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A New Three-Parameter Lifetime Model: Properties and Applications

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Abstract

A new three-parameter flexible version from the Nadarajah Haghighi model based on Lemonte (2013) is proposed and studied. Statistical properties of the new version are derived. A numerical analysis for the variance, skewness and kurtosis is presented as well as three-dimensional plots are sketched for discovering the flexibility of the new model. A simple type copula based construction is presented for deriving many bivariate and multivariate type distributions. Parameter estimates process are conducted by the well-known method of maximum likelihood. Numerical illustration of real data set is employed to compare the new model with other competitive models. A numerical simulations are executed to test performance of the used method.

Keywords: Burr X family, Morgenstern family, Clayton copula, Nadarajah-Haghighi model, MomentsL modeling, simulations, failure rate, order statistics

1. Introduction

Lemonte (2013) proposed a new three-parameter exponential-type family of distributions which can be used in modeling survival data called the exponentiated Nadarajah Haghighi (ENH). Focusing on a special case of it, the cumulative distribution function (CDF) of the two-parameter ENH model is given by.

$$\boldsymbol{W_{\beta,\delta}(x) = \left\{1 - \exp\left[1 - (1 + x)^{\delta}\right]\right\}^{\beta}}$$

and the corresponding probability density function (PDF) is

$$w_{\beta,\delta}(x) = \beta\delta(1+x)^{\delta-1}\exp\left[1-(1+x)^{\delta}\right]\left\{1-\exp\left[1-(1+x)^{\delta}\right]\right\}^{\beta-1},$$

where $\beta > 0$ and $\delta > 0$ are shape parameters. Clearly, when $\beta = \delta = 1$, we have the standard exponential (Exp) model. When $\delta = 1$; we have the exponentiated exponential (ExpExp) model. When $\beta = 1$, we have the one parameter Nadarajah and Haghighi (NH) model (Nadarajah and Haghighi (2011)).

We will refer to the new distribution as the Burr X exponentiated Nadarajah Haghighi (BuX-ENH) model. Following Yousof et al. (2017), the CDF and the PDF of the BuX-G class of distributions can be written as

$$F_{\eta,\underline{\Psi}}(x)=\left\{1-\exp\left[-\left(\frac{\mathbf{W}_{\underline{\Psi}}(x)}{\overline{\mathbf{W}}_{\underline{\Psi}}(x)}\right)^2\right]\right\}^{\eta}, \quad (1)$$

and

$$\begin{aligned} f_{\eta,\underline{\Psi}}(x) = 2\eta\frac{w_{\underline{\Psi}}(x)\mathbf{W}_{\underline{\Psi}}(x)}{\overline{\mathbf{W}}_{\underline{\Psi}}(x)^3}\exp\left[-\left(\frac{\mathbf{W}_{\underline{\Psi}}(x)}{\overline{\mathbf{W}}_{\underline{\Psi}}(x)}\right)^2\right] \\ \times\left\{1-\exp\left[-\left(\frac{\mathbf{W}_{\underline{\Psi}}(x)}{\overline{\mathbf{W}}_{\underline{\Psi}}(x)}\right)^2\right]\right\}^{\eta-1}, \end{aligned} \quad (2)$$

respectively, where $\eta > 0$ is the shape parameter, $w_{\underline{\Psi}}(x)$ and $\boldsymbol{W}_{\underline{\Psi}}(\boldsymbol{x})$ denote the PDF and the CDF of any baseline model with parameter vector $\underline{\Psi}$, $\overline{\boldsymbol{W}}_{\underline{\Psi}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{W}_{\underline{\Psi}}(\boldsymbol{x})$ is the survival (reliability) function (SF or RF) of the baseline model and $\boldsymbol{w}_{\underline{\Psi}}(\boldsymbol{x}) = \boldsymbol{d}\ \boldsymbol{W}_{\underline{\Psi}}(\boldsymbol{x})/\boldsymbol{dx}$. Using $\boldsymbol{W}_{\beta,\delta}(x), w_{\beta,\delta}(x)$ and (1), we can obtain the new three-parameter BuXENH PDF as

$$F(x) = \left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(1+x)^\delta]\}^\beta}{1-\{1-\exp[1-(1+x)^\delta]\}^\beta}\right)^2\right]^\eta\right\}\Big|_{(x>0)}, \quad (3)$$

with corresponding PDF

$$\begin{aligned} f(x) =& 2\,\eta\,\beta\,\delta\,(1+x)^{\delta-1} \\ &\times\frac{\{1-\exp[1-(1+x)^\delta]\}^{2\beta-1}}{\exp[1-(1+x)^\delta]\left(\{1-\exp[1-(1+x)^\delta]\}^\beta\right)^3} \\ &\times\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(1+x)^\delta]\}^\beta}{1-\{1-\exp[1-(1+x)^\delta]\}^\beta}\right)^2\right]\right\}^{\eta-1} \\ &\times\exp\left[-\left(\frac{\{1-\exp[1-(1+x)^\delta]\}^\beta}{1-\{1-\exp[1-(1+x)^\delta]\}^\beta}\right)^2\right]\Big|_{(x>0)}. \end{aligned} \quad (4)$$

The RF, hazard rate function (HRF), reversed hazard rate function (RHRF) and cumulative hazard rate function (CHRF) of X can be derived with the well-known relationships. For $\eta = \delta = 1$, we have the Rayleigh exponentiated exponential (REE) model. For $\delta = 1$, we have the BuX exponentiated exponential (BuXEE) model. For $\beta = \eta = 1$, we have the one-parameter Rayleigh NH (RNH) model. For $\eta = 1$ we have the Rayleigh exponentiated (RENH) model. For $\beta = \delta = 1$, we have the BuX exponential (BuXE) model. The statistical literature review contains some useful NH extensions such as Nascimento et. al. (2019) converted the NH model to NH family. Elsayed and Yousof (2019) presented the Burr X Nadarajah Haghighi model which is a special case from (3) when $\beta = 1$. Alizadeh et al., (2018) presented the extended exponentiated NH model relevant properties, characterizations and applications. Finally, Ibrahim (2020) presented the generalized odd Log-logistic NH distribution with statistical properties and different methods of estimation.

Figure 1 proves that the new PDF can be a unimodal, symmetric and the right (left) skewed. Figure 2 proves that the HRF can be decreasing or increasing or J- shape or bathtub (U) failure rate function.

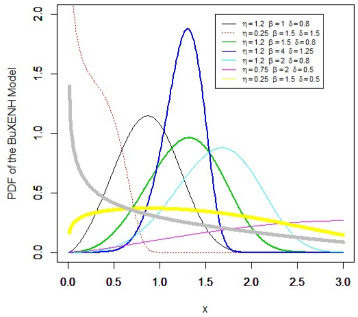

Figure 1 Plots of the BuXENH PDF at some parameters value.

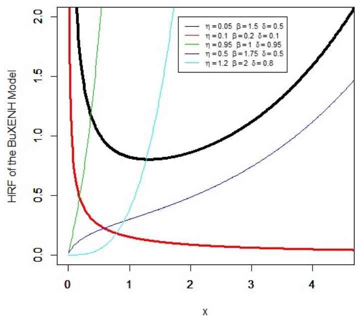

Figure 2 Plots of the BuXENH HRF at some parameters value.

