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Inverse Lomax Log-Logistic Distribution with Applications

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Abstract
In this article, an extension of the inverse Lomax (IL) distribution with the log-logistic distribution called iInverse Lomax log-logistic (IL-LL) distribution is considered. Various statistical properties of IL-LL distribution which include moment generating function, order statistics, moments, and Rényi entropy are derived. The results of a Monte Carlo simulation study to evaluate the performance of the maximum likelihood parameter estimates of the model are presented. The IL-LL distribution is applied to three different data sets comprising the proportion of toxicity for chromium in marine waters; Number of air conditioning faults in jet aircraft; and relief times of patients being given an analgesic. Using the maximum likelihood estimation method, the analytical criteria, suggests that the new model fits the data sets better than the existing competitors.

Keywords: Inverse Lomax-G family, log-logistic distribution, inverse Lomax log-logistic distribution, Monte Carlo simulation, maximum likelihood estimation

1. Introduction

There are greater interests in developing new classes of continuous univariate distributions. The inclusion of additional parameters has proved useful in exploring skewness and tail properties as well improving the goodness-of - fit of the baseline distributions.

The established generators of distributions include: Inverse Lomax G by Falgore and Doguwa (2020a), the Top-Leone Exponentiated G by Ibrahim et. al. (2020), the Inverse Lomax-Exponentiated G by Falgore and Doguwa (2020b), the Burr X Exponential G by Sanusi et. al. (2020), Kumaraswamy-Odd Rayleigh G by Falgore and Doguwa (2020c), the Power Lindley-G by Hassan and Nassr (2019), the Odd Frechet G by UlHaq and Elgarhy (2018), the beta transmuted H by Afify et. al. (2017), the New Weibull G by Tahir et. al. (2016), the transmuted geometric G by Afify et. al. (2016), the Kumaraswamy Marshal-Olkin by Alizadeh et. al. (2015), the Lomax G by Cordeiro et. al. (2014), the transformed-transformer (T-X) by Alzaatreh et. al. (2013), the Kumaraswamy G by Cordeiro and de Castro (2011), the beta G by Eugene et. al. (2002), as well as the Marshall-Olkin family by Marshall and Olkin (1997), among others.

In the last few decades, the log-logistic (LL) distribution, known as the Risk distribution in economics has been widely used especially for applications in survival and reliability analysis. Log-logistic distribution is an alternative to the log-normal distribution because it provides an initially increasing and decreasing fault rate function. The cumulative distribution function (cdf) and probability density function (pdf) of the Log-Logistic distribution are given by

$$G(x;\lambda) = x^{\lambda}(1+x^{\lambda})^{-1} \qquad x, \lambda > 0, \tag{1}$$

$$g(x;\lambda)=\lambda x^{\lambda-1}(1+x^{\lambda})^{-2} \qquad x,\lambda>0, \tag{2}$$

where λ is the shape parameter. The motivation for this research is to extend the log-logistic distribution in order to improve its flexibility and characteristics, and increase its usage in modeling real life applications.

The rest of the paper is organized as follows. Section 2 presents the proposed inverse Lomax log logistic distribution. Some of the statistical properties of the IL-LL distribution like moments, moment generating function, Renyi entrophy, and order statistics are considered in Section 3. While the estimation technique is discussed in Section 4, Section 5 presents the results of the Monte Carlo Simulation study which should numerically show the accuracy and consistency of the Maximum Likelihood estimators. The application of the proposed distribution to three different data sets are discussed in Section 6. Section 7 concludes the paper.

2. The Inverse Lomax-Log Logistic Distribution

Recently, Falgore and Doguwa (2020a) proposed inverse Lomax-G generator of distributions with the cumulative density function (cdf) given by

$$F(x;\zeta)=\left(1+\frac{\beta\bar{G}(x;\sigma)}{G(x;\sigma)}\right)^{-\alpha}; \qquad x>0,\alpha,\beta,\sigma>0 \tag{3}$$

where $\zeta=(\sigma,\alpha,\beta)^T$, $\bar{G}(x;\sigma)=1-G(x;\sigma)$ and also β and α are the two additional parameters that are added to make the baseline distribution much more flexible. The corresponding pdf $f(x;\zeta)$ of IL-G family obtained by differentiating Equation 3 as given below

$$f(x;\zeta)=\frac{\beta\alpha g(x;\sigma)}{[G(x;\sigma)]^2}\left(1+\frac{\beta\left(1-G(x;\sigma)\right)}{G(x;\sigma)}\right)^{-(1+\alpha)}. \tag{4}$$

