



Thailand Statistician
January 2023; 21(1): 93-109
<http://statassoc.or.th>
Contributed paper

On Interval Estimation of the Geometric Parameter in a Zero-inflated Geometric Distribution

Patchanok Srisuradetchai* and Kittima Dangsupa

Department of Mathematics and Statistics, Faculty of Science and Technology,
Thammasat University, Rangsit Campus, Pathum Thani, Thailand.

*Corresponding author; e-mail: patchanok@mathstat.sci.tu.ac.th

Received: 29 May 2021

Revised: 7 July 2022

Accepted: 7 July 2022

Abstract

Many real-life situations contain count data with an excessive number of zeros. Such situations can be explained by a zero-inflated geometric distribution (ZIG). The score confidence intervals using observed and expected Fisher information based on profile likelihood and the Wald confidence interval using profile likelihood (WCP) are all derived for the geometric parameter of the ZIG in this paper. These proposed intervals are compared with the Wald confidence interval using full likelihood (WC), the most commonly used interval. Then, Monte-Carlo simulations are used to compare all the intervals, and the results show that the score intervals outperform the Wald-type intervals in terms of coverage probability. Also, proposed score intervals and the WCP have closed forms and can be computed without the use of a computer, whereas the WC does not. As a result, the proposed formulas are beneficial in practice.

Keywords: Profile likelihood, Score interval, Wald interval, Monte-Carlo simulations.

1. Introduction

The geometric distribution is a discrete probability distribution that shows how many independent, identical Bernoulli trials it takes to get the first success. Thus, the random values are non-negative integers. This distribution has been widely used to model real-life data in many applications. For example, the number of claims presented to a certain insurance company (Ismail et al. 2018), the strategy space of games involving word sense disambiguation (Tripodi and Pelillo 2017), the duration of a state in a hidden semi-Markov model (Yu 2010), and the number of drilling attempts in an area before finding a productive oil well all have a geometric distribution (Wackerly et al. 2008). The corresponding probability distribution for geometric random variable is given by

$$\Pr(X = x) = (1 - p)^x p, \quad x = 0, 1, 2, \dots,$$

where X denotes the number of failures until the first success. This distribution has mean $(1 - p)/p$ and variance $(1 - p)/p^2$ and is the only model with non-negative integer support that can be characterized by the “Markovian property” (Joshi 2015). However, count data are generally prone to

being affected by random fluctuations and at some point, extra zeros may occur (Pandya et al. 2012). When a dataset appears to have the excess number of zeros, the zero-inflated Poisson (ZIP) distribution is a popular choice for modelling such data (Lambert 1992, Hall 2000). Many applications, including applications in manufacturing (Lambert 1992), horticulture (Hall 2000), demography (Aryal 2011), and zoology (Zipkin et al. 2014), have employed the ZIP distribution. The “heterogeneity problem,” in which the observed variance is lower or higher than the variance of the theoretical model, can be caused by an excessive number of zeros. The latter situation (sometimes called over-dispersion) is commonly found in data analyses. To overcome the over-dispersion problem, the zero-inflated negative binomial (ZINB), of which the zero-inflated geometric (ZIG) distribution is a special case, is the most commonly choice. The ZIG is a mixture of the geometric and Bernoulli probability mass function. The probability mass function is thus:

$$\Pr(X = x) = \begin{cases} \pi + (1 - \pi)p, & x = 0, \\ (1 - \pi)(1 - p)^x p, & x = 1, 2, \dots, \end{cases}$$

where $p > 0, 0 < \pi < 1$ (Johnson et al. 2005). Parameter p is probability of success in the geometric distribution and π is the mixture weight for the Bernoulli component. The mixing parameter π is chosen such that the value of $\Pr(X = x)$ contains $(0, 1)$, i.e. it varies over the interval $-p/(1-p) < \pi < 1$ (Joshi 2015). Figure 1 illustrated the parameter space of (p, π) . The population mean and population variance are defined as $E(X) = (1 - \pi)(1 - p)/p$ and $Var(X) = (1 - \pi)[1 + \pi(1 - p)](1 - p)/p^2$, respectively. Furthermore, because $\partial E(X)/\partial \pi < 0$ and $\partial E(X)/\partial p < 0$, the mean of the ZIG distribution decreases as both parameters increase. The variance-to-mean ratio is always greater than 1, and then ZIG distribution seems to be always over-dispersed (Mallick and Joshi 2018).

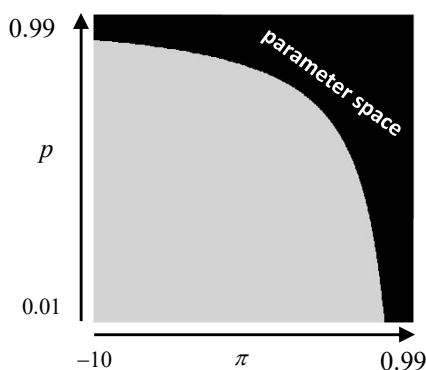


Figure 1 Parameter space of the ZIG distribution

The ZIG regression is an alternative model in various applications. For example, Hussein and Hamodi (2017) studied the influencing factors on the number of Iraqi patients under the age of five years old infected with pneumonia using the geometric, hurdle-geometric, and ZIG distributions, Kolev et al. (2001) applied the ZIG distribution to over-dispersed insurance data, and Mallick and Joshi (2018) analyzed Swedish fertility data. ZIG regression models can also be analyzed with the vector generalized linear and additive models (VGAM) package in R (Yee 2015).