2. Useful Representation

In this section, we provide a very useful linear representation for the BuXENH density function. If $|\omega| < 1$ and $\vartheta > 0$ is a real non-integer, the power series holds

$$(1-\omega)^{\vartheta} = \sum_{\ell_3=0}^{\infty} \frac{(-\omega)^{\ell_3}\Gamma(1+\vartheta)}{\ell_3!\Gamma(1+\vartheta-\ell_3)}. \tag{5}$$

Applying (5) to term

$$A(x) = \left\{1 - \exp\left[-\left(\frac{\{1-\exp[1-(1+x)^{\delta}]\}^{\beta}}{1-\{1-\exp[1-(1+x)^{\delta}]\}^{\beta}}\right)^2\right]\right\}^{\eta-1},$$

in (4) we have

$$A(x) = \sum_{\ell_3=0}^{\infty} \frac{(-)^{\ell_3}\Gamma(\eta)}{\ell_3!\Gamma(\eta-1)}\exp\left[-\ell_3\left(\frac{\{1-\exp[1-(1+x)^{\delta}]\}^{\beta}}{1-\{1-\exp[1-(1+x)^{\delta}]\}^{\beta}}\right)^2\right],$$

compiling A(x) with (4) we get

$$
\begin{aligned}
f(x) = & \, 2\,\eta\,\beta\,\delta\,(1+x)^{\delta-1}\text{exp}[1-(1+x)^{\delta}] \\
& \times \sum_{\ell_3=0}^{\infty} \frac{(-)^{\ell_3}\Gamma(\eta)}{\ell_3!\Gamma(\eta-\ell_3)} \, \frac{\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{2\beta-1}}{(1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta})^{3}} \\
& \times \text{exp}\Bigg[-\dot{\ell_3}\left(\frac{\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}{1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}\right)^{2}\Bigg].
\end{aligned} \tag{6}
$$

Applying the power series to the term

$$
B(x) = \text{exp}\Bigg[-\dot{\ell_3}\left(\frac{\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}{1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}\right)^{2}\Bigg],
$$

then,

$$
B(x) = \sum_{\ell_1=0}^{\infty} \frac{[-\dot{\ell_3}]^{\ell_1}}{\ell_1!} \left(\frac{\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}{1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta}}\right)^{2\ell_1}.
$$

Equation (6) becomes

$$
\begin{aligned}
f(x) \; = & \, 2\,\eta\,\beta\,\delta\,(1+x)^{\delta-1}\text{exp}[1-(1+x)^{\delta}] \\
& \times \sum_{\ell_3,\ell_1=0}^{\infty} \frac{(-1)^{\ell_3+\ell_1}(\dot{\ell_3})^{\ell_1}\Gamma(\eta)}{\ell_3!\ell_1!\Gamma(\eta-\ell_3)} \\
& \times \frac{(\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\})^{\beta(2\ell_1+2)-1}}{(1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta})^{2\ell_1+3}}.
\end{aligned} \tag{7}
$$

Consider the series expansion

$$
(1-\omega)^{-\vartheta} = \sum_{\ell_2=0}^{\infty} \frac{\Gamma(\vartheta+\ell_2)}{\ell_2!\Gamma(\vartheta)}\omega^{\ell_2}|_{(|\omega|<1 \text{ and } \vartheta>0)}. \tag{8}
$$

Applying the expansion in (8) to (7) for the term

$$
C(x) = (1-\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta})^{(2\ell_1+3)},
$$

then,

$$
C(x) = \sum_{\ell_2=0}^{\infty} \frac{\Gamma(2\ell_1+3+\ell_2)}{\ell_2!\Gamma(2\ell_1+3)}\left\{1-\text{exp}\left[1-(1+x)^{\delta}\right]\right\}^{\beta\ell_2}.
$$

Inserting C(x) ; Equation (7) becomes

$$
\begin{aligned}
f(x) \;\; = \;\; & 2\eta \sum_{\ell_3,\ell_1,\ell_2=0}^{\infty} \frac{(-1)^{\ell_3+\ell_1}\,(\dot{\ell_3})^{\ell_1}\,\Gamma(\eta)\,\Gamma(2\ell_1+\ell_2+3)}{\ell_3!\,\ell_1!\ell_2!\Gamma(\eta-\ell_3)\,\Gamma(2\ell_1+3)\,\beta^{*}} \\
& \times \underbrace{\begin{pmatrix} \beta^{*}\delta\beta \\ (1+x)^{\delta-1}\text{exp}\left[1-(1+x)^{\delta}\right] \\ \left\{1-\text{exp}\left[1-(1+x)^{\delta}\right]\right\}^{\beta^{*}-1} \end{pmatrix}}_{w_{\beta^{*}}(x;\delta,\beta)}|_{\beta^{*}=[\beta(2\ell_1+\ell_2+2)]}.
\end{aligned}
$$

This can be written as

$$f(x)=\sum_{\ell_1,\ell_2=0}^{\infty}\phi_{(\ell_1,\ell_2)}\omega_{\beta^*,\delta,\beta(x)}, \tag{9}$$

where

$$\phi_{(\ell_1,\ell_2)}=\frac{2\eta(-1)^{\ell_1}\Gamma(\eta)\Gamma(2\ell_1+\ell_2+3)}{\ell_1!\ell_2!\Gamma(2\ell_1+3)(2\ell_1+\ell_2+2)}\sum_{\ell_3=0}^{\infty}\frac{(-1)^{\ell_3}(1+\ell_3)^{\ell_1}}{(\ell_3!)\Gamma(\eta-\ell_3)}$$

and

$$\begin{aligned}\omega_{[\beta(2j+\ell_2+2)],\delta,\beta}(x)=\beta^*\delta\beta&\\ &\times(1+x)^{\delta-1}\text{exp}\left[1-(1+x)^{\delta}\right]\\ &\times\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta^*-1},\end{aligned}$$

is the PDF of the ENH model with power parameter β^*. (9) reveals that the density of X can be expressed as a linear mixture of ENH densities. So, several mathematical properties of the new family can be obtained by knowing those of the ENH distribution. Similarly, the CDF of the BuXENH model can also be expressed as a mixture of ENH CDFs given by

$$F(x)=\sum_{\ell_1,\ell_2=0}^{\infty}\phi_{(\ell_1,\ell_2)}\boldsymbol{W}_{\beta^*,\delta,\beta}(x), \tag{10}$$

where

$$\boldsymbol{W}_{\beta^*,\delta,\beta}(x)=\left\{1-\text{exp}[1-(1+x)^{\delta}]\right\}^{\beta^*}$$

is the CDF of the ENH model with power parameter β^*.

