	0.6598	0.8077
		10.0	0.2372	0.5475	0.5343	0.5799	0.6538
		1.0	0.2069	0.4762	1.6270	2.1435	2.9295
		1.5	0.2069	0.4762	1.4846	1.8290	2.3530
2.5	1.2	2.0	0.2069	0.4762	1.4135	1.6717	2.0647
		5.0	0.2069	0.4762	1.2854	1.3887	1.5459
		10.0	0.2069	0.4762	1.2427	1.2943	1.3729
		1.0	0.1969	0.4530	1.9605	2.4967	3.2960
		1.5	0.1969	0.4530	1.8070	2.1645	2.6973
3.0	1.5	2.0	0.1969	0.4530	1.7302	1.9983	2.3980
		5.0	0.1969	0.4530	1.5921	1.6993	1.8592
		10.0	0.1969	0.4530	1.5460	1.5996	1.6796
		1.0	0.1726	0.3958	2.5633	3.1476	3.9758
		1.5	0.1726	0.3958	2.3755	2.7650	3.3172
5.0	2.0	2.0	0.1726	0.3958	2.2816	2.5738	2.9879
		5.0	0.1726	0.3958	2.1126	2.2295	2.3951
		10.0	0.1726	0.3958	2.0563	2.1147	2.1975

4.2. The APOSE moments

The r^{th} moments of the APOSE distribution is expressed as

$$\mu(m)=\begin{cases}\int_0^\infty\sum_{b=0}^\infty\sum_{d=0}^b\frac{(-1)^d\lambda(\log\alpha)^{b+1}\exp(-\lambda(m-\theta)(d+1))}{(\alpha-1)(b-d)!d!}, & \text{if } \alpha\in R^+-\{1\}\\ \lambda\int_0^\infty m^r\exp(-\lambda(m-\theta))\,dm, & \text{otherwise } \alpha=1.\end{cases} \tag{18}$$

On integrating and simplifying using Lemma 3.2, we have

$$\mu(m)=\begin{cases}\sum_{b=0}^\infty\sum_{d=0}^b\sum_{w=0}^r(-1)^d\lambda^{-r}\binom{r}{w}\left[\theta\lambda(d+1)\right]^{r-w}(d+1)^{-(r+1)}\frac{(\log\alpha)^{b+1}}{(\alpha-1)(b-d)!d!}\Gamma(w+1), & \text{if } \alpha\in R^+-\{1\}\\ \sum_{w=0}^r\binom{r}{w}\lambda^{-w}\theta^{r-w}\Gamma(w+1), & \text{otherwise } \alpha=1.\end{cases} \tag{19}$$

More so, the APOSE probability generating function (mgf) is expressed as

$$M(t)=\begin{cases}\sum_{b=0}^\infty\sum_{d=0}^b\sum_{w=0}^r(-1)^d\lambda^{1-p}\binom{p}{w}\left[\theta\lambda(d+1)\right]^{p-w}(d+1)^{-(p+1)}\frac{(\log t)^p(\log\alpha)^{b+1}}{(\alpha-1)(b-d)!d!p!}\Gamma(w+1), & \text{if } \alpha\in R^+-\{1\}\\ \sum_{w=0}^r\binom{r}{w}\frac{(\log t)}{p!}\lambda^{-w}\theta^{r-w}\Gamma(w+1), & \text{otherwise } \alpha=1.\end{cases} \tag{20}$$

4.3. The probability weighted moments (PWM)

The PWM function could be used to obtain the parameters and the quantile of the APOSE distribution that may not be obtained explicitly. Thus, the $(n,r)^{\text{th}}$ PWM of the APOSE random variable M is defined as

$$p(n,r)=\begin{cases}\sum_{b=0}^\infty\sum_{d=0}^b\sum_{w=0}^r(-1)^{n+d-k}\binom{r}{w}\binom{n}{k}\lambda(\alpha-1)^{n-1}\frac{(\log\alpha)^{b(k+1)+1}\left[\theta(dk+d+1)\right]^{r-w}}{((b-d)!d!)^{k+1}\left[\lambda(dk+d+1)\right]^{r+1}}\Gamma(w+1), & \text{if } \alpha\in R^+-\{1\}\\ \sum_{k=0}^n\sum_{w=0}^r(-1)^{n-k}\binom{n}{k}\binom{r}{w}\frac{\left[\lambda\theta(k+1)\right]^{r-w}}{(\lambda(k+1))^r}\Gamma(w+1), & \text{otherwise } \alpha=1.\end{cases} \tag{21}$$