Based on Equations (3) and (4), we can insert Equations (1) and (2) of the baseline log-logistic distribution and come up with the Inverse Lomax-Log logistic (IL-LL) distribution. The cdf of IL-LL is

$$F_{IL-LL}(x;\lambda,\alpha,\beta)=\left(1+\beta x^{-\lambda}\right)^{-\alpha} \qquad x,\alpha,\beta,\lambda>0 \tag{5}$$

$$\lim_{x\to 0}F_{IL-LL}(x;\alpha,\beta,\lambda)=0$$
$$\lim_{x\to\infty}F_{IL-LL}(x;\lambda,\alpha,\beta)=1,$$

where $\lambda,\alpha>0$ are the shape parameters and $\beta>0$ is the scale parameter, respectively. These shows that

$$0\le F_{IL-LL}(x;\lambda,\alpha,\beta)\le 1$$

$\forall$ x. And the pdf of IL-LL is given by

$$f_{IL-LL}(x;\lambda,\alpha,\beta)=\alpha\lambda\beta x^{-\lambda-1}\left(1+\beta x^{-\lambda}\right)^{-\alpha-1} \qquad x>0,\alpha,\beta,\lambda>0. \tag{6}$$

2.1. Validity of IL-LL distribution

$$\int_0^{\infty}f_{IL-LL}(x;\lambda,\alpha,\beta)dx=\int_0^{\infty}\alpha\beta\lambda x^{-(\lambda+1)}\left(1+\beta x^{-\lambda}\right)^{-\alpha-1}dx \tag{7}$$

let $y=\beta x^{-\lambda}$, and $dx=-\frac{dy}{\beta\lambda x^{-\lambda-1}}$. By considering the limits and to substitute for x and dx in Equation 7

$$\alpha\int_0^{\infty}\frac{dy}{(1+y)^{1+\alpha}}=\alpha\left(\frac{1}{\alpha}\right)=1.$$

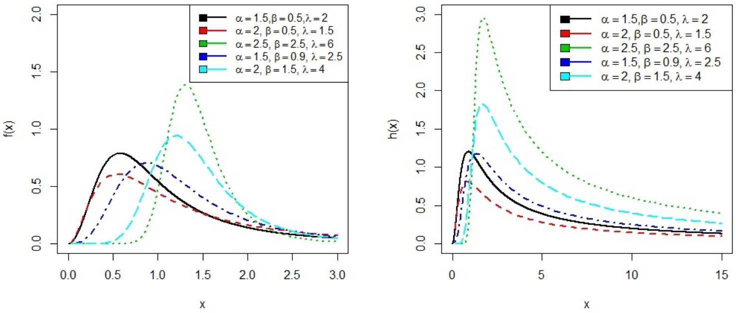

Figure 1 The IL-IL distribution pdf and hazard function plots with varying parameter values.

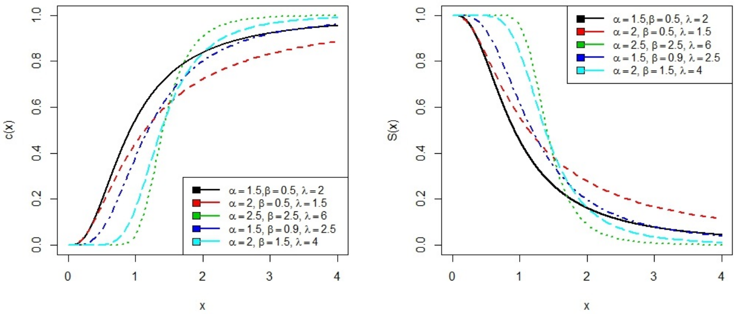

Figure 2 The cdf and survival function plots of IL-IL distribution with various parameter values.

3. Statistical Properties of Inverse Lomax-Log Logistic (IL-LL) Distribution

In this section, we considered some of the statistical properties of the IL-LL distribution like moments, moment generating function, Rényi entrophy, and order statistics.