3. Statistical Properties

The $\boldsymbol{r}^{th}$ ordinary moment of X is given by

$$\acute{\mu}_{\boldsymbol{r}}=\boldsymbol{E}(X^{\boldsymbol{r}})=\int_{-\infty}^{\infty}f(x)x^{\boldsymbol{r}}dx.$$

Then we obtain

$$\acute{\mu}_{\boldsymbol{r}}=\sum_{\ell_1,\ell_2,\ell_3=0}^{\infty}\sum_{\ell_4=0}^{\boldsymbol{r}}\mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}\Gamma\left(1+\frac{\ell_4}{\delta},\dot{\ell_3}\right)|_{\dot{\ell_3}=1+\ell_3}, \tag{11}$$

where

$$\mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}=\phi_{(\ell_1,\ell_2)}\,\mathbb{C}_{\ell_3,\ell_4}^{\beta^*},$$

$$\mathbb{C}_{\ell_3,\ell_4}^{(\omega,\mathbf{r})}=\omega\beta^{-\mathbf{r}}\frac{(-1)^{\mathbf{r}+\ell_3-\ell_4}\ \exp\left(\dot{\ell_3}\right)}{\left(\dot{\ell_3}\right)^{1+\frac{\ell_4}{\omega}}}\binom{\mathbf{r}}{\ell_4}\binom{-1+\omega}{\ell_3},$$

or

$$\acute{\mu}_r=\sum_{\ell_1,\ell_2=0}^{\infty}\sum_{\zeta=0}^{\beta^*-1}\sum_{\ell_4=0}^{r}\mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}\Gamma\left(1+\frac{\ell_4}{\delta},\dot{\ell_3}\right)|_{(\beta^*>0\text{ and integer})}.$$

Setting $\boldsymbol{r}=1$ in (11), we have the mean of X

$$\acute{\mu}_1=\sum_{\ell_1,\ell_2,\ell_3=0}^{\infty}\sum_{\ell_4=0}^{1}\mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}\Gamma\left(1+\frac{\ell_4}{\delta},\dot{\ell_3}\right),$$

where

$$\mathbb{C}^{\beta^*}_{\ell_1,\ell_2,\ell_3,\ell_4},=\phi_{(\ell_1,\ell_2)}\mathbb{C}^{\beta^*}_{\ell_3,\ell_4},$$

$$\mathbb{C}^{[\eta,1]}_{\ell_3,\ell_4}=\frac{\eta}{\beta}\frac{(-1)^{\mathbf{r}+\ell_3-\ell_4}\;\exp\left(\dot{\ell_3}\right)}{\left(\dot{\ell_3}\right)^{1+\frac{\ell_4}{\delta}}}\binom{-1+\eta}{\ell_3}\binom{1}{\ell_4},$$

and

$$\acute{\mu}_r=\sum_{\ell_1,\ell_2=0}^{\infty}\sum_{\ell_3=0}^{\beta^*-1}\sum_{\ell_4=0}^{1}\mathbb{C}^{\beta^*}_{\ell_1,\ell_2,\ell_3,\ell_4}\Gamma\left(1+\frac{\ell_4}{\delta},\dot{\ell_3}\right)|_{(\beta^*>0\text{ and integer})},$$

where

$$\Gamma(\eta,\nu)=\int_{\nu}^{\infty}z^{-1+\eta}\exp(z)dz$$

denotes the complementary incomplete gamma function, which can be evaluated in MATHEMATICA, R, etc. The variance (V(X)), skewness (S(X)) and kurtosis (Ku(X)) measures can be calculated from the ordinary moments using well-known relationships (see the numerical analysis given in Table 1).

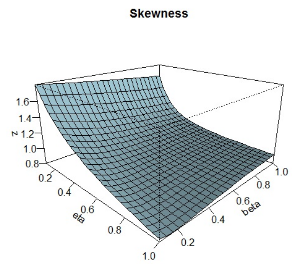

Figure 3 3-D plot for skewness of the new model when $\delta=1.25$

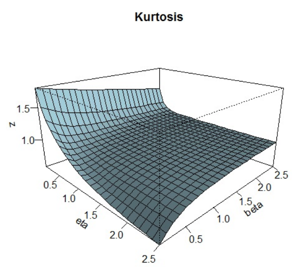

Figure 4 3-D plot for kurtosis of the new model when $\delta=1.25$

Here, we provide a formula for the moment generating function (MGF) $M_X\left(t\right)=\mathbf{E}\left(e^{t\,X}\right)$ of X. Clearly, the MGF can be derived from (9) as

$$M_x(t)=\sum_{\ell_1,\ell_2,\ell_3,\boldsymbol{r}=0}^{\infty}\sum_{\ell_4=0}^{\boldsymbol{r}}\frac{t^{\boldsymbol{r}}}{\boldsymbol{r}!}\mathbb{C}^{\beta^*}_{\ell_1,\ell_2,\ell_3,\ell_4}\Gamma\left(1+\frac{\ell_4}{\delta},\dot{\ell_3}\right),$$

and

$$M_x(t) = \sum_{\ell_1,\ell_2,\boldsymbol{r}=0}^{\infty} \sum_{\ell_3=0}^{\beta^*-1} \sum_{\ell_4=0}^{\boldsymbol{r}} \frac{t^{\boldsymbol{r}}}{\boldsymbol{r}!} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \Gamma\left(1+\frac{\ell_4}{\delta}, \dot{\ell_3}\right)|_{(\beta^*>0 \text{ and integer})}.$$

3.1. Incomplete moments

The s^{th} incomplete moment, say $\boldsymbol{\Upsilon}_s\,(t)$, of X can be expressed from (9) as

$$\boldsymbol{\Upsilon}_s\,(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{\ell_1,\ell_2,\ell_3,r=0}^{\infty} \sum_{\ell_4=0}^{s} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \begin{bmatrix} \Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}) \\ -\Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}(1+\beta t)^{\delta}) \end{bmatrix}, \tag{12}$$

and

$$\boldsymbol{\Upsilon}_s\,(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{\ell_1,\ell_2=0}^{\infty} \sum_{\zeta=0}^{\beta^*-1} \sum_{\ell_4=0}^{s} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \begin{bmatrix} \Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}) \\ -\Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}(1+\beta t)^{\delta}) \end{bmatrix}|_{(\beta^*>0 \text{ and integer})}.$$

The mean deviations about the mean$[\ell_{11} = \boldsymbol{E}(|X - \acute{\mu}_1|)]$ and about the median $[\ell_{12} = \boldsymbol{E}(|X - Q(\frac{1}{2})|)]$ of X are given by

$$\ell_{11} = -2\boldsymbol{\Upsilon}_1(\acute{\mu}_1) + 2\acute{\mu}_1 F(\acute{\mu}_1) \text{ and } \ell_{12} = -2\boldsymbol{\Upsilon}_2\left(Q\left(\frac{1}{2}\right)\right) + \acute{\mu}_1,$$

respectively, where $\acute{\mu}_1 = \boldsymbol{E}(X)$, Median $(X) = Q(\frac{1}{2})$ is the median, $F(\acute{\mu}_1)$ is easily calculated from (3) and $\boldsymbol{\Upsilon}_1\,(t)$ is the first incomplete moment given by (12) with $s = 1$. The $\boldsymbol{\Upsilon}_1\,(t)$ can be derived from (12) as

$$\boldsymbol{\Upsilon}_1\,(t) = \sum_{\ell_1,\ell_2,\ell_3=0}^{\infty} \sum_{\ell_4=0}^{1} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \begin{bmatrix} \Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}) \\ -\Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}(1+\beta t)^{\delta}) \end{bmatrix},$$

and

$$\boldsymbol{\Upsilon}_1\,(t) = \sum_{\ell_1,\ell_2=0}^{\infty} \sum_{\ell_3=0}^{\beta^*-1} \sum_{\ell_4=0}^{1} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \begin{bmatrix} \Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}) \\ -\Gamma(1+\frac{\ell_4}{\delta}, \dot{\ell_3}(1+\beta t)^{\delta}) \end{bmatrix}|_{(\beta^*>0 \text{ and integer})}.$$

3.2. Probability weighted moments (PWMs)

The $(s, \mathbf{r})^{\text{th}}$ PWM of X following the BuXENH, say $\mu_{s,r}$ is formally defined by

$$\mu_{s,\mathbf{r}} = \boldsymbol{E}\{X^s F(X)^{\mathbf{r}}\} = \int_{-\infty}^{\infty} x^s F(x)^{\mathbf{r}} f(x)dx.$$