4.4. The APOSE entropy

The APOSE Renyi entropy for a random variable M is expressed as

$$AR(M)=\begin{cases}\frac{1}{1-\delta}\log\left[\sum_{b=0}^\infty\sum_{d=0}^b\frac{\lambda(-1)^d(\log\alpha)^{b+1}}{(b-d)!d!(\alpha-1)}\right]^\delta\int_{-\infty}^\infty\exp\left(-\lambda\delta(m-\theta)(d+1)\right)dm, & \text{if } \alpha\in R^+-\{1\},\delta>0,\delta\neq0\\ \frac{1}{1-\delta}\log\int_{-\infty}^\infty\left[\lambda^\delta\exp\left(-\lambda\delta(m-\theta)\right)\right]dm, & \text{otherwise } \alpha=1,\delta>0,\delta\neq0.\end{cases} \tag{22}$$

This implies that

$$AR(M)=\begin{cases}\dfrac{1}{1-\delta}\log\left[\left[\displaystyle\sum_{b=0}^{\infty}\sum_{d=0}^{b}\frac{\lambda(-1)^{d}(\log\alpha)^{b+1}}{(b-d)!d!(\alpha-1)}\right]^{\delta}\dfrac{1}{\lambda\delta(d+1)}\right], & \text{if } \alpha\in R^{+}-\{1\},\delta>0,\delta\neq 0\\ \dfrac{1}{1-\delta}\log\left(\dfrac{\lambda^{\delta-1}}{\delta}\right), & \text{otherwise } \alpha=1,\delta>0,\delta\neq 0.\end{cases} \tag{23}$$

4.5. The APOSE order statistics

Let $M_1,M_2,\ldots,M_3$ be APOSE random sample. Also, let $M_{(1)},M_{(2)},\ldots,M_{(3)}$ be the order statistics that corresponds. Thus, the k^{th} order statistics is expressed as

$$u(m)=\frac{n!}{(k-1)!(n-k)!}\left[U(m)\right]^{k-1}\left[S(m)\right]^{n-k}u(m),\quad -\infty<m<\infty. \tag{24}$$

However, substituting and simplifying gives

$$u(m)=\begin{cases}\dfrac{n!}{(k-1)!(n-k)!}\left[\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}\right]^{k-1}\left[1-\dfrac{\alpha^{1-\exp(-\lambda(m-\theta))}-1}{(\alpha-1)}\right]\\ \times\alpha^{1-\exp(-\lambda(m-\theta))}\dfrac{\lambda\exp(-\lambda(m-\theta))}{(\alpha-1)}\log\alpha, & \text{if } \alpha\in R^{+}-\{1\},\\ \dfrac{\lambda n!}{(k-1)!(n-k)!}\left[1-\exp(-\lambda(m-\theta))\right]^{k-1}\left[\exp(-\lambda(m-\theta))\right]^{n-k}\\ \times\exp(-\lambda(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{25}$$

The maximum order statistics for the APOSE distribution is obtained when $k=n.$ More so, the minimum order statistics is obtained when $k=1.$ Final, the median order statistics is obtained when n is odd, $n=2v+1,$ and $k=v+1,$ when even and $n=2v.$

4.6. The APOSE moments of the residual life

The APOSE n^{th} moment of the residual life, say $p_n(t)=E\left[(M-t)^n\,|M>t\right]$ for $n=1,2,3,4\ldots$ can be uniquely determines $U(m)$ (see Navarro et al. 1998). Thus, it can be expressed as

$$p_n(t)=\frac{1}{1-U(m)}\sum_{b=0}^{\infty}\sum_{d=0}^{b}\sum_{k=0}^{\beta}\sum_{w=0}^{n}\binom{n}{w}\binom{\beta}{k}(-1)^{n+d+\beta-k-w}\lambda\frac{(\log\alpha)^{b+1}t^{n-w}}{(\alpha-1)(b-d)!d!}$$
$$\times(\theta(d+1))^{\beta-k}(\lambda(d+1))^{-(\beta+1)}\Gamma(\beta+1). \tag{26}$$