Point estimation for the ZIG distribution has been of interest in the literature. Sharma (1985) originated the idea of including the inflation parameter in modeling trends for rural out-migration data. The method of moments has been used to obtain estimators for the ZIG parameters, and asymptotic expressions of variances and covariances between estimators have been obtained via the multivariate central limit theorem. Later, Iwunor (1995) used the maximum likelihood approach and the mean-zero-frequency method to derive the ZIG parameter estimators, but neither provided closed-form estimates for π and p . The corresponding variance and covariance matrices of the parameters were also given in Iwunor's paper. Sastry et al. (2016) proposed a new generalized geometric distribution permitting the inflation/deflation of the zero-count probability and studied some of its properties. Mallick and Joshi (2018) used the method of moments estimation (MME), empirical probability-generating-function-based (PG) method, and maximum likelihood estimation (MLE) to find point estimators for the generalized inflated geometric (GIG) distribution. This distribution is useful because the zero-one-inflated geometric (ZOIG) and Zero-one-two-inflated geometric (ZOTIG) distribution are special cases of the GIG. The ZIG distribution is also a special case of the GIG distribution, and estimators of the ZIG distribution are $\hat{p}_{MME} = (2\bar{X}) / (S^2 + \bar{X} + \bar{X}^2)$, $\hat{\pi}_{MME} = (S^2 - \bar{X} - \bar{X}^2) / (S^2 - \bar{X} + \bar{X}^2)$, $\hat{p}_{PG} = (1 - f_0) / \bar{X}$, and $\hat{\pi}_{PG} = (\bar{X}f_0 + f_0 - 1) / (\bar{X} + f_0 - 1)$, where f_0 is the relative frequency of $X = 0$. The MLE is recognized to be the most accurate, but it lacks a closed-form expression in this case, making it the most difficult estimate to achieve. Adarabioyo and Ipinyomi (2019) used the ZIP, ZINB, and ZIG distributions on data that has a positively skewed distribution with a mean of 1.25 and a zero-inflation parameter of 0.10. The simulated data were analyzed to find MLE, and the results showed that the ZIP distribution outperformed the ZINB and ZIG distributions.

Because the score test only needs the computation of the limited estimator and the estimate of the likelihood function under the null hypothesis, it is known to be superior to the Wald test (Fahrmeir et al. 2013). The score confidence interval could be another way to improve the interval estimation, and it is based on Fisher information (FIM), which typically represents the expected value of observed Fisher information. To make a clear distinction between the two FIMs, the terms "expected FIM" and "observed FIM" are used in this paper. However, we are only interested in p , and π can be considered a nuisance parameter, which can be eliminated by using the profile likelihood.

The profile likelihood is often used when a typical way of interval construction, i.e., Wald intervals (using full likelihood), is difficult to use in the presence of a nuisance parameter(s) (Cole et al. 2014). This type of likelihood will be used to remove the nuisance parameter, π , for the Wald and score statistics in this study. When the sample size is small, confidence intervals made with profile likelihood are often better than those made with asymptotic standard errors based on the full likelihood (Venzon and Moolgavkar 1988).

Based on the literature, explicit interval estimations of parameter p for the ZIG distribution have not been found and have no formulas that can be utilized. In this paper, four confidence intervals are derived for p when the value of π is unknown. These are the Wald confidence interval (WC), the Wald confidence interval using profile likelihood (WCP), the score confidence intervals using observed FIM (SCI), and the expected FIM (SCJ). Note that Iwunor (1995) already provided point estimators and Fisher information (FIM) which can be easily used for the WC construction. The main contribution to the WC is to show that the estimated values of both observed and expected Fisher information are the same and to give a simple formula. The other three intervals will utilize the profile likelihood to eliminate π . Furthermore, both observed and expected FIMs are employed for the

construction of WCP and score intervals. The performances of the intervals are studied through Monte-Carlo simulations. Coverage probabilities (CPs) of the confidence intervals and interval lengths (ILs) are investigated to compare four types of intervals.

2. Parameter Estimations for the ZIG Distribution

Suppose $X = (X_1, X_2, \dots, X_n)$ is a random sample from the ZIG distribution, then the full likelihood function of (p, π) given the observed data $x = (x_1, x_2, \dots, x_n)$ is

$$L(p, \pi | x) = \prod_{i=1}^n [\pi + (1 - \pi)p]^{I(x_i=0)} [(1 - \pi)(1 - p)^{x_i} p]^{I(x_i \geq 1)},$$

$$= [\pi + (1 - \pi)p]^{n_0} [(1 - \pi)p]^{n_1} (1 - p)^{\sum x_i},$$

where n_0 and n_1 are the numbers of zero counts and positive counts, respectively. The relative likelihood function is defined as $L(p, \pi | x) / \max L(p, \pi | x)$, and the characteristics of the contour plot depend on the parameter values of the ZIG distribution, as shown in Figure 2. The contour shapes are not far from being elliptical, making the assumption of asymptotic multivariate normal distributions for the MLEs more reliable. The following equations are derived by differentiating $\log L(p, \pi | x)$ with respect to p and π , respectively, evaluating at $p = \hat{p}_{ML}$ and $\pi = \hat{\pi}_{ML}$, and setting the expression equal to zero:

$$\frac{\sum x_i}{1 - \hat{p}_{ML}} - \frac{n_1}{\hat{p}_{ML}} = \frac{n_0(1 - \hat{\pi}_{ML})}{\hat{\pi}_{ML} + (1 - \hat{\pi}_{ML})\hat{p}_{ML}} \quad \text{and} \quad \frac{n_1}{1 - \hat{\pi}_{ML}} = \frac{n_0(1 - \hat{p}_{ML})}{\hat{\pi}_{ML} + (1 - \hat{\pi}_{ML})\hat{p}_{ML}}. \quad (1)$$

In addition, $\hat{\pi}_{ML}$ can be derived in terms of $\hat{p}_{ML}$: $\hat{\pi}_{ML} = (n_0 - n\hat{p}_{ML}) / (n - n\hat{p}_{ML})$. Although there is no closed form for p , (1) can be further simplified to the quadratic equation as follows

$$(\sum x_i + n)(1 - \hat{\pi}_{ML})(\hat{p}_{ML})^2 + [(\sum x_i + n_1)\hat{\pi}_{ML} - n(1 - \hat{\pi}_{ML})]\hat{p}_{ML} - n_1\hat{\pi}_{ML} = 0. \quad (2)$$

The root of (2), i.e. the MLE, can be written in a term of $\hat{\pi}_{ML}$:

$$\hat{p}_{ML} = \frac{-[(\sum x_i + n + n_1)\hat{\pi}_{ML} - n] \pm \sqrt{[(\sum x_i + n_0)^2 \hat{\pi}_{ML}^2 + 2[(n_1 - n_0)\sum x_i - nn_0]\hat{\pi}_{ML} + n^2]}}{2(\sum x_i + n)(1 - \hat{\pi}_{ML})}. \quad (3)$$

Using substitution, $\hat{\pi}_{ML} = (n_0 - n\hat{p}_{ML}) / (n - n\hat{p}_{ML})$ can be plugged into (3), resulting in a nonlinear equation with the single parameter $\hat{p}_{ML}$ that can be easily solve with functions in R such as `uniroot()` (R Core Team 2020). Solving one nonlinear equation is much easier than solving a system of nonlinear equations, which could lead to interesting phenomena, including chaos and complexity (Baofu 2010). In this paper, function `maxLik` using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm in package `maxLik` (Henningesen and Toomet 2011) was employed to calculate $\hat{p}_{ML}$ and $\hat{\pi}_{ML}$.