3.1. Moments

Suppose X is a random variable with density function defined in Equation (6), the r^{th} non-central moments of X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f_{IL-LL}(x;\alpha,\beta,\lambda)dx \tag{8}$$
$$= \alpha\beta^{\frac{r}{\lambda}} \frac{\Gamma\left(1-\frac{r}{\lambda}\right)\Gamma\left(\alpha+\frac{r}{\lambda}\right)}{\Gamma(1+\alpha)}, \; r = 1, 2, 3, 4, ...$$

where $\frac{r}{\lambda} < 1$ and $r < \lambda$.

The first non-central moment is given by $\mu_1' = \alpha\beta^{\frac{1}{\lambda}} \frac{\Gamma\left(1-\frac{1}{\lambda}\right)\Gamma\left(\alpha+\frac{1}{\lambda}\right)}{\Gamma(1+\alpha)}$,

The second non-central moment is given by $\mu_2' = \alpha\beta^{\frac{2}{\lambda}} \frac{\Gamma\left(1-\frac{2}{\lambda}\right)\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\Gamma(1+\alpha)}$, The third non-central moment is given by $\mu_3' = \alpha\beta^{\frac{3}{\lambda}} \frac{\Gamma\left(1-\frac{3}{\lambda}\right)\Gamma\left(\alpha+\frac{3}{\lambda}\right)}{\Gamma\left(1+\alpha\right)}$,

The fourth non-central moment is given by $\mu_4' = \alpha\beta^{\frac{4}{\lambda}} \frac{\Gamma\left(1-\frac{3}{\lambda}\right)\Gamma\left(\alpha+\frac{3}{\lambda}\right)}{\Gamma\left(1+\alpha\right)}$.

Table 1 The Variance, skewness, mean, and kutosis based on the moments of IL-LL distribution for some parameter values

parameters	mean	variance	skewness	kurtosis
α=3, β= 0.4, λ= 8	1.0939	0.0457	1.1376	1.2072
α=3, β= 0.8, λ= 8	1.1929	0.0543	1.1376	1.2072
α=4, β= 4, λ= 15	1.2449	0.0146	1.0304	1.0422
α=1, β= 0.5, λ= 23	0.9733	0.0059	1.0191	1.0256
α=0.5, β= 5, λ= 20	1.0193	0.0157	1.0438	1.0577
α=0.5, β= 5, λ= 9	1.0647	0.0803	1.2158	1.2975
α=2, β= 1, λ= 5	1.2828	0.2044	1.5887	2.2492
α=1, β= 1, λ= 5	1.0689	0.1786	1.7029	2.4491
α=0.1, β= 0.4, λ= 5	0.3048	0.0994	4.2888	6.6989
α=0.5, β= 0.5, λ= 5	0.7423	0.1295	1.9584	2.8836

Table 2 The Variance, skewness, mean, and kutosis based on the moments of LL distribution for some parameter values

parameters	mean	variance	skewness	kurtosis
λ= 8	1.0262	0.0577	1.1866	1.2732
λ= 8	1.0262	0.0577	1.1866	1.2732
λ= 15	1.0073	0.0151	1.0462	1.0629
λ= 23	1.0031	0.0063	1.0191	1.0256
λ= 20	1.0041	0.0084	1.0254	1.0343
λ= 9	1.0206	0.0445	1.1413	1.2019
λ= 5	1.0689	0.1786	1.7029	2.4491
λ= 5	1.0689	0.1786	1.7029	2.4491
λ= 5	1.0689	0.1786	1.7029	2.4491
λ= 5	1.0689	0.1786	1.7029	2.4491

Tables 1 and 2 presents the mean, variances, skewness, and kurtosis for the IL-LL and LL distributions, respectively. The mean for the IL-LL distribution ranges from 0.3048 to 1.2828 while that of LL ranges from 1.0031 to 1.0689, the variance of IL-LL is as small as 0.0059 while that of LL is 0.0084. For the skewness, IL-LL distribution ranges from 1.0191 to 4.2888. Whereas, LL distribution ranges from 1.0191 to 1.7029. Lastly, the kurtosis of IL-LL distribution ranges from 1.0256 to 6.6989, while LL distribution ranges from 1.0256 to 2.4491, respectively. All these indicate the strength of distribution of IL-LL over distribution of LL.