5. The APOSE Parameter Estimation

In this section, the maximum likelihood method is employed to obtain the parameters of the APOSE distribution. Let $m=(m_1,m_2,\ldots,m_n)$ be the APOSE random sample with unknown parameter vector $K=(\alpha,\lambda,\theta)^T$. Then, the log-likelihood function ℓ is given as

$$\ell=n\log(\log\alpha)+n\log\lambda-\lambda\sum_{i=1}^{n}(m_i-\theta)-n\log(\alpha-1)+\sum_{i=1}^{n}\left(1-\exp(-\lambda(m_i-\theta))\right)\log\alpha. \tag{27}$$

Thus, the partial derivatives of (27) with respect to the unknown parameters and equating to zero are expressed as

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha \log \alpha}-\frac{n}{(\alpha-1)}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\exp\left(-\lambda\left(m_i-\theta\right)\right)\right)=0, \tag{28}$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\left(m_i-\theta\right)+\sum_{i=1}^{n}\rho'_{i,\lambda}=0, \tag{29}$$

where $\rho_i=\left(1-\exp\left(-\lambda\left(m_i-\theta\right)\right)\right)\log \alpha,$ and

$$\frac{\partial \ell}{\partial \theta}=n\lambda+\sum_{i=1}^{n}\rho'_{i,\theta}=0. \tag{30}$$

The parameter estimates of the unknown can be obtained numerically by Newton-Raphson algorithm in MATHEMATICAL, R, MATLAB and MAPLE. However, in this study, R software was used (see R Core Team 2013).

5.1. The simulation study with the APOSE distribution

A Monte Carlo simulation study is performed to examine the performance, applicability and flexibility of the APOSE model. The simulation is performed as follows

- Random data are generated using the APOSE quantile function defined as $m=\theta-\lambda^{-1}\log\left(1-\left(\log \alpha\right)^{-1}\log\left[v\left(\alpha-1\right)+1\right]\right).$
- The parameter values are set as follow $\alpha=0.7$, $\theta=$ 0.5 and $\lambda=1.5$.
- The random sample sizes are taken as follows: 5, 10, 30, 50,100, 150, and 200.
- Each of the random sample size is replicated 5000 times.

The simulation investigated the average estimates (AEs), biases, variances and mean squared errors (MSEs) of the APOSE model maximum likelihood estimates. The maximum likelihood estimates (MLE) is estimated as $\widehat{MSE}_M=\frac{1}{5000}\sum_{i=1}^{5000}(\hat{M}_i-M)^2$. The bias is obtained as $\widehat{Bias}_M=\frac{1}{5000}\sum_{i=1}^{5000}(\hat{M}_i-M).$

Table 2 shows the Monte Carlo simulation results. Table 2 shows that the MSE, variance and biases of the parameter estimates decreases as the sample sizes increases.

5.2. Bivariate extension

This section introduces the bivariate concepts for the APOSE distribution. Let $M_1\sim APOSE\left(\lambda_1,\theta,\alpha\right)$, $M_2\sim APOSE\left(\lambda_2,\theta,\alpha\right)$, and $M_3\sim APOSE\left(\lambda_3,\theta,\alpha\right)$ be independent random variables. Also, let $X=\min\left\{M_1,M_3\right\}$ and $Y=\min\left\{M_2,M_3\right\}$. Then, the cdf of the APOSE bivariate random variable $\left(X,Y\right)$ can be defined as

$$U(m)=\begin{cases}\dfrac{\alpha^{1-\exp(-\lambda_1(m-\theta))\exp(-\lambda_2(m-\theta))\exp(-\lambda_3(m-\theta))}-1}{(\alpha-1)}, & \text{if } \alpha\in R^+-\{1\},\\ 1-\exp\left(-\lambda_1(m-\theta)\right)\exp\left(-\lambda_2(m-\theta)\right)\exp\left(-\lambda_3(m-\theta)\right), & \text{otherwise } \alpha=1.\end{cases} \tag{31}$$

The bivariate pdf is given as

$$u(m)=\begin{cases}\alpha^{1-\exp(-\lambda_1(m-\theta))\exp(-\lambda_2(m-\theta))\exp(-\lambda_3(m-\theta))} \\ \times\dfrac{(\lambda_1\lambda_2\lambda_3)\exp(-\lambda_1(m-\theta))\exp(-\lambda_2(m-\theta))\exp(-\lambda_3(m-\theta))}{(\alpha-1)}\log\alpha, & \text{if } \alpha\in R^+-\{1\}, \\ \lambda_1\lambda_2\lambda_3\exp(-\lambda_1(m-\theta))\exp(-\lambda_2(m-\theta))\exp(-\lambda_3(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{32}$$