Using equations (3) and (4), we can write

$$f(x)F(X)^{\mathbf{r}} = \sum_{\ell_1,\ell_2=0}^{\infty} q_{\ell_1,\ell_2}\omega_{\beta^*,\delta,\beta}(x),$$

where

$$q_{\ell_1,\ell_2} = \frac{2\eta(-1)^{\ell_1}\Gamma(2\ell_1+\ell_2+3)}{\ell_1!\ell_2!\Gamma(2\ell_1+3)\beta^*} \times \sum_{m=0}^{\infty}(-1)^m(1+m)^{\ell_1}\binom{\eta(\mathbf{r}+1)-1}{m}.$$

Then, the $(s,\mathbf{r})^{\text{th}}$ PWM of X can be expressed as

$$\mu_{s,\mathbf{r}} = \sum_{\ell_1,\ell_2,\ell_3=0}^{\infty}\sum_{\ell_4=0}^{r} V_{\ell_3,\ell_4}^{\beta^*}\Gamma(1+\frac{\ell_4}{\delta},\ell_3),$$

where

$$V_{\ell_3,\ell_4}^{\beta^*} = q_{\ell_1,\ell_2}\mathbb{C}_{\ell_3,\ell_4}^{\beta^*}$$

and also

$$\mu_{s,\mathbf{r}} = \sum_{\ell_1,\ell_2=0}^{\infty}\sum_{\ell_3=0}^{\beta^*-1}\sum_{\ell_4=0}^{\mathbf{r}} V_{\ell_3,\ell_4}^{\beta^*} \times \Gamma(1+\frac{\ell_4}{\delta},\ell_3)\,|_{(\beta^*>0 \text{ and integer})}.$$

3.3. Residual and reversed residual life

The n^{th} moment of the residual life, say

$$\eta(t) = \boldsymbol{E}[(X-t)^n]|_{(X>t \text{ and } n=1,2,\cdots)},$$

which uniquely determine the $F(x)$. The n^{th} moment of the residual life of X is given by

$$\eta_n(t) = \frac{\int_t^{\infty}(x-t)^n dF(x)}{1-F(t)}.$$

Therefore,

$$\eta_n(t) = \frac{1}{1-F(t)}\sum_{\ell_1,\ell_2,\ell_3=0}^{\infty}\sum_{\ell_4=0}^{n}\sum_{\mathbf{r}=0}^{n}\binom{n}{\mathbf{r}}(-t)^{n-\mathbf{r}} \times \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}\Gamma\left(1+\frac{\ell_4}{\delta},\ell_3\right),$$

or

$$\eta_n(t) = \frac{1}{1-F(t)}\sum_{\ell_1,\ell_2=0}^{\infty}\sum_{\ell_4=0}^{n}\sum_{\mathbf{r}=0}^{n}\binom{n}{\mathbf{r}}(-t)^{n-\mathbf{r}} \times \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*}\Gamma\left(1+\frac{\ell_4}{\delta},\ell_3\right)|_{(\beta^*>0 \text{ and integer})}.$$

The mean residual life (MRL) function or the life expectation at age t can be defined by

$$\eta_{n=1}(t) = \boldsymbol{E}[(X-t)|X>t],$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n=1$ in the last equation.

The n^{th} moment of the reversed residual life, say

$$\eta_n(t) = \boldsymbol{E}[(t-X)^n]|_{(X\le t,\, t>0 \text{ and } n=1,2,\cdots)},$$

then, we obtain

$$\eta_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$\eta_n(t) = \frac{1}{F(t)} \sum_{\ell_1,\ell_2,\ell_3=0}^{\infty} \sum_{\ell_4=0}^{n} \sum_{\mathbf{r}=0}^{n} (-1)^{\mathbf{r}} \binom{n}{\mathbf{r}} t^{n-\mathbf{r}} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \left[\Gamma\left(1+\frac{\ell_4}{\delta}, \ell_3^{\cdot}\right) - \Gamma\left(1+\frac{\ell_4}{\delta}, \ell_3^{\cdot}(1+\beta t)^{\delta}\right) \right],$$

or

$$\eta_n(t) = \frac{1}{F(t)} \sum_{\ell_1,\ell_2=0}^{\infty} \sum_{\ell_3=0}^{\beta^*-1} \sum_{\ell_4=0}^{n} \sum_{r=0}^{n} (-1)^{r} \binom{n}{\mathbf{r}} t^{n-\mathbf{r}} \mathbb{C}_{\ell_1,\ell_2,\ell_3,\ell_4}^{\beta^*} \times \begin{bmatrix} \Gamma(1+\frac{\ell_4}{\delta}, \ell_3^{\cdot}) \\ -\Gamma(1+\frac{\ell_4}{\delta}, \ell_3^{\cdot}(1+\beta t)^{\delta}) \end{bmatrix} |_{(\beta^*>0 \text{ and integer})}.$$

The mean waiting time (MWT) or the mean inactivity time (MIT) which also called the mean reversed residual life function, is given by

$$\eta_{n=1}(t) = \boldsymbol{E}[(t-X)^n]|_{(X\leq t,\ t>0\, and\, n=1)},$$

and it represents the waiting time elapsed since the failure of an item on condition had occurred in $(0,t)$. The MIT of the BuXENH distribution can be obtained easily by setting $n = 1$ in the above equation of $\eta_n(t)$.

3.4. Order statistics

Let $X_1, X_2, \cdots, X_n$ be any random sample (RS) from the BuXENH of distribution and let $X_{(1:n)}, X_{(2:n)}, \cdots, X_{(n:n)}$ be the corresponding order statistics. The PDF of ℓ_4^{th} order statistic, say $X_{(i:n)}$, can be written as

$$f(x_{i:n}) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{13}$$

where $\beta(\cdot,\cdot)$ is the beta function. Using (3) , (4) in Equation (13) we get

$$f(x)F(x)^{j+i-1} = \sum_{\ell_1,\ell_2=0}^{\infty} \boldsymbol{\Upsilon}_{\ell_1,\ell_2} \omega_{\beta^*,\delta,\beta}(x),$$

where

$$\Upsilon_{\ell_1,\ell_2} = \frac{2\eta(-1)^{\ell_1}\Gamma(2\ell_1+\ell_2+3)}{\ell_1!\ell_2!\Gamma(2\ell_1+3)\beta^*} \times \sum_{m=0}^{\infty} (-1)^m (1+m)_1^{\ell} \binom{\eta(j+i)-1}{m}.$$

The PDF of $X_{i:n}$ can be expressed as

$$f(x_{i:n}) = \sum_{\ell_1,\ell_2=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j} \boldsymbol{\Upsilon}_{\ell_1,\ell_2}}{B(i, n-i+1)} \omega_{\beta^*,\delta,\beta}(x). \tag{14}$$

Then, the density function of the BuXENH order statistics is a mixture of ENH density. Based on (14), the moments of $X_{i:n}$ can be expressed as

$$\boldsymbol{E}(X_{i:n}^{q}) = \sum_{\ell_1,\ell_2,h=0}^{\infty}\sum_{d=0}^{\mathbf{r}}\sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)}W_{\ell_1,\ell_2,h,d}^{[\beta^*,q]}\Gamma(1+\delta^{-1}d,1+h),$$

where

$$W_{\ell_1,\ell_2,h,d}^{[\beta^*,q]} = \boldsymbol{\Upsilon}_{\ell_1,\ell_2}\mathbb{C}_{h,d}^{[\beta^*,q]}.$$

Or

$$\boldsymbol{E}(X_{i:n}^{q}) = \sum_{\ell_1,\ell_2=0}^{\infty}\sum_{h=0}^{\beta^*-1}\sum_{d=0}^{\mathbf{r}}\sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)} \times W_{\ell_1,\ell_2,h,d}^{[\beta^*,q]}\Gamma(1+\delta^{-1}d,1+h)|_{(\beta^*>0 \text{ and integer})}.$$

4. Numerical Calculations

Numerical calculations for the E(X); V(X), S(X) and K(X) are listed in Table 1. Based on Table 1, we note that:

1) The S(X) of the BuXENH distribution is always positive.
2) The K(X) of the BuXENH distribution is only more than 3.
3) The mean of the BuXENH distribution increases as η increases.
4) The mean of the BuXENH distribution increases as β increases.
5) The mean of the proposed model decreases as δ increases.