3.2. The moment generating function

Moment generating function (mgf) of the IL-LL distribution can be given in terms of (8) as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\alpha\beta^{\frac{r}{\lambda}} \frac{\Gamma\left(1-\frac{r}{\lambda}\right)\Gamma\left(\alpha+\frac{r}{\lambda}\right)}{\Gamma\left(1+\alpha\right)}.$$

3.3. Rényi entropy

If X is random variable with density function $f(x)$ defined in Equation (6), then the Rényi entropy of X is given by

$$R_\tau(x) = \frac{1}{1-\tau}\left[\int_{-\infty}^{\infty} f(x)^\tau dx\right], \qquad \tau > 0, \tau \neq 1; x \in \Re. \tag{9}$$

The function $f(x)^\tau$ in Equation (9) can be

$$f(x)^\tau = (\alpha\beta\lambda)^\tau\, x^{-\tau(\lambda+1)}\left(1+\beta x^{-\lambda}\right)^{-\tau(1+\alpha)}. \tag{10}$$

By inserting Equation (10) back in Equation (9), we have

$$\int_{-\infty}^{\infty} f(x)^\tau dx = (\alpha\beta\lambda)^\tau \int_0^{\infty} x^{-\tau(\lambda+1)}\left(1+\beta x^{-\lambda}\right)^{-\tau(1+\alpha)} dx.$$

Let $w = \beta x^{-\lambda}$ and substituting back. Then,

$$\int_{-\infty}^{\infty} f(x)^\tau dx = \frac{(\alpha\beta\lambda)}{\beta\lambda}\int_0^{\infty} \frac{\left(\frac{w}{\beta}\right)^{[\tau\lambda+\tau-\lambda]-1}}{(1+w)^{\tau+\alpha\tau}} dw.$$

Finally, the Rényi entropy is given by

$$R_\tau(x) = \alpha\lambda^{\tau-1}\beta^{\lambda-\tau\lambda}\frac{\Gamma\left(\tau\lambda+\tau-\lambda\right)\Gamma\left(2\alpha\tau+\tau+\lambda-\tau\lambda\right)}{\Gamma\left(2\alpha\tau+2\tau\right)}.$$

3.4. Order statistic

Let $X_1, X_2, X_3, \ldots, X_n$ be the random samples of size n from probability distribution with pdf $f(x)$ and cdf $F(x)$ as defined in Equations (6) and (5), respectively. Suppose $X_{1:n}, X_{2:n}, X_{3:n}, \ldots, X_{n:n}$ denoted the corresponding order statistics derived from this samples. Then, the p^{th} order statistic can be defined as

$$f_{p:n}(x) = \frac{n!f(x)}{(p-1)!(n-p)!}F(x)^{p-1}\left[1-F(x)\right]^{n-p}. \tag{11}$$

Equation (11) can be

$$f_{p:n}(x) = \frac{2\alpha\beta\lambda n! f(x)F(x)^{i(p-1)}}{(p-1)!(n-p)!}\sum_{i=1}^{n}(-1)^i\binom{n-p}{i}.$$

Therefore, the order statistics can be given by

$$f_{p:n}(x) = \Omega_i x^{-\lambda-1}\left(1+\beta x^{-\lambda}\right)^{-(\alpha+i+1)+ip},$$

where $\Omega_j = \frac{2\alpha\beta\lambda n!}{(p-1)!(n-p)!}\sum_{i=1}^{n}(-1)^i\binom{n-p}{i}$.

4. Estimation

In this section, the parameters of the proposed IL-LL distribution will be estimated using maximum likelihood method. Let $x_1, x_2, x_3, \ldots, x_n$ be random samples of n observations drawn from the IL-LL distribution with vector of parameter $\Theta = (\alpha, \beta, \lambda)^T$. Then the log-likelihood function of Equation (6) denoted by $L(\eta)$ can be written as

$$l(\Theta) = n\log(\alpha\beta\lambda) - (\lambda+1)\sum_{i=1}^{n}\log(x) - (\alpha+1)\sum_{i=1}^{n}\log(1+\beta x^{-\lambda}). \tag{12}$$