The marginal cdf of X is expressed as

$$U_{x(m)}=\begin{cases}\dfrac{\alpha^{1-\exp(-(\lambda_1+\lambda_3)(m-\theta))}-1}{(\alpha-1)}, & \text{if } \alpha\in R^+-\{1\}, \\ 1-\exp(-(\lambda_1+\lambda_3)(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{33}$$

Also, the marginal cdf of Y is defined as

$$U_{y(m)}=\begin{cases}\dfrac{\alpha^{1-\exp(-(\lambda_2+\lambda_3)(m-\theta))}-1}{(\alpha-1)}, & \text{if } \alpha\in R^+-\{1\}, \\ 1-\exp(-(\lambda_2+\lambda_3)(m-\theta)), & \text{otherwise } \alpha=1.\end{cases} \tag{34}$$

Table 2 Monte Carlo simulation results for mean estimates, estimated biases, variance and mean squared errors

n	Parameter	AE	Variance	Bias	MSE
	$\alpha=0.7$	1.0917	0.0032	–0.0206	0.0036
5	$\theta=0.5$	0.2910	0.0728	–0.2833	0.1530
	$\lambda=1.5$	2.8474	0.0224	0.2871	0.1048
	$\alpha=0.7$	1.0109	0.0032	–0.0167	0.0034
10	$\theta=0.5$	0.3291	0.0540	–0.2769	0.1306
	$\lambda=1.5$	2.4831	0.0138	0.2724	0.0880
	$\alpha=0.7$	1.0000	0.0031	–0.0107	0.0032
30	$\theta=0.5$	0.3853	0.0495	–0.2671	0.1208
	$\lambda=1.5$	2.1001	0.0108	0.1547	0.0347
	$\alpha=0.7$	0.9825	0.0029	–0.0115	0.0030
50	$\theta=0.5$	0.4517	0.0415	–0.2706	0.1147
	$\lambda=1.5$	1.6619	0.0103	0.1284	0.0267
	$\alpha=0.7$	0.7319	0.0026	–0.0124	0.0027
100	$\theta=0.5$	0.4927	0.0302	–0.2736	0.1050
	$\lambda=1.5$	1.4001	0.0084	0.0855	0.0157
	$\alpha=0.7$	0.7109	0.0023	–0.0135	0.0024
150	$\theta=0.5$	0.5010	0.0172	–0.2756	0.0931
	$\lambda=1.5$	1.4921	0.0045	0.0362	0.0058
	$\alpha=0.7$	0.6991	0.0021	–0.0137	0.0022
200	$\theta=0.5$	0.5000	0.0074	–0.2792	0.0853
	$\lambda=1.5$	1.5010	0.0027	0.0161	0.0029

6. Applications

In this section, the application to real-life data is provided and investigated to enhance the flexibility and applicability of the proposed model. The goodness-of-fit statistics of the APOSE distribution is compared to the shifted exponential (SE) distribution, exponential (E), exponentiated shifted exponential (ExSE) distribution (Agu et al. 2020), generalized Lindley (GL) distribution (Ekhosuehi and Opone 2018), power Lindley (PL) distribution (Ghitany et al. 2013) , Lindley-exponential (LE) distribution Bhati et al. (2015), Inverted exponential (IE) distribution, alpha power inverted exponential (APIE) distribution (Ünal et al. 2018) and generalized inverted generalized exponential (GIGE) distribution (Oguntunde and Adejumo 2015). The goodness-of-fit are based on test statistics of Alkaike information criteria (AIC), Hanniquin information criteria (HQIC), Bayesian information criteria (BIC), consistent Akaike information criteria (CAIC), Cramer-von Mises (W), Anderson Darling (A), and the p values (p-value).

The first data consist of 72 exceedances of flood peaks in m^3/s of the Wheaton River near Carcross in Yukon Territory; as used in (Agu et al. 2020), (see Ekhosuehi and Opone 2018, and Akinsete et al. 2008). Table 3 provides the results of the different model test statistics for the Wheaton river data.