Table 1 E(X), V(X), S(X) and K(X) of the BuXENH distribution

η	β	δ	$E(X)$	$V(X)$	$S(X)$	$K(X)$
0.1	1.2	0.25	1.825306	12.9764	3.240258	16.17459
0.5			6.180584	30.77811	1.419628	5.318784
1			9.101343	34.88533	1.034272	4.220169
10			19.73202	27.84087	0.708228	3.824648
50			26.26252	21.2879	0.7626281	4.016543
100			28.81213	19.20417	0.7906129	4.098009
500			34.25282	15.60443	0.8460061	4.261613
10	0.1	0.5	0.02196148	0.00029232	1.898168	8.825817
	0.2		0.2185484	0.00852704	0.9203215	4.295162
	0.3		0.5312317	0.02878662	0.6748841	3.68801
	0.4		0.8821722	0.05651321	0.5615921	3.473251
	0.5		1.241497	0.08827399	0.4938769	3.364005
5	0.5	0.1	46.36367	1884.3660	3.097874	21.31205
		0.2	5.365082	6.8494040	1.231972	5.521063
		0.3	2.375216	0.8218181	0.7939127	3.960367
		0.4	1.473930	0.2447502	0.5918894	3.47713
		0.5	1.057536	0.1075985	0.4745227	3.262254

5. Simple Type Copula Based Construction

In this Section, we consider several approaches to construct the bivariate and the multivariate BuXENH type distributions via copula (or with straightforward bivariate CDFs form, in which we just need to consider two different BuXENHCDFs).For more details Farlie (1960), Gumbel (1961& 1960), Johnson and Kotz (1975 & 1977), Al-babtain et al. (2020), Mansour et al. (2020 a-f), Elgohari and Yousof (2020 a,b), Salah et al. (2020), Yousof et al. (2020), Ibrahim et al. (2020) and Ali et al. (2021 a,b).

5.1. Via Morgenstern family

First, we start with CDF for Morgenstern family of two random variables(X_1, X_2) which has the following form

$$F_\theta(x_1,x_2)|_{(|\theta|\leq 1)} = F_1(x_1)F_2(x_2)\{1+\theta[1-F_1(x_1)][1-F_2(x_2)]\},$$

setting

$$F_{\delta_1,\beta_1,\eta_1}(x_1) = \left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(x_1+1)^{\delta_1}]\}^{\beta_1}}{1-\{1-\exp[1-(x_1+1)^{\delta_1}]\}^{\beta_1}}\right)^2\right]\right\}\right)^{\eta_1},$$

and

$$F_{\delta_2,\beta_2,\eta_2}(x_2) = \left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(x_2+1)^{\delta_2}]\}^{\beta_2}}{1-\{1-\exp[1-(x_2+1)^{\delta_2}]\}^{\beta_2}}\right)^2\right]\right\}\right)^{\eta_2},$$

then we have a five dimension parameter model.

5.2. Via Clayton copula

5.2.1 The bivariate extension

The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

$$\mathbb{C}_{\theta(\ell_{2(x)},\upsilon_{(y)})} = \left[\ell_{2(x)}^{-(\theta_1+\theta_2)} + \upsilon_{(y)}^{-(\delta_1+\delta_2)} - 1\right]^{-\frac{1}{\theta_1+\theta_2}}.$$

This is indeed a valid copula. Next, let us assume that $X \sim$ BuXENH$(\beta_1,\delta_1,\eta_1)$ and $Y \sim$ BuXENH$(\beta_2,\delta_2,\eta_2)$ .Then, setting

$$\ell_{2(x)} = \left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(x+1)^{\delta_1}]\}^{\beta_1}}{1-\{1-\exp[1-(x+1)^{\delta_1}]\}^{\beta_1}}\right)^2\right]\right\}\right)^{\eta_1},$$

and

$$\upsilon_{(y)} = \left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(y+1)^{\delta_2}]\}^{\beta_2}}{1-\{1-\exp[1-(y+1)^{\delta_2}]\}^{\beta_2}}\right)^2\right]\right\}\right)^{\eta_2},$$

the associated CDF bivariate BuXENH type distribution will be

$$\mathbb{C}_\theta(x,y) = \left(\begin{array}{c} \left[\left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(x+1)^{\delta_1}]\}^{\beta_1}}{1-\{1-\exp[1-(x+1)^{\delta_1}]\}^{\beta_1}}\right)^2\right]\right\}\right)^{\eta_1}\right]^{-(\delta_1+\delta_2)} \\ +\left[\left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(y+1)^{\delta_2}]\}^{\beta 2}}{1-\{1-\exp[1-(y+1)^{\delta_2}]\}^{\beta_2}}\right)^2\right]\right\}\right)^{\eta_2}\right]^{-(\delta_1+\delta_2)} \\ -1 \end{array}\right)^{-\frac{1}{\theta_1+\theta_2}}.$$

5.2.2 The multivariate extension

A straightforward m-dimensional extension IRom the above will be

$$H(\underline{X}) = \left\{ \sum_{i=1}^{m} \left[\left(\left\{ \begin{array}{c} 1- \\ \exp\left[-\left(\frac{\{1-\exp[1-(x_i+1)^{\delta_i}]\}^{\beta_i}}{1-\{1-\exp[1-(x_i+1)^{\delta_i}]\}^{\beta_i}} \right)^2 \right] \\ +1-m \end{array} \right\} \right)^{\eta_i} \right]^{-(\theta_1+\theta_2)} \right\}^{-1/(\theta_1+\theta_2)},$$

where $\underline{X} = x_1, x_2, \cdots, x_m$ Further future works could be allocated for studying the bivariate and the multivariate extensions of the BuXENH model.

6. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a rs from BuXENH distribution with parameter vector $\underline{\boldsymbol{\Psi}} = (\eta, \delta, \beta)^T$. The log-likelihood function for $\underline{\boldsymbol{\Psi}}$,say$\ell(\underline{\boldsymbol{\Psi}})$ is given by

$$\begin{aligned} \ell(\underline{\boldsymbol{\Psi}}) &= n\log\eta + n\log\beta + n\log\delta \\ &\quad + (2\beta-1)\sum_{i=0}^{n}\log\{1-\exp[1-(\beta x_i+1)^{\delta}]\} \\ &\quad + n\log 2 - 3\sum_{i=0}^{n}\log\{1-\exp[1-(\beta x_i+1)^{\delta}]\} \\ &\quad + (\eta-1)\sum_{i=0}^{n}\log\left(\left\{1-\exp\left[-\left(\frac{\{1-\exp[1-(\beta x_i+1)^{\delta}]\}^{\beta}}{1-\{1-\exp[1-(\beta x_i+1)^{\delta}]\}^{\beta}}\right)^2\right]\right\}\right) \\ &\quad + \sum_{i=0}^{n}[(1+x)^{\delta}-1] + \sum_{i=0}^{n}\left[-\left(\frac{\{1-\exp[1-(\beta x_i+1)^{\delta}]\}^{\beta}}{1-\{1-\exp[1-(\beta x_i+1)^{\delta}]\}^{\beta}}\right)^2\right]. \end{aligned} \tag{15}$$

The function $\ell(\underline{\boldsymbol{\Psi}})$ can be maximized either by using the different programs like R (optim function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating 15. The score vector elements,

$$\boldsymbol{U}(\underline{\boldsymbol{\Psi}}) = \left(\frac{\partial\ell(\underline{\boldsymbol{\Psi}})}{\partial\eta}, \frac{\partial\ell(\underline{\boldsymbol{\Psi}})}{\partial\beta}, \frac{\partial\ell(\underline{\boldsymbol{\Psi}})}{\partial\delta} \right)^T,$$

are easily to be derived.