By differentiating Equation (12) partially with respect to α, β, and λ, we derived the components of score vector $U(\Theta)$ presented as follows

$$U_\alpha(\Theta) = \frac{n}{\alpha} - \sum_{i=1}^{n} log(1+\beta x^{-\lambda}) \tag{13}$$

$$U_\beta(\Theta) = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{x^{-\lambda}}{(1+\beta x^{-\lambda})} \tag{14}$$

$$U_\lambda(\Theta) = \frac{n}{\lambda} - \sum_{i=1}^{n} log(x) + (1+\alpha)\beta \sum_{i=1}^{n} \frac{x^{-\lambda} log(x)}{(1+\beta x^{-\lambda})}. \tag{15}$$

By setting Equations (13), (14), and (15) to zero and also solving them simultaneously yields the maximum likelihood estimators of the IL-IL distribution. However, the above equations are not tractable and cannot be solved analytically. As such, appropriate statistical software can be used to solve them numerically using iterative technique.

5. Monte Carlo Simulation

Here, a Monte Carlo simulation study is conducted and the results presented to show the parameter estimates performance at various true parameter values. The study is described as follows:

1. For known parameter values i.e $\Theta = (\alpha, \beta, \lambda)^T$, we simulated a random sample of size n is simulated from the IL-LL distribution using Equation (16).
2. We then Estimate the parameters of the IL-LL distribution using MLE.
3. Perform N= 1,000 replications of steps 1 through 2.
4. For each of the three estimated parameters of the IL-LL, we compute the Bias, MSE, from the 1,000 replicates for thr true $\alpha = 0.3$, $\beta = 0.4$, and $\lambda = 0.5$. The statistics are given by

$$\hat{\Theta} = \frac{1}{N}\sum_{i=1}^{N} \hat{\Theta}_i, \quad Bias(\hat{\Theta}) = \hat{\Theta} - \Theta, \quad MSE(\hat{\Theta}) = \sum_{i=1}^{N} \left(\hat{\Theta}_i - \Theta\right)^2$$

where $\hat{\Theta_i} = (\hat{\lambda}, \hat{\alpha}, \hat{\beta})$ are the maximum likelihood estimates for the i^{th} iteration and chosen sample size n $(n = 30, 70, 150, 300, 500, 1000)$.The quantile function for IL-LL is giving as

$$Q_{IL-LL}(u) = \left(\frac{u^{\frac{-1}{\alpha}} - 1}{\beta}\right)^{-\frac{1}{\lambda}}. \tag{16}$$

The numerical results are presented in Tables 3, 4, and 5. The simulation study has shown that irrespective of the parameter values chosen, the Bias and MSE of the parameter estimates decay as the sample size n increases. Thus, the larger the sample size, the more accurate and consistent are the parameter estimates. The estimates are good as they approach the true parameter values as the sample size increases.

Table 3 A Monte Carlo simulation results for IL-LL distribution at $\lambda = 0.5$, $\beta = 0.4$, and $\alpha = 0.3$

Parameter	n	Bias	MSE	Estimate
α	30	0.0592	0.1282	0.3592
	70	0.0178	0.0199	0.3178
	150	0.0071	0.0079	0.3071
	300	0.0017	0.0033	0.3017
	500	0.0005	0.0019	0.3005
	1000	0.0004	0.0009	0.3004
β	30	0.3835	1.9855	0.7835
	70	0.064	0.1952	0.464
	150	0.0174	0.0254	0.4174
	300	0.0091	0.0097	0.4091
	500	0.0069	0.0059	0.4069
	1000	0.0032	0.0029	0.4032
λ	30	0.3047	0.7818	0.8047
	70	0.0633	0.0463	0.5633
	150	0.0256	0.0128	0.5256
	300	0.0136	0.0049	0.5136
	500	0.0081	0.0027	0.5081
	1000	0.0039	0.0012	0.5039

Table 4 A Monte Carlo simulation results for IL-LL distribution at $\lambda = 1$, $\beta = 0.4$, and $\alpha = 0.3$

Parameter	n	Bias	MSE	Estimate
α	30	0.0748	0.2044	0.3748
	70	0.0179	0.0203	0.3179
	150	0.0074	0.0081	0.3074
	300	0.0015	0.0033	0.3015
	500	0.0007	0.0018	0.3007
	1000	-5.7478	0.0008	0.2999
β	30	0.3664	1.7470	0.7664
	70	0.0631	0.1781	0.4631
	150	0.0179	0.0258	0.4179
	300	0.0087	0.0096	0.4087
	500	0.0066	0.0058	0.4066
	1000	0.0036	0.0029	0.4036
λ	30	0.4786	1.4829	1.4786
	70	.1265	.1838	1.1265
	150	0.0517	0.0523	1.0517
	300	0.0278	0.0199	1.0278
	500	0.0158	0.0106	1.0158
	1000	0.0087	0.0047	1.0087