The second data set represents the number of vehicle fatalities for 39 counties in South Carolina for 2012 (www-fars.nhtsa.dot.gov/States) as used in Mann (2016). The dataset is as follow:

22, 26, 17, 4, 48, 9, 9, 31, 27, 20, 12, 6, 5, 14, 9, 16, 3, 33, 9, 20, 68, 13, 51, 13, 2, 4, 17, 16, 6, 52, 50, 48, 23, 12, 13, 10, 15, 8, 1.

Table 4 provides the results of the different model test statistics for the vehicle fatalities data. However, the performance of any statistical model is determined by the value that corresponds to the lowest AIC test statistic. Alternatively, the model with the highest value of log-likelihood test statistic. Figures 3 and 4 show the empirical density and cdf of the APOSE model for vehicle fatalities.

Table 3 Results of test statistics for fitted models to Wheaton river data with standard errors (in parentheses)

Models	MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
APOSE	$\hat{\lambda}=0.0026(0.004)$ $\hat{\theta}=0.2815(0.262)$ $\hat{\alpha}=4.475(0.069)$	271.57	271.92	278.40	274.28	0.021	0.179	0.556
LE	$\hat{\lambda}=0.062(0.012)$ $\hat{\theta}=1.121(0.141)$	507.07	507.24	511.62	508.88	0.154	0.845	0.237
PL	$\hat{\lambda}=0.700(0.057)$ $\hat{\theta}=0.339(0.056)$	508.44	508.61	512.99	510.25	0.153	0.877	0.020
GL	$\hat{\alpha}=0.870(0.110)$ $\hat{\lambda}=16.51(35.714)$ $\hat{\theta}=0.154(0.083)$	508.730	509.08	515.56	511.44	0.141	0.808	0.104
E	$\hat{\lambda}=0.0819(0.009)$	506.30	506.31	508.53	507.16	0.1305	0.75	0.108
	$\hat{\alpha}=0.2250(0.088)$							

ExSE	$\hat{\lambda}=149.00(2.458)$ $\hat{\theta}=-12.871(0.751)$	525.212	525.56	532.042	527.931	0.257	0.890	0.110
APIE	$\hat{\lambda}=59.623(48.411)$ $\hat{\theta}=0.8194(0.182)$	537.11	537.28	541.66	538.92	0.4238	2.430	0.002

Table 3 (Continued)

Models	MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
IE	$\hat{\lambda}=1.8968(0.223)$	571.62	571.72	573.92	572.56	25.390	1.441	0.001
GIGE	$\hat{\alpha}=2.7177(0.055)$ $\hat{\lambda}=4.1438(2.650)$ $\hat{\theta}=1.0715(1.295)$	654.508	654.86	661.338	657.23	0.059	0.205	0.028
SE	$\hat{\lambda}=0.8023(4.609)$ $\hat{\theta}=4001.2(4.339)$	110089.9	110090.1	110096.8	110092	0.529	1.082	0.004

Table 4 Results of test statistics for fitted models to vehicle fatalities data with standard errors (in parentheses)

Models	MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
APOSE	$\hat{\lambda}=0.0907(0.0131)$ $\hat{\theta}=-0.0243(0.6897)$ $\hat{\alpha}=18.8451(0.001)$	311.846	311.954	313.509	312.442	0.0505	0.3453	0.5019
LE	$\hat{\lambda}=0.0629(0.0130)$ $\hat{\theta}=1.9787(0.3922)$	315.02	315.19	319.57	316.83	0.0510	0.1474	0.3019
PL	$\hat{\lambda}=0.9120(0.0958)$ $\hat{\theta}=0.1285(0.1285)$	313.70	313.873	318.25	315.512	0.0273	0.3187	0.00154
GL	$\hat{\alpha}=1.240(0.1297)$ $\hat{\lambda}=3.413(2.929)$ $\hat{\theta}=0.0253(0.0086)$	312.91	313.262	319.74	315.62	0.0097	0.7320	0.0739
E	$\hat{\lambda}=154.923(0.0081)$	315.52	316.207	320.5119	317.3119	0.0817	0.5480	0.4442
ExSE	$\hat{\alpha}=0.0684(0.0155)$ $\hat{\lambda}=0.4924(0.0844)$	329.87	330.22	336.70	332.58	0.0920	0.2816	0.0528