A $(1 - \alpha)100\%$ confidence interval for p is defined by two statistics $U_n = U(X_1, X_2, \dots, X_n)$ and $V_n = V(X_1, X_2, \dots, X_n)$, and the interval must satisfy the following properties:

$$\Pr(U(X_1, X_2, \dots, X_n) \leq p \leq V(X_1, X_2, \dots, X_n)) \geq 1 - \alpha,$$

for all p 's in the parameter space. The statistics U_n and V_n are called the limits of the confidence interval, and $U_n \leq V_n$. The confidence level $(1 - \alpha)$ is sometimes referred to as the coverage probability (Held and Bové 2014). Before obtaining the confidence intervals, the FIM will be

described in the following section.

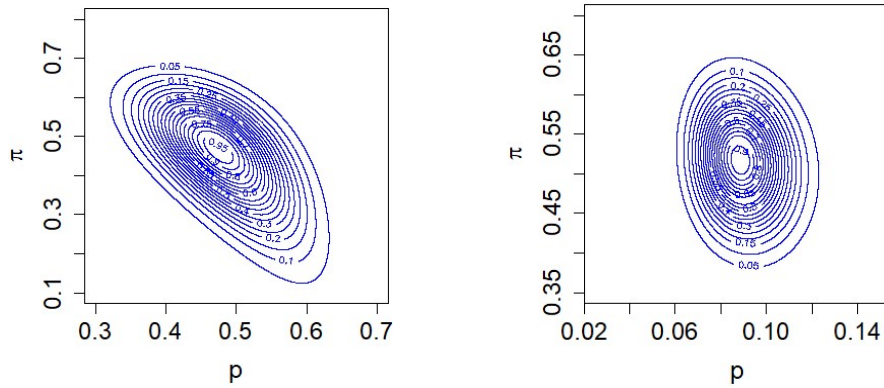


Figure 2 Contour plots of relative likelihood functions of (p, π) given that $(x_1, \dots, x_{100})$ is sampled from (left) $ZIG(p = 0.5, \pi = 0.5)$ and (right) $ZIG(p = 0.1, \pi = 0.5)$

3. Wald Approach

Under the regularity condition, the MLE $\hat{\boldsymbol{\theta}} = (\hat{p}_{ML}, \hat{\pi}_{ML})^T$ has an approximate multivariate normal distribution with mean $E(\hat{\boldsymbol{\theta}}) = (p, \pi)^T$ and variance given by the inverse of the FIM matrix, i.e. $\hat{\boldsymbol{\theta}} \sim N_2(\boldsymbol{\theta}, I^{-1}(\boldsymbol{\theta}))$. There are two types of FIM matrices: the observed and expected FIMs that can be used here.

The observed FIM $I(p, \pi)$ is a matrix in which each element is the negative second derivative of the likelihood. For the ZIG distribution, the observed FIM with two parameters is given by

$$\begin{aligned}
 I(p, \pi) &= \begin{bmatrix} I_{11} = \frac{-\partial^2}{\partial p^2} L(p, \pi | x) & I_{12} = \frac{-\partial^2}{\partial p \partial \pi} L(p, \pi | x) \\ I_{22} = \frac{-\partial^2}{\partial p \partial \pi} L(p, \pi | x) & I_{22} = \frac{-\partial^2}{\partial \pi^2} L(p, \pi | x) \end{bmatrix}, \\
 &= \begin{bmatrix} \frac{n_0(1-\pi)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{p^2} + \frac{\sum_{i=1}^n x_i}{(1-p)^2} & \frac{n_0}{[\pi + (1-\pi)p]^2} \\ \frac{n_0}{[\pi + (1-\pi)p]^2} & \frac{n_0(1-p)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{(1-\pi)^2} \end{bmatrix}. \quad (4)
 \end{aligned}$$

Furthermore, it is not difficult to show that expected FIM, $J(p, \pi)$, which is not a function of a particular observation, has the following form:

$$J(p, \pi) = \begin{bmatrix} J_{11} = E \left(\frac{N_0(1-\pi)^2}{[\pi + (1-\pi)p]^2} + \frac{N_1}{p^2} + \frac{\sum_{i=1}^n X_i}{(1-p)^2} \right) & J_{12} = E \left(\frac{N_0}{[\pi + (1-\pi)p]^2} \right) \\ J_{21} = E \left(\frac{N_0}{[\pi + (1-\pi)p]^2} \right) & J_{22} = E \left(\frac{N_0(1-p)^2}{[\pi + (1-\pi)p]^2} + \frac{N_1}{(1-\pi)^2} \right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n(1-\pi)^2}{\pi + (1-\pi)p} + \frac{n(1-\pi)(1-p+p^2)}{p^2(1-p)} & \frac{n}{\pi + (1-\pi)p} \\ \frac{n}{\pi + (1-\pi)p} & \frac{n(1-p)}{[\pi + (1-\pi)p](1-\pi)} \end{bmatrix}, \quad (5)$$

where N_0 and N_1 are random variables representing the numbers of zero counts and positive counts, respectively. Note that $E(N_0) = n[\pi + (1-\pi)p]$, $E(N_1) = n(1-\pi)(1-p)$, and $E(X_i) = (1-\pi)(1-p)/p$, $i = 1, 2, \dots, n$. Matrices $I(p, \pi)$ and $J(p, \pi)$ are essential for Wald intervals, but only the element in the 1st row and 1st column will be used in each matrix because only p is of interest.

3.1. Wald confidence interval

Here, a Wald confidence interval is based on the Wald statistic for testing the null hypothesis $H_0 : p = p_0$ versus the alternative hypothesis $H_0 : p \neq p_0$, where p_0 is a specified value. The two test statistics, both of which have asymptotically standard normal distributions under the null hypothesis, are as follows:

$$\sqrt{I_{11}(\hat{p}_{ML}, \hat{\pi}_{ML})}(\hat{p}_{ML} - p) \sim N(0, 1) \quad \text{and} \quad \sqrt{J_{11}(\hat{p}_{ML}, \hat{\pi}_{ML})}(\hat{p}_{ML} - p) \sim N(0, 1).$$

Accordingly, the corresponding $(1-\alpha)100\%$ Wald confidence intervals for p are

$$\hat{p}_{ML} \pm z_{1-\alpha/2} \sqrt{I^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})} \quad \text{and} \quad \hat{p}_{ML} \pm z_{1-\alpha/2} \sqrt{J^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})},$$

where $I^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$ and $J^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$ correspond to the elements in the first row and the first column of the inverse matrices of $I(\hat{p}_{ML}, \hat{\pi}_{ML})$ and $J(\hat{p}_{ML}, \hat{\pi}_{ML})$, respectively.