Table 5 A Monte Carlo simulation results for IL-LL distribution at $\lambda = 1.5$, $\beta = 0.4$, and $\alpha = 0.3$

Parameter	n	Bias	MSE	Estimate
α	30	0.0682	0.1619	0.3682
	70	0.0180	0.0202	0.3180
	150	0.0066	0.0079	0.3066
	300	0.0019	0.0033	0.3019
	500	0.0012	0.0018	0.3012
	1000	0.0	0.0008	0.3006
β	30	0.3669	1.8558	0.7669
	70	0.0644	0.2022	0.4644
	150	0.0190	0.0257	0.4190
	300	0.0084	0.0095	0.4084
	500	0.0053	0.0058	0.4053
	1000	0.0032	0.0028	0.4032
λ	30	0.6402	2.4253	2.1402
	70	0.1886	0.4095	1.6886
	150	0.0793	0.1169	1.5793
	300	0.0406	0.0444	1.5406
	500	0.0225	0.0239	1.5225
	1000	0.0129	0.0105	1.5129

6. Application

We demonstrate the applicability of the IL-IL distribution to three data sets. The first data set represents 36 proportion of toxicity for chromium in marine waters. The data has an outliers. Details about the data can be seen in Shao (2000) and Wang and Lee (2011). The second data set reflect the amount of failures for the air conditioning system of jet airplanes as reported by Cordeiro and Lemonte (2011). The third data set represents the Relief Times of some patients receiving an analgesic as reported by Clark and Gross (1975) and Shanker et. al. (1975). The summary of the three data sets are in Table 6.

Table 6 The summary of the three data sets analyzed

Data	n	Maximum	Minimum	Mean	Median	Mode	Skewness	Kurtosis
Relief	20	4.1	1.1	1.9	1.7	1.75	1.7198	2.9241
Marine	36	10000	2.4	1374.284	602.56	1000	2.8443	8.0032
Air condition	179	603	1	89.1341	51	14	2.2333	5.7353

We used Adequacy Model package by Marinho et. al. (2019) in R software developed by Core Team (2017). The goodness of fit statistics used in comparing the performances are as reported by Marinho et. al. (2019). The smaller the value of the goodness of fit measures the better the fit to the data.

As comparators, we use the Marshall-Olkin Log-logistic (MLL) distribution by Gui (2013); the Zografos-Blakrishnan Log-logistic (ZBLL) distribution by Hamedani (2013); and the odds exponential log-logistic (OELL) distribution by Rosaiah et. al. (2017). The cdf of these three comparators are

Table 7 MLEs and log-likelihoods for the data sets

		MLEs			
Data Sets	Models	α	β	λ	-ll
Relief	IL-LL	12.5319	0.4739	4.0031	15.5088
	MLL	1.3178	5.9358	5.4093	16.4787
	ZBLL	1.7857	1.9162	0.6894	29.9657
	OELL	1.9107	1.7118	1.7325	23.5664
Marine Data	IL-LL	2.1991	11.1509	0.5698	292.872
	MLL	1.8154	0.4146	7.0095	300.6979
	ZBLL	0.7134	3.4282	3.8476	294.7173
	OELL	0.321	9.6098	0.7086	295.9606
Air Condition Data	IL-LL	2.5874	12.1413	0.9729	988.8015
	MLL	1.7794	0.7905	6.6285	1034.731
	ZBLL	1.07	1.9286	9.2974	995.4114
	OELL	0.535	1.9286	9.2974	1031.363