7. Simulation Studies

In this Section, we simulate the BuXENH model by taking n = 50; 100; 250; 500 and 1000: For each sample size (n), we evaluate the ML estimations (MLEs) of the parameters. Then, we repeat the process 1000 times (i.e. N = 1000) and compute the averages of the estimates (AEs) and the mean squared errors (MSEs). Table 3 gives all numerical results of the simulation experiments. The numerical results in Table 3 indicate that the MSEs and the bias of $\hat{\eta}, \hat{\delta}$ and $\hat{\beta}$ decay towards zero when n increases for all settings of η, δ and β as expected under the asymptotic theory or large sample theory. The AEs of the parameters tend to be closer to the true parameter values

$$\begin{aligned} \boldsymbol{I}: &\quad \eta = 3.5, \beta = 2, \delta = 3.5 \\ \boldsymbol{II}: &\quad \eta = 0.5, \beta = 4, \delta = 0.5 \end{aligned}$$

when n increases. These results support that the asymptotic normal model provides good approximation to the finite sample model of the MLEs.

Table 2 AEs and MSE for $N = 1000$

n	Θ	AE	MSE	Θ	AE	MSE
		I			**II**	
50	η	3.9140431	0.4946601	η	0.8072650	0.4467096
	β	1.7987872	1.5511413	β	3.7729865	0.5562624
	δ	3.3611143	0.8064241	δ	0.8968461	1.2250001
100	η	3.8350883	0.3895662	η	0.7467954	0.3020398
	β	1.8402122	1.3076565	β	3.7900983	0.5000916
	δ	3.4522802	0.3918004	δ	0.7450011	0.4888196
250	η	3.7601383	0.3007621	η	0.5952243	0.1509002
	β	1.9127442	0.3632541	β	3.8587660	0.2671801
	δ	3.4922332	0.0931274	δ	0.6297001	0.2990991
	β	0.9609656	0.1054192	β	4.6400833	0.3281886
500	η	3.5134365	0.0031623	η	0.5013011	0.0022600
	β	1.9897431	0.0019876	β	3.9590981	0.0011198
	δ	3.5004343	0.0076872	δ	0.5009512	0.0099211
1000	η	3.5000231	0.0001421	η	0.5000661	0.0002461
	β	1.9998776	0.0001923	β	3.9911910	0.0012630
	δ	3.5004311	0.0000650	δ	0.5002443	0.0004702

8. Data Analysis

In this section, we present an application based on the real data set to show the flexibility of the BuXENH distribution. First, we compare BuXENH with the RNH, the odd Lindley NH distribution (OLNH) (Yousof et al., (2017)), Proportional reversed hazard rate (PRHRNH) (new), exponentiated Weibull NH (New), the Gamma-NH (GaNH) (Ortega et al., (2015)), Marshall-Olkin NH (MONH) (Lemonte et al., (2016)), generalized NH (ENH) (Lemonte (2013)), beta-NH (BNH) (Dias et al., (2018)), the standard NH distributions. Other useful extension of the NH model such as the Topp-Leone NH distribution (Yousof and Korkmaz (2017)) and extended exponentiated NH model (Alizadeh et al., (2018)). The model selection is applied using the estimated log-likelihood $\left(\widehat{\ell\left(\underline{\mathbf{\Psi}}\right)}\right)$, Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion ($\mathrm{AI_C}$), Consistent Akaike information criteria ($\mathrm{CAI_C}$), Bayesian information criterion ($\mathrm{BI_C}$), and Hannan-Quinn information criterion ($\mathrm{HQI_C}$). $\mathrm{AI_C}$, $\mathrm{CAI_C}$, $\mathrm{BI_C}$ and $\mathrm{HQI_C}$.

$$\begin{aligned} AI_C &= -2\widehat{\ell(\underline{\mathbf{\Psi}})} + 2n_{(p)}, \\ CAI_C &= -2\widehat{\ell(\underline{\mathbf{\Psi}})} + \frac{2n_{(p)}}{n - [n_{(p)}] - 1}, \\ BI_C &= -2\widehat{\ell(\underline{\mathbf{\Psi}})} + p\log[n_{(p)}], \end{aligned}$$

and

$$HQI_C = -2\widehat{\ell(\underline{\mathbf{\Psi}})} + 2[n_{(p)}]\log(\log n)$$

where $n_{(p)}$ is the number of the estimated model parameters and n is sample size. In general, the smaller values of $\mathrm{AI_C}$, $\mathrm{CAI_C}$, $\mathrm{BI_C}$, $\mathrm{HQI_C}$ and K-S indicate to the better fit to the data set and the biggest log-likelihood and p values of the K-S statistics is chosen. The used data corresponds to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. These data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place (see Choulakian and Stephens (2001)). This data also have been applied by Lemonte (2013) for the ENH distribution. Second, total time on test (TTT) plot (see Aarset (1987)) is given for the used data set (see Figure 5 (left)). The TTT plot for the exceedances of flood peaks data in Figure 5 denotes that the failure rate function of these data is a bathtub-shaped (U) function. The box plot is plotted in Figure 5 (right). Finally, we present the estimated PDF, estimated CDF, estimated HRF in Figure 6. The probability-probability (P-P) plot and the Kaplan-Meier survival plot are a graphical technique for assessing whether or not a data set follows a given distribution. Figure 7 (left) gives the probability-probability (P-P). Based on Figure 7 (left) it is noted that the exceedances data follows a the BuXENH distribution. Figure 7 (right) gives the Kaplan-Meier survival plot. Based on Figure 7 (right) it is noted that the exceedances data follows a the BuXENH distribution.