To obtain $I^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$, the general form of the inverse matrix of $I(p, \pi)$ is needed first as follows:

$$I^{-1}(p, \pi) = \frac{1}{\det(I(p, \pi))} \begin{bmatrix} \frac{n_0(1-p)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{(1-\pi)^2} & \frac{-n_0}{[\pi + (1-\pi)p]^2} \\ \frac{-n_0}{[\pi + (1-\pi)p]^2} & \frac{n_0(1-\pi)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{p^2} + \frac{\sum_{i=1}^n x_i}{(1-p)^2} \end{bmatrix}.$$

The determinant of matrix $I(p, \pi)$ can be simplified as

$$\det(I(p, \pi)) = \left(\frac{n_0(1-\pi)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{p^2} + \frac{\sum_{i=1}^n x_i}{(1-p)^2} \right) \left(\frac{n_0(1-p)^2}{[\pi + (1-\pi)p]^2} + \frac{n_1}{(1-\pi)^2} \right) - \left(\frac{n_0}{[\pi + (1-\pi)p]^2} \right)^2,$$

$$= \frac{C_1(p, \pi) + C_2(p, \pi)}{[\pi + (1-\pi)p]^3 p^2 (1-p)^2 (1-\pi)^2},$$

where $C_1(p, \pi) = (1-p)^2 (1-\pi)^2 \left\{ p^2 \left[\left(n_0 n_1 + n_0 \sum_{i=1}^n x_i - n_0^2 \right) (\pi + (1-\pi)p) - 2n_0^2 (1-p)(1-\pi) \right] \right.$

$$\left. + n_0 n_1 (1-p)^2 (\pi + (1-\pi)p) \right\},$$

and $C_2(p, \pi) = n_1 [\pi + (1-\pi)p]^3 \left[n_1 (1-p)^2 + p^2 \sum_{i=1}^n x_i \right].$

Following the acquisition of $I^{-1}(p, \pi)$, which contains unknown p and π , the estimate of $I^{-1}(p, \pi)$ will be $I^{-1}(\hat{p}, \hat{\pi})$. Consequently, the $(1-\alpha)100\%$ Wald confidence interval (WC) for p using the observed FIM is

$$\hat{p}_{ML} \pm z_{1-\alpha/2} \sqrt{\widehat{\text{var}}(\hat{p}_{ML})},$$

where $\widehat{\text{var}}(\hat{p}_{ML}) = \frac{[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^3 (\hat{p}_{ML})^2 (1-\hat{p}_{ML})^2 (1-\hat{\pi}_{ML})^2}{C_1(\hat{p}_{ML}, \hat{\pi}_{ML}) + C_2(\hat{p}_{ML}, \hat{\pi}_{ML})}$

$$\times \left\{ \frac{n_0(1-\hat{p}_{ML})^2}{[\hat{\pi}_{ML} + (1-\hat{p}_{ML})\hat{p}_{ML}]^2} + \frac{n_1}{(1-\hat{\pi}_{ML})^2} \right\}.$$

Because there is no guarantee for a Wald-typed interval to be in $(0,1)$, we need to modify the resulting outcome. If the lower limit is less than zero and/or the upper limit is greater than one, the values can be truncated at zero and/or one, respectively. Accordingly, the WC will be

$$\left(\max \left\{ 0, \hat{p}_{ML} - z_{1-\alpha/2} \sqrt{\widehat{\text{var}}(\hat{p}_{ML})} \right\}, \min \left\{ 1, \hat{p}_{ML} + z_{1-\alpha/2} \sqrt{\widehat{\text{var}}(\hat{p}_{ML})} \right\} \right).$$

In the following sub-section, the inverse matrix of $J(p, \pi)$ will be discussed and it will be shown that $I(\hat{p}, \hat{\pi})$ is equal to $J(\hat{p}, \hat{\pi})$.

3.2. Estimates of observed and expected Fisher information

The estimated observed FIM will be further investigated and found to be equal to the estimated expected FIM by verifying element in the matrices. Note that the forms of two FIMs are different, but their estimates are the same. From the second term of (1), that equation is equivalent to

$$n_1 [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}] - n_0 (1-\hat{p}_{ML})(1-\hat{\pi}_{ML}) = 0. \quad (6)$$

Evaluating the difference between the estimates of I_{12} and J_{12} (in (4) and (5), respectively), it arrives at

$$\hat{I}_{12} - \hat{J}_{12} = \left[n_0 - n(\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}) \right] / [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2,$$

$$= \left[n_0 (1-\hat{p}_{ML})(1-\hat{\pi}_{ML}) - n_1 (\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}) \right] / [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2.$$

From (6), $\hat{I}_{12} - \hat{J}_{12}$ will be zero. Similarly, $\hat{I}_{12} - \hat{J}_{12}$ and $\hat{I}_{22} - \hat{J}_{22}$ can be proved to be zero by using (6). Appendix 1 contains the details of the proofs. This means that the WC created with $J^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$ will be identical to the WC created with $I^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$ although $J^{-1}(p, \pi)$ and $I^{-1}(p, \pi)$ are different.

4. Profile Likelihood Approach

The profile likelihood approach is employed to eliminate the nuisance parameter π and can act as an “ordinary” likelihood function (Pawitan 2013). Let $\tilde{\pi}$ denote the MLE with respect to $L(p, \pi | x)$ for fixed p . It is easy to show that $\tilde{\pi} = (n_0 - np)/(n - np)$. The log of the profile likelihood function of p can be expressed as

$$\begin{aligned} \log L_p(p) &= \log L(p, \tilde{\pi}) = n_0 \log[\tilde{\pi} + (1 - \tilde{\pi})p] + n_1 \log(1 - \tilde{\pi}) + n_1 \log p + \sum_{i=1}^n x_i \log(1 - p), \\ &= n_0 \log\left(\frac{n_0}{n}\right) + n_1 \log\left(\frac{1 - n_0/n}{1 - p}\right) + n_1 \log p + \sum_{i=1}^n x_i \log(1 - p). \end{aligned}$$