	$\hat{\theta} = 0.9999(0.0071)$							
APIE	$\hat{\lambda} = 28.9353(59.1893)$ $\hat{\theta} = 3.1462(1.8065)$	321.23	321.57	324.5666	322.4333	0.0942	0.6221	0.2705

Table 4 (Continued)

Models	MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
IE	$\hat{\lambda} = 8.1993(1.3129)$	322.3282	322.4363	323.9917	322.925	12.9751	77.8365	0.0000
GIGE	$\hat{\alpha} = 1.239(0.232)$ $\hat{\lambda} = 3.066(0.001)$ $\hat{\theta} = 3.084(0.005)$	325.49	325.84	332.32	328.20	7.9724	27.0198	0.0001
SE	$\hat{\lambda} = -0.3747(2.1983)$ $\hat{\theta} = 8.1260(7.9104)$	319.03	319.203	323.58	320.84	0.2893	0.5291	0.0045

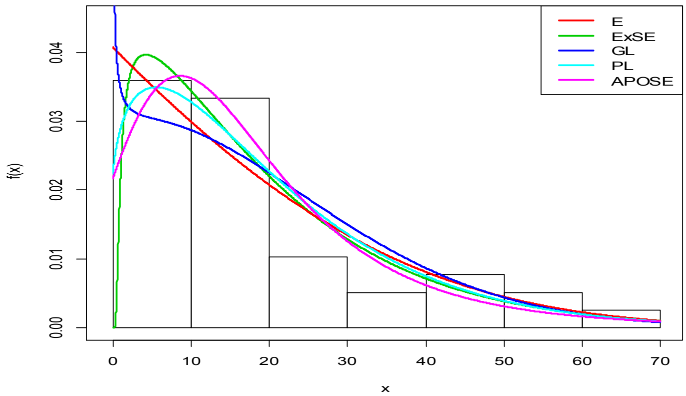

Figure 3 The APOSE estimated density plot for vehicle fatalities

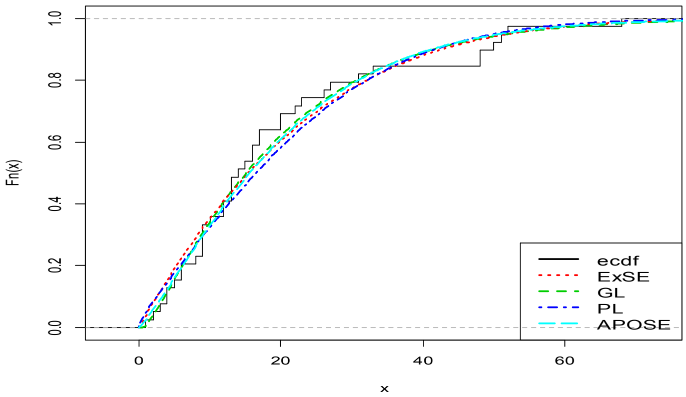

Figure 4 The APOSE estimated cdf plot for vehicle fatalities

7. Discussion

In the two real life cases considered in Tables 3 and 4 in this study, the APOSE distributions have the lowest AIC value in Wheaton river data and vehicle fatalities data respectively. Hence, the proposed model competes favorably with other existing models for the data used.

8. Conclusions

This study proposes a new class of the shifted exponential distribution named alpha power shifted exponential (APOSE) distribution. The new model extends the shifted exponential distribution by providing a more flexibility to analyzing real-life data. Some of its statistical structural properties were examined. The APOSE distribution was also expressed as a linear function of the shifted exponential distribution. The model parameters were obtained by maximum likelihood method. A simulation study was used to illustrate the performance of the proposed model. A two real-life application was further used to investigate the flexibility and efficiency of the proposed model. It was observed that the proposed model provide a better fit compared to some existing statistical models like shifted exponential (SE) distribution, exponential (E), exponentiated shifted exponential (ExSE), generalized Lindley (GL) distribution, power Lindley (PL) distribution, Lindley-exponential (LE) distribution, inverted exponential (IE) distribution, Alpha power inverted exponential (APIE) distribution and generalized inverted generalized exponential (GIGE) distribution. The proposed model can also be applied to the field of survival lifetime data, economics, hydrology, growth rate in modeling and others.

Conflict of interest

The authors declare that there is no conflict of interest.

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