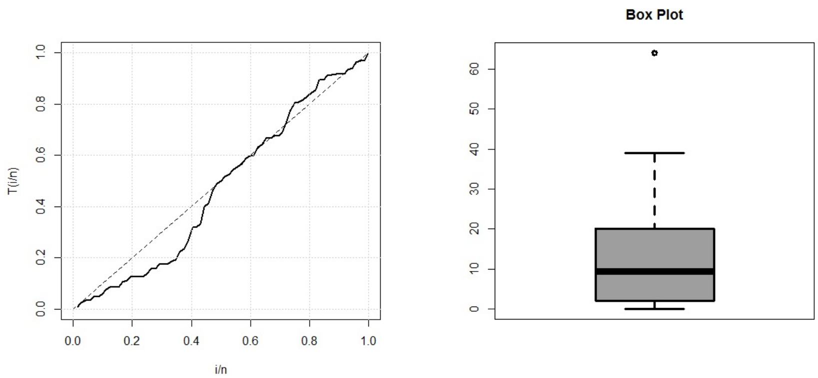

Figure 5 TTT plot and box plot of the exceedances of flood peaks data

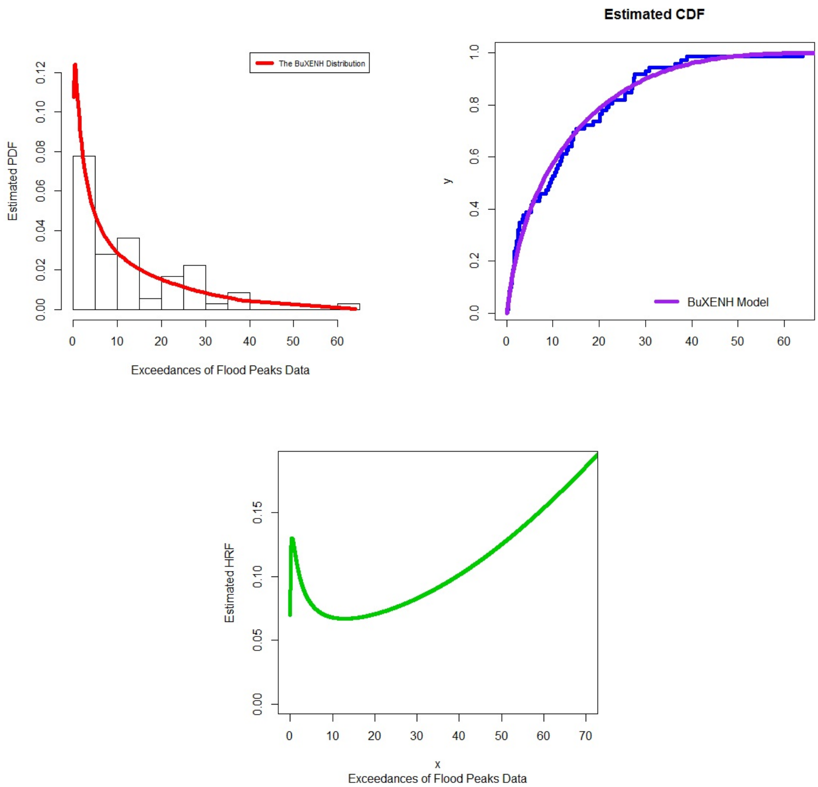

Figure 6 The estimated PDF, CDF, and HRF for the BuXENH model

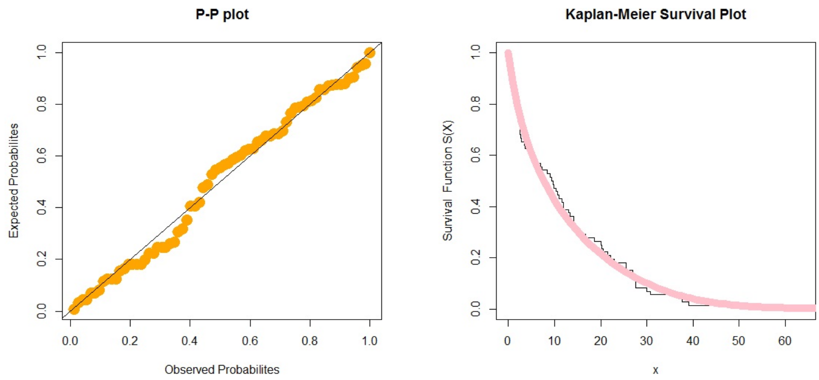

Figure 7 P-P and Kaplan-Meier survival plots for the BuXENH model

All results of this application are listed in Table 4 and Table 5. These results show that the OLNH distribution has the lowest values for AI_C, CAI_C, BI_C, HQI_C and K-S values and also has the biggest estimated log-likelihood and p-value for the K-S statistics among all the fitted models. Thus, it could be chosen as the best model under these criteria and compared to the other fitted models. Finally, we plot estimated functions for the density, CDF, P-P, Kaplan-Meier survival plots of the BuXENH for the exceedances of flood peaks data in Figure 4. Clearly, the BuXENH distribution provides a closer fit to the empirical PDF and CDF. Also, from these figures, we get a bathtub-shaped (**U**-shaped) for the estimated HRF for the exceedances of flood peaks data, which coincide with the TTT plot given in Figure 3.

Table 3 Estimates of the competitive models fitted to the Choulakian and Stephens data

Model	Estimates (SD)			
NH(β, δ)	0.841	0.1094		
	(0.259)	(0.059)		
RNH(β, δ)	0.125	6.28		
	(0.012)	(2.919)		
OLNH(η, δ, β)	0.7293	0.2519	1.8065	
	(0.6059)	(0.052)	(3.355)	
PRHRNH(η, δ, β)	0.364	1.714	0.031	
	(0.068)	(1.191)	(0.031)	
GaNH(η, δ, β)	0.7286	1.9299	0.0242	
	(0.1385)	(1.7591)	(0.0312)	
MONH(η, δ, β)	23.77	0.0011	0.2660	
	(5.5053)	(0.0003)	(0.0895)	
ENH(η, β, δ)	0.7289	1.7126	0.0309	
	(0.1404)	(1.2607)	(0.0330)	
BNH(η, β, δ)	0.8381	316.0285	0.6396	0.0003
	(0.1215)	(4.2194)	(0.8227)	(0.0004)
EWNH(η, β, δ)	2.7591	0.3989	0.4732	0.6129
	(1.742)	(0.167)	(0.158)	(0.959)
BuXENH(η, β, δ)	0.21107	3.09625	0.27634	
	(0.155)	(2.642)	(0.0616)	

Table 4 Statistics of the competitive models fitted to the Choulakian and Stephens data

Model	loglike	AI_C	CAI_C	BI_C	HQI_C	K-S(p-value)
BuXENH	**−249.2157**	**504.43**	**504.78**	**511.26**	**507.15**	**0.07945 (0.7537)**
RNH	−251.722	507.44	507.62	513.99	509.7	0.10629 (0.3901)
NH	−251.9874	507.97	508.15	515.53	509.79	0.12444 (0.2148)
OLNH	−250.589	507.18	507.53	514.01	509.9	0.1009 (0.4565)
PRHRNH	−300.83	607.66	608.02	614.49	610.38	0.24985 (0.0003)
GaNH	−250.917	507.834	508.187	514.66	510.55	0.1065 (0.3880)
MONH	−251.087	508.175	508.53	515.005	510.894	0.1074 (0.377)
EWNH	−250.032	508.064	508.66	517.17	511.69	0.0974 (0.50)
ENH	−250.925	507.849	508.202	514.679	510.57	0.1067 (0.386)
BNH	−251.356	510.713	511.31	519.82	514.34	0.1044 (0.4127)

9. Conclusions

A new three-parameter flexible version from the Nadarajah Haghighi model based on (Lemonte, 2013) is proposed and studied. Statistical properties of the new version are derived. A numerical analysis for the variance, skewness and kurtosis is presented and we found that:

1) The S(X) of the BuXENH distribution is always positive.
2) The K(X) of the BuXENH distribution is only more than 3.
3) The mean of the BuXENH distribution increases as η increases.
4) The mean of the BuXENH distribution increases as β increases.
5) The mean of the proposed model decreases as δ increases.