The first derivative of $\log L_p(p)$ with respect to p yields

$$\begin{aligned} \frac{\partial}{\partial p} \log L_p(p) &= \frac{\partial}{\partial p} \left[n_0 \log\left(\frac{n_0}{n}\right) + n_1 \log\left(1 - \frac{n_0}{n}\right) - n_1 \log(1 - p) + n_1 \log p + \sum_{i=1}^n x_i \log(1 - p) \right], \\ &= \frac{n_1}{1 - p} + \frac{n_1}{p} - \frac{\sum x_i}{1 - p} = \frac{n_1 - \sum x_i}{1 - p} + \frac{n_1}{p} = \frac{n_1 - p \sum x_i}{p(1 - p)} = S(p, \tilde{\pi}). \end{aligned}$$

The quantity $S(p, \tilde{\pi}) = (n_1 - p \sum x_i) / [p(1 - p)]$ is also called the score function for the profile likelihood $L_p(p)$. Setting $\frac{\partial}{\partial p} \log L_p(p) = 0$, it is found that the MLE is $\hat{p}_{ML}^p = n_1 / \sum_{i=1}^n x_i$.

4.1. Fisher information using profile likelihood

The corresponding observed FIM, $I(p, \tilde{\pi})$, can be derived as follows

$$\begin{aligned} I(p, \tilde{\pi}) &= -\frac{\partial^2}{\partial p^2} \log L_p(p) = -\left[\frac{n_1 - \sum x_i}{(1 - p)^2} - \frac{n_1}{p^2} \right] = \frac{(\sum x_i)p^2 - (2n_1)p + n_1}{p^2(1 - p)^2}, \text{ and} \\ I(\hat{p}_{ML}^p, \tilde{\pi}) &= \frac{(\sum x_i)(\hat{p}_{ML}^p)^2 - (2n_1)\hat{p}_{ML}^p + n_1}{(\hat{p}_{ML}^p)^2(1 - \hat{p}_{ML}^p)^2} = \frac{\sum x_i}{\hat{p}_{ML}^p(1 - \hat{p}_{ML}^p)}. \end{aligned} \quad (7)$$

In addition, the expected FIM, $J(p, \tilde{\pi})$, can be derived as follows

$$J(p, \tilde{\pi}) = E[I(p, \tilde{\pi})] = E\left[\frac{(\sum X_i)p^2 - (2N_1)p + N_1}{p^2(1 - p)^2} \right] = \frac{p^2 \sum E(X_i) - 2pE(N_1) + E(N_1)}{p^2(1 - p)^2}.$$

Because $E(X_i) = (1 - \pi)(1 - p)/p$ and $E(N_1) = n(1 - \pi)(1 - p)$, we arrive at

$$J(p, \tilde{\pi}) = \frac{n(1 - \pi)}{p^2} = \frac{n}{p^2} \left(1 - \frac{n_0 - np}{n - np} \right) = \frac{n - n_0}{p^2(1 - p)} = \frac{n_1}{p^2(1 - p)}.$$

Therefore,

$$J(\hat{p}_{ML}^p, \tilde{\pi}) = \frac{n_1}{(\hat{p}_{ML}^p)^2 (1 - \hat{p}_{ML}^p)} = \frac{\sum x_i}{\hat{p}_{ML}^p (1 - \hat{p}_{ML}^p)}, \quad (8)$$

which is the same as $I(\hat{p}_{ML}^p, \tilde{\pi})$. Note that $J(p, \tilde{\pi})$ and $I(p, \tilde{\pi})$ have different forms, but the estimates $I(\hat{p}_{ML}^p, \tilde{\pi})$ and $J(\hat{p}_{ML}^p, \tilde{\pi})$ are equal.

4.2. Wald confidence intervals using profile likelihood

Wald confidence intervals using profile likelihood (WCPs) are similar to WCs, but $I^{-1}(\hat{p}_{ML}, \tilde{\pi})$ is employed instead of $I^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$, and $J^{-1}(\hat{p}_{ML}, \tilde{\pi})$ is used instead of $J^{11}(\hat{p}_{ML}, \hat{\pi}_{ML})$. Hence, the Wald-type confidence intervals using the profile likelihood are

$$\hat{p}_{ML}^p - z_{1-\alpha/2} \sqrt{I^{-1}(\hat{p}_{ML}^p, \tilde{\pi})} \quad \text{and} \quad \hat{p}_{ML}^p - z_{1-\alpha/2} \sqrt{J^{-1}(\hat{p}_{ML}^p, \tilde{\pi})}. \quad (9)$$

Consequently, WCPs using either $I(\hat{p}_{ML}^p, \tilde{\pi})$ or $J(\hat{p}_{ML}^p, \tilde{\pi})$ yield the same form. Substituting (7) or (8) into (9) leads to the same formula,

$$\left(\max \left\{ 0, \hat{p}_{ML}^p - z_{1-\alpha/2} \sqrt{\frac{\hat{p}_{ML}^p (1 - \hat{p}_{ML}^p)}{\sum x_i}} \right\}, \min \left\{ 1, \hat{p}_{ML}^p + z_{1-\alpha/2} \sqrt{\frac{\hat{p}_{ML}^p (1 - \hat{p}_{ML}^p)}{\sum x_i}} \right\} \right),$$

where $\hat{p}_{ML}^p = n_1 / \sum_{i=1}^n x_i$. Because the lower and upper limits of p must be in the range (0,1), the functions min and max are used to truncate the result.

4.3. Score confidence intervals using profile likelihood

Under the Fisher regularity conditions, the score statistic $S(p, \tilde{\pi}) / \sqrt{I(p, \tilde{\pi})}$, where $\tilde{\pi} = (n_0 - np) / (n - np)$, has an approximately normal distribution and can be considered an approximate pivot for p . Thus, the score confidence interval of p using $\sqrt{I(p, \tilde{\pi})}$ is a set of values of p satisfying the condition $|S(p, \tilde{\pi}) / \sqrt{I(p, \tilde{\pi})}| \leq z_{1-\alpha/2}$, which can also be expressed as the following

$$\frac{\left(n_1 - p \sum_{i=1}^n x_i \right)^2}{n_1 - 2pn_1 + p^2 \sum_{i=1}^n x_i} \leq \chi_{1,(1-\alpha)}^2,$$

$$\left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n x_i - \chi_{1,(1-\alpha)}^2 \right) p^2 + 2n_1 \left(\chi_{1,(1-\alpha)}^2 - \sum_{i=1}^n x_i \right) p + n_1 (n_1 - \chi_{1,(1-\alpha)}^2) \leq 0.$$