Table 8 The Goodness-of-fit statistics for the data sets

Data Sets	Models	AIC	CAIC	BIC	HQIC
Relief	**IL-LL**	**37.0176**	**38.5176**	**40.0048**	**37.6007**
	MLL	38.9551	40.4551	41.9423	39.5382
	ZBLL	65.9313	67.4313	68.9185	66.5144
	OELL	53.1329	54.6329	56.12	53.7159
Marine Data	**IL-LL**	**591.7441**	**592.4941**	**596.4946**	**593.4021**
	MLL	607.3959	608.1459	612.1464	609.054
	ZBLL	595.8346	596.5846	596.5846	600.5851
	OELL	597.9213	598.6713	602.6718	599.5793
Air Condition Data	**IL-LL**	**1983.603**	**1983.74**	**1993.165**	**1987.48**
	MLL	2075.462	2075.599	2085.024	2079.34
	ZBLL	1996.823	1996.96	2006.385	2000.7
	OELL	2068.726	2068.864	2078.289	20772.6

respectively given as:

$$F_{MLL}(x;\alpha,\beta,\lambda) = 1 - \frac{\beta^{\alpha}\lambda}{x^{\alpha}+\beta^{\alpha}\lambda} \qquad x,\lambda,\beta,\alpha > 0$$

$$F_{ZBLL}(x,\beta,\lambda,\alpha) = \frac{\gamma\left(\beta, log\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]\right)}{\Gamma\left(\beta\right)} \qquad x,\beta,\alpha,\lambda > 0$$

$$F_{OELL}(x;\beta,\lambda,\alpha) = 1 - exp\left\{\frac{1}{\beta}\left(\frac{x}{\lambda}\right)^{\alpha}\right\} \qquad \alpha,\lambda,\beta,x > 0.$$

As shown in Table 8, the IL-IL model has the best fit with minimum values of the analytical criteria, thus outperforming the other comparator models.

7. Discussion

The estimates of the parameters are good and also follows the law of large number; as the sample size increases, the estimates approaches the true values. This is clear from Tables 3, 4, and 5, respectively. The inverse Lomax log-logistic distribution has proven to be more flexible compared with the other comparators with a minimum values of the goodness-of-fit statistics as shown in Table 8.

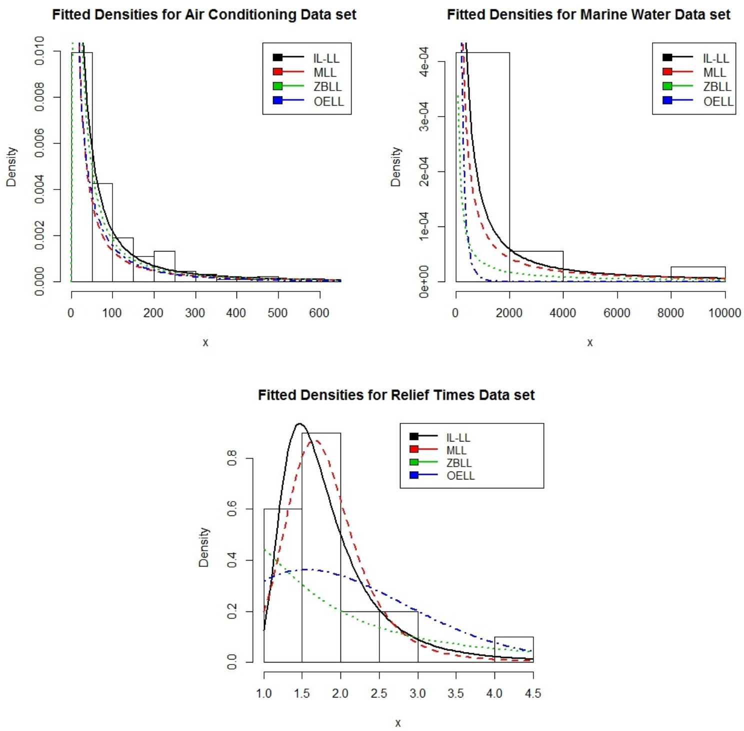

Figure 3 Fitted densities plots for the three data sets.

8. Conclusion

In conclusion, we proposed and study a new model based on inverse Lomax-G family called the inverse Lomax-log logistic (IL-IL) distribution. We investigated some of the IL-LL's statistical properties including Moments and Moment generating function, entropy and order statistics. We estimated the parameters using maximum likelihood method and showed numerically through simulation that the parameter estimates are consistent. The pdf plots in Figiure 1 indicates that the shape can be skewed to the right whereas, the hazard function plots explains the shape as skewed to the right and decreasing. An application to the three datasets namely: the proportion of toxicity for chromium in marine waters; the number of failures for the air conditioning system of jet airplanes; and relief-times of patients receiving an analgesic empirically showed the significance and relevance of the new model.

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