As well as three-dimensional plots are sketched for discovering the flexibility of the new model. A simple type Copula based construction is presented for deriving many bivariate and multivariate type distributions using the Morgen- stern family and Clayton copula. Parameter estimates process are conducted by the well-known method of maximum likelihood. Numerical illustration of real data set is employed to compare the new model with other competitive models. A numerical simulations are executed to test performance of the used method.

As a future work, we can apply many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test, modified Nikulin-Rao-Robson goodness-of-fit test, Bagdonaviius-Nikulin goodness-of-fit test, modified Bagdonaviius-Nikulin goodness-of-fit test, to the new BuXENH model as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020 a,d), Yadav et al. (2020) and Goual and Yousof (2020), among others.

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References

Aarset MV. How to identify bathtub hazard rate. IEEE Trans Reliab. 1987; 36: 106-108.

Al-babtain AA, Elbatal I, Yousof HM. A New Flexible Three-Parameter Model: Properties, Clayton Copula, and Modeling Real Data. Symmetry. 2020; 12(3): 440.

Ali MM, Yousof HM, Ibrahim M. A new version of the generalized Rayleigh distribution with copula, properties, applications and different methods of estimation. In: Irfan Ali et al., editors. Optimal Decision Making in Operations Research and Statistics: Methodologies and Applications. New York: CRC Press; 2021a; 1: 1-25.

Ali MM, Yousof HM, Ibrahim M. Expanding the Burr X model: properties, copula, real data modeling and different methods of estimation. In: Irfan Ali et al., editors. Optimal Decision Making in Operations Research and Statistics: Methodologies and Applications. New York: CRC Press; 2021b; 1: 26-49.

Alizadeh M, Rasekhi M, Yousof HM, Ramires TG, Hamedani GG. Extended exponentiated Nadarajah-Haghighi model: mathematical properties, characterizations and applications. Stud Sci Math Hung. 2018; 55: 498-522.

Choulakian V, Stephens MA. Goodness-of fit for the generalized Pareto distribution. Technometrics. 2001; 43: 478-484.

Dias CRB, Alizadeh M, Cordeiro GM.The beta Nadarajah-Haghighi distribution. Hacet J Math Stat. 2018; 47: 1302-1320.

Elgohari H, Yousof HM. A generalization of lomax distribution with properties, copula and real data applications. Pak J Stat Oper Res. 2020; 16(4): 697-711.

Elgohari H, Yousof HM. New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. Stat Optim Inf Comput. 2020; 8(4): 972-993.

Elsayed HAH, Yousof HM. The Burr X Nadarajah Haghighi distribution: statistical properties and application to the exceedances of flood peaks data. J Math Stat. 2019; 15: 146-157.

Farlie DJG. The performance of some correlation coefficients for a general bivariate distribution. Biometrika. 1960; 47: 307-323.

Goual H, Yousof HM, Ali MM. Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pak J Stat Oper Res. 2019; 15(3): 745-771.

Goual H, Yousof HM. Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. J Appl Stat. 2020; 47(3): 393-423.

Goual H, Yousof HM, Ali MM. Lomax inverse Weibull model: properties, applications, and a modified chi-squared goodness-of-fit test for validation. J Nonlinear Sci Appl. 2020; 13(6): 330-353.

Gumbel EJ. Bivariate exponential distributions. J Am Stat Assoc. 1960; 55: 698-707.

Gumbel EJ. Bivariate logistic distributions. J Am Stat Assoc. 1961; 56: 335-349.

Ibrahim M, Yadav A S, Yousof HM, Goual H, Hamedani GG. A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Commun Stat Appl Meth. 2019; 26(5): 473-495.

Ibrahim M. The generalized odd Log-logistic Nadarajah Haghighi distribution: statistical properties and different methods of estimation. J Appl Prob Stat. 2020; 15(2): 61-84.

Ibrahim M, Sharma VK, Korkmaz MC, Ali MM, Chesneau C, Yousof HM. A generalization of Lindley distribution: copula, properties and different methods of estimation. MATHSTATICA. 2020; 1: 27.

Johnson NL, Kotz S. On some generalized Farlie-Gumbel-Morgenstern distributions. Commun Stat Theory. 1975; 4: 415-427.

Johnson NL, Kotz S. On some generalized Farlie-Gumbel- Morgenstern distributions- II: Regression, correlation and further generalizations. Commun Stat Theory. 1977; 6: 485-496.

Lemonte AJ. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Comput Stat Data Anal. 2013; 62: 149-170.

Lemonte AJ, Cordeiro GM, Moreno-Arenas G. A new useful three-parameter extension of the exponential distribution. Statistics 2016; 50: 312-337.

Mansour MM, Ibrahim M, Aidi K, Butt NS, Ali MM, Yousof HM, Hamed MS. A new log-logistic lifetime model with mathematical properties, copula, modified goodness-of-fit test for validation and real data modeling. Mathematics. 2020a; 8(9): 1508.

Mansour MM, Butt NS, Ansari SI, Yousof HM, Ali MM, Ibrahim M. A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications. Contrib Maths. 2020b; 1: 57-66.

Mansour MM, Korkmaz MC, Ali MM, Yousof HM, Ansari SI, Ibrahim M. A generalization of the exponentiated Weibull model with properties, copula and application. Eurasian Bull Math. 2020c; 3(2): 84-102.

Mansour M, Rasekhi M, Ibrahim M, Aidi K, Yousof H M, Elrazik EA. A new parametric life distribution with modifed Bagdonavicius-Nikulin goodness-of-fit test for censored validation, properties, applications, and different estimation methods. Entropy. 2020d; 22(5): 592.

Mansour M, Yousof HM, Shehata WA, Ibrahim M. A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. J Nonlinear Sci Appl. 2020e; 13(5), 223-238.

Mansour MM, Butt NS, Yousof HM, Ansari SI, Ibrahim M. A generalization of reciprocal exponential model: Clayton copula, statistical properties and modeling skewed and symmetric real data sets. Pak J Stat Oper Res. 2020f; 16(2): 373-386.

Nadarajah S, Haghighi F. An extension of the exponential distribution. Statistics. 2011; 45: 543-558.

Nascimento AD, Silva KF, Cordeiro GM, Alizade M, Yousof HM, Hamedani GG. The odd Nadarajah-Haghighi family of distributions: properties and applications. Stud Sci Math Hung. 2019; 56(2): 185-210.

Ortega EM, Lemonte AJ, Silva GO, Cordeiro GM. 2015. New flexible models generated by gamma random variables for lifetime modeling. J Appl Stat. 2015; 42(10), 2159-2179.

Salah MM, El-Morshedy M, Eliwa MS, Yousof HM. Expanded Fréchet model: mathematical properties, copula, different estimation methods, applications and validation testing. Mathematics. 2020; 8(11), 1949.

Yadav AS, Goual H, Alotaibi RM, Ali MM, Yousof HM. Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry. 2020; 12(1): 57.

Yousof HM, Afify AZ, Hamedani GG, Aryal G. The Burr X generator of distributions for lifetime data. J Stat Theory Appl. 2017; 16, 288-305.

Yousof HM, Korkmaz MC. Topp-Leone Nadarajah-Haghighi distribution. J Stat: Stat Act Sci. 2017; 2, 119-128.

Yousof HM, Hamedani GG, Ibrahim M. The two-parameter Xgamma Fréchet distribution: characterizations, copulas, mathematical properties and different classical estimation methods. contrib Math. 2020; 2, 32-41.

Yousof HM, Korkmaz MC, Hamedani GG. (2017). The odd Lindley Nadarajah- Haghighi distribution. J Math Comput Sci. 2017; 7, 864-882.