The expression on the left-hand side above is in quadratic form; therefore, the $(1-\alpha)100\%$ score confidence interval using $\sqrt{I(p, \tilde{\pi})}$ (or called SCI) can be written as

$$\left(\frac{-G_2 - \sqrt{G_2^2 - 4G_1G_3}}{2G_1}, \frac{-G_2 + \sqrt{G_2^2 - 4G_1G_3}}{2G_1} \right),$$

where $G_1 = \left(\frac{n_1}{\hat{p}_{ML}^p} \right) \left(\frac{n_1}{\hat{p}_{ML}^p} - \chi_{1,(1-\alpha)}^2 \right)$, $G_2 = 2n_1 \left(\chi_{1,(1-\alpha)}^2 - \frac{n_1}{\hat{p}_{ML}^p} \right)$, and $G_3 = n_1 \left(n_1 - \chi_{1,(1-\alpha)}^2 \right)$. Note that the values of the lower and upper endpoints must be bounded at 0 and 1, respectively. Then, the above confidence interval can be modified to

$$\left(\max \left\{ 0, \frac{-G_2 - \sqrt{G_2^2 - 4G_1G_3}}{2G_1} \right\}, \min \left\{ 1, \frac{-G_2 + \sqrt{G_2^2 - 4G_1G_3}}{2G_1} \right\} \right).$$

Using the same idea, the score confidence interval can be also constructed by using the expected FIM, where the score confidence interval using $\sqrt{J(p, \tilde{\pi})}$ (SCJ) is defined as $\left\{ p \mid S(p, \tilde{\pi}) / \sqrt{J(p, \tilde{\pi})} \leq z_{1-\alpha/2} \right\}$. It is further simplified as

$$\frac{\left(n_1 - p \sum_{i=1}^n x_i \right)^2}{n_1 (1-p)} \leq \chi_{1,(1-\alpha)}^2,$$

$$\left(\sum_{i=1}^n x_i \right)^2 p^2 + n_1 \left(\chi_{1,(1-\alpha)}^2 - 2 \sum_{i=1}^n x_i \right) p + n_1 \left(n_1 - \chi_{1,(1-\alpha)}^2 \right) \leq 0.$$

The expression on the left-hand side above is in quadratic form, so its roots will be the lower and upper endpoints of the SCJ. If the interval is constrained to be in (0, 1), the approximate $(1-\alpha)100\%$ SCJ can be written as

$$\left(\max \left\{ 0, \frac{-H_2 - \sqrt{H_2^2 - 4H_1H_3}}{2H_1} \right\}, \min \left\{ 1, \frac{-H_2 + \sqrt{H_2^2 - 4H_1H_3}}{2H_1} \right\} \right),$$

where $H_1 = \left(\frac{n_1}{\hat{p}_{ML}^p} \right)^2$, $H_2 = n_1 \left(\chi_{1,(1-\alpha)}^2 - \frac{2n_1}{\hat{p}_{ML}^p} \right)$, and $H_3 = n_1 \left(n_1 - \chi_{1,(1-\alpha)}^2 \right)$.

5. Simulation Study

In the previous section, four confidence intervals, namely, the WC, WCP, SCI, and SCJ, are derived. To evaluate the performances of these intervals, Monte-Carlo simulations are conducted by employing R 4.0.3 (R Core Team 2020). Samples of size n are generated from $ZIG(p, \pi)$, where $p = 0.2, 0.4, 0.6, 0.8$, $\pi = -4, -3, -2, -1, 0, 0.2, 0.4, 0.6, 0.8$, and $n = 10, 30, 50, 100, 200, 500$. Each (p, π) must be in the parameter space $\{(p, \pi) \mid -p/(1-p) < \pi < 1, 0 < p < 1\}$ shown in Figure 2. After 10,000 iterations, the coverage probabilities (CPs) and interval lengths (ILs) of the 95% confidence intervals are estimated, and the results are summarized in Tables 1 to 4. Note that the IL is calculated from the difference between the original lower and upper limits (without truncation) to obtain true interval length.

The WC, which is commonly used, tends to have the lowest CP compared to the others. Both the SCI and SCJ produce greater CPs than the others, but this is not surprising, as these intervals have greater ILs than WC and WCP. Usually, the higher a CP is, the greater the IL will be. Both parameters p and π affect the CP such that for given p and n , when parameter π has a small value, the CP tends to approach the desirable confidence level, 0.95. Also, when π is fixed, the CP will decrease as p increases. The worst case arises when π and p are both high; one sign that indicates this

concern is that the dataset will mostly contain zeros and very few positive integers. Under such circumstances, when the sample size is small, the score and Wald approaches are distinct; the score confidence intervals are noticeably conservative or have CPs of far greater than 0.95, while the Wald-type intervals are liberal or have CPs of less than the nominal level. For example, when a sample of size 30 drawn from $ZIG(\pi = 0.8, p = 0.8)$, the CPs of the WC and WCP are 0.202 and 0.297, respectively, while they are 0.993 and 0.961 for SCI and SCJ, respectively. For general cases in which π and p are not simultaneously high and n is small, the CPs of the WC and WCP are below 0.95, while the CPs of the SCI and SCJ are above 0.95. Nonetheless, as n increases, the CP increases and approaches 0.95, and when n is large enough, all methods produce approximately the same CPs and ILs in most cases.

6. Real Data Analysis

The real data used in this study concern the number of precision farming technology components adopted by Ohio farmers. Isgin et al. (2008) employed ZIP and negative binomial regression analyses to determine factors influencing farmers' decisions to adopt a greater number of precision technologies. The frequency distribution of the data is depicted in Figure 3. Note that 314 farmers out of 491 had not adopted any technology, indicating zero inflation. This data is tested to be ZIG by using the goodness-of-fit tests with Chi-squared statistic of 17.39 and p-value of 0.1355.

Using the maximum likelihood method, $\hat{p}_{ML} = 0.2369$ and $\hat{\pi}_{ML} = 0.5276$, and the estimated $\Pr(X = x)$ can be calculated from $\widehat{\Pr(X = x)} = 0.6395^{I_{\{0\}}(x)} \left[0.1119(0.7631)^x \right]^{I_{\{1,2,\dots\}}(x)}$, where $I_{\{0\}}(x) = 1$ for $x = 0$ and $I_{\{0\}}(x) = 0$ otherwise, and $I_{\{1,2,\dots\}}(x)$ equals 1 for positive integers and 0 otherwise. The estimates and observed proportions are illustrated in Figure 4. As can be seen, $ZIG(p = 0.2369, \pi = 0.5276)$ is capable of explaining the number of technologies adopted. The 95% confidence intervals for p calculated are the following: the WC is (0.2064553, 0.2674401), the WCP is (0.2064554, 0.2674402), the SCI is (0.2063767, 0.2675189), and SCJ is (0.2058401, 0.2668370). The length of all intervals is approximately 0.06. In this case, all the methods produce similar confidence intervals because the values of p and π are not high. The results from the real data agree with those from the simulation study in Table 4 where data follows either $ZIG(\pi = 0.4, p = 0.2)$ or $ZIG(\pi = 0.6, p = 0.2)$ with $n = 500$. In such cases, all CPs are near 0.95, and all ILs are around 0.06. Consequently, simulation results provide a rough idea about CPs and ILs when the point estimates are calculated for a given dataset.

Table 1 Coverage probabilities [interval lengths] for $X \sim \text{ZIG}(\pi, p = 0.8)$

π		Sample size (n)					
		10	30	50	100	200	500
-4	WC	0.6386[0.2876]	0.5800[0.1271]	0.6103[0.1028]	0.6030[0.0693]	0.6690[0.0488]	0.7246[0.0306]
	WCP	0.8808[0.3784]	0.8861[0.2470]	0.9378[0.1942]	0.9443[0.1389]	0.9447[0.0987]	0.9473[0.0626]
	SCI	0.9946[0.5071]	0.9094[0.2612]	0.9395[0.2004]	0.9478[0.1411]	0.9484[0.0994]	0.9503[0.0627]
	SCJ	0.9392[0.4835]	0.9563[0.2612]	0.9627[0.2006]	0.9434[0.1410]	0.9484[0.0995]	0.9501[0.0627]
-3	WC	0.8136[0.4450]	0.9027[0.2730]	0.9234[0.2162]	0.9350[0.1548]	0.9458[0.1103]	0.9457[0.0699]
	WCP	0.8219[0.4020]	0.9128[0.2739]	0.9280[0.2164]	0.9375[0.1549]	0.9460[0.1102]	0.9479[0.0699]
	SCI	0.9938[0.6203]	0.9263[0.2951]	0.9328[0.2244]	0.9400[0.1581]	0.9467[0.1113]	0.9493[0.0703]
	SCJ	0.9597[0.5667]	0.9578[0.2962]	0.9522[0.2251]	0.9542[0.1580]	0.9488[0.1113]	0.9516[0.0018]
-2	WC	0.7867[0.4931]	0.9014[0.3120]	0.9248[0.2479]	0.9326[0.1783]	0.9427[0.1272]	0.9490[0.0808]
	WCP	0.7148[0.4164]	0.8994[0.3092]	0.9144[0.2476]	0.9362[0.1785]	0.9427[0.1270]	0.9467[0.0807]
	SCI	0.9965[0.8870]	0.9344[0.3493]	0.9334[0.2629]	0.9379[0.1829]	0.9458[0.1289]	0.9552[0.0811]
	SCJ	0.9584[0.7238]	0.9553[0.3479]	0.9499[0.2637]	0.9517[0.1833]	0.9502[0.1288]	0.9535[0.0811]
-1	WC	0.6165[0.4934]	0.8766[0.3641]	0.9004[0.2982]	0.9261[0.2171]	0.9363[0.1553]	0.9460[0.0987]
	WCP	0.5650[0.4092]	0.8767[0.3571]	0.9047[0.2979]	0.9226[0.2167]	0.9397[0.1551]	0.9458[0.0989]
	SCI	0.9964[1.5526]	0.9762[0.4628]	0.9324[0.3286]	0.9378[0.2265]	0.9419[0.1582]	0.9457[0.0993]
	SCJ	0.9584[1.1015]	0.9527[0.4476]	0.9566[0.3299]	0.9519[0.2271]	0.9526[0.1584]	0.9502[0.0996]
0	WC	0.3307[0.4018]	0.6979[0.4334]	0.8467[0.3827]	0.9003[0.2974]	0.9252[0.2177]	0.9350[0.1392]
	WCP	0.3662[0.3570]	0.7059[0.4117]	0.8526[0.3810]	0.9088[0.2997]	0.9265[0.2171]	0.9380[0.1391]
	SCI	0.9947[2.6539]	0.9955[0.9550]	0.9870[0.5416]	0.9300[0.3315]	0.9329[0.2258]	0.9455[0.1411]
	SCJ	0.9607[1.9667]	0.9567[0.7761]	0.9559[0.1272]	0.9567[0.3309]	0.9511[0.2271]	0.9486[0.0092]
0.2	WC	0.2675[0.3621]	0.6053[0.4368]	0.7863[0.4067]	0.8922[0.3287]	0.9233[0.2419]	0.9370[0.1551]
	WCP	0.3312[0.3409]	0.6134[0.4068]	0.7930[0.2201]	0.8929[0.3252]	0.9201[0.2411]	0.9338[0.1552]
	SCI	0.9932[2.9482]	0.9966[1.2458]	0.9929[0.6806]	0.9434[0.3813]	0.9397[0.2557]	0.9420[0.1587]
	SCJ	0.9602[2.2197]	0.9550[0.9571]	0.9606[0.6106]	0.9561[0.3791]	0.9539[0.2551]	0.9484[0.1587]
0.4	WC	0.2108[0.3138]	0.5108[0.4495]	0.6946[0.4308]	0.8701[0.3639]	0.9104[0.2747]	0.9284[0.1785]
	WCP	0.2951[0.3262]	0.5230[0.4010]	0.6945[0.4074]	0.8676[0.3594]	0.9100[0.2751]	0.9304[0.1785]
	SCI	0.9865[3.1698]	0.9947[1.6772]	0.9967[0.9301]	0.9737[0.4704]	0.9303[0.2987]	0.9425[0.1839]
	SCJ	0.9625[2.4875]	0.9579[1.2717]	0.9597[0.7845]	0.9568[0.4555]	0.9583[0.2991]	0.9490[0.1838]
0.6	WC	0.1436[0.2389]	0.3791[0.4356]	0.5414[0.4395]	0.7865[0.4062]	0.8909[0.3295]	0.9227[0.2181]
	WCP	0.2461[0.2929]	0.4090[0.3695]	0.5443[0.3938]	0.7890[0.3926]	0.8962[0.3274]	0.9247[0.2172]
	SCI	0.9868[3.4021]	0.9930[2.2907]	0.9953[1.4947]	0.9935[0.6898]	0.9463[0.3827]	0.9339[0.2261]
	SCJ	0.9635[2.7756]	0.9592[1.0656]	0.9582[1.1490]	0.9569[0.6172]	0.9587[0.3762]	0.9527[0.2280]
0.8	WC	0.0781[0.1418]	0.2018[0.3452]	0.3239[0.4163]	0.5438[0.4304]	0.7841[0.4046]	0.9027[0.2981]
	WCP	0.2368[0.2986]	0.2969[0.3242]	0.3763[0.3577]	0.5638[0.4112]	0.7875[0.3960]	0.8991[0.2976]
	SCI	0.9588[3.5400]	0.9930[3.0443]	0.9926[2.5185]	0.9956[1.5243]	0.9928[0.6869]	0.9318[0.3327]
	SCJ	0.9655[3.0886]	0.9607[2.4660]	0.9607[1.9662]	0.9607[1.1663]	0.9577[0.6210]	0.9533[0.3314]

Table 2 Coverage probabilities [interval lengths] for $X \sim \text{ZIG}(\pi, p = 0.6)$

π		Sample size (n)					
		10	30	50	100	200	500
-1	WC	0.9085[0.4953]	0.9342[0.2997]	0.9431[0.2337]	0.9478[0.1659]	0.9460[0.1174]	0.9490[0.0744]
	WCP	0.9074[0.4952]	0.9365[0.2999]	0.9433[0.2337]	0.9414[0.1658]	0.9440[0.1174]	0.9531[0.0743]
	SCI	0.9611[0.6168]	0.9440[0.3165]	0.9522[0.2410]	0.9519[0.1683]	0.9493[0.1182]	0.9503[0.0746]
	SCJ	0.9646[0.5503]	0.9519[0.3067]	0.9478[0.2365]	0.9511[0.1667]	0.9512[0.1178]	0.9476[0.0744]
0	WC	0.8152[0.6161]	0.9175[0.4207]	0.9342[0.3292]	0.9447[0.2345]	0.9473[0.1662]	0.9469[0.1049]
	WCP	0.8089[0.5929]	0.9234[0.4201]	0.9319[0.3311]	0.9432[0.2348]	0.9434[0.1660]	0.9484[0.1050]
	SCI	0.9864[1.3543]	0.9476[0.4805]	0.9448[0.3528]	0.9533[0.2421]	0.9481[0.1686]	0.9498[0.1057]
	SCJ	0.9495[0.9652]	0.9597[0.4467]	0.9527[0.3395]	0.9517[0.2375]	0.9503[0.1673]	0.9464[0.1054]
0.2	WC	0.7227[0.6121]	0.9138[0.4654]	0.9260[0.3680]	0.9386[0.2624]	0.9460[0.1857]	0.9499[0.1177]
	WCP	0.7427[0.5960]	0.9109[0.4652]	0.9294[0.3681]	0.9372[0.2618]	0.9433[0.1857]	0.9477[0.1176]
	SCI	0.9853[1.6768]	0.9550[0.5669]	0.9496[0.4022]	0.9503[0.2732]	0.9515[0.1893]	0.9448[0.1184]
	SCJ	0.9513[1.1826]	0.9612[0.5102]	0.9576[0.3828]	0.9527[0.2667]	0.9514[0.1868]	0.9510[0.1179]
0.4	WC	0.6195[0.5955]	0.8923[0.5218]	0.9184[0.4232]	0.9348[0.3031]	0.9456[0.2147]	0.9443[0.1359]
	WCP	0.6557[0.5847]	0.8996[0.5183]	0.9197[0.4234]	0.9356[0.3035]	0.9482[0.2146]	0.9511[0.1359]
	SCI	0.9808[2.0436]	0.9686[0.7442]	0.9540[0.4824]	0.9493[0.3185]	0.9480[0.2196]	0.9515[0.1370]
	SCJ	0.9485[1.4869]	0.9594[0.6200]	0.9610[0.4498]	0.9519[0.3095]	0.9511[0.2169]	0.9537[0.1363]
0.6	WC	0.4740[0.5250]	0.8290[0.5847]	0.8993[0.5028]	0.9249[0.3703]	0.9383[0.2621]	0.9454[0.1668]
	WCP	0.5643[0.5661]	0.8379[0.5760]	0.8980[0.4993]	0.9278[0.3686]	0.9452[0.2625]	0.9461[0.1662]
	SCI	0.9783[2.4667]	0.9826[1.1485]	0.9570[0.6659]	0.9437[0.4042]	0.9485[0.2730]	0.9511[0.1684]
	SCJ	0.9465[1.8906]	0.9522[0.8620]	0.9600[0.5802]	0.9616[0.3842]	0.9553[0.2673]	0.9538[0.1675]
0.8	WC	0.2678[0.3527]	0.6035[0.5974]	0.7819[0.5999]	0.9001[0.5041]	0.9329[0.3701]	0.9434[0.2356]
	WCP	0.4605[0.5210]	0.6479[0.5782]	0.7975[0.5884]	0.8973[0.5008]	0.9342[0.3705]	0.9378[0.2354]
	SCI	0.9715[2.8409]	0.9864[1.9809]	0.9844[1.3545]	0.9654[0.6838]	0.9459[0.4055]	0.9497[0.2429]
	SCJ	0.9397[2.3408]	0.9487[1.5157]	0.9516[1.0230]	0.9631[0.5839]	0.9539[0.3846]	0.9496[0.2378]

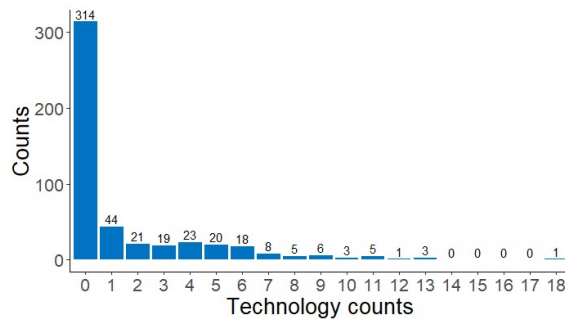
**Figure 3** Technology adoption frequency distribution

Table 3 Coverage probabilities [interval lengths] for $X \sim \text{ZIG}(\pi, p = 0.4)$

π		Sample size (n)					
		10	30	50	100	200	500
0	WC	0.9271[0.4881]	0.9399[0.2910]	0.9430[0.2240]	0.9512[0.1575]	0.9495[0.1111]	0.9475[0.0702]
	WCP	0.9205[0.5063]	0.9439[0.2912]	0.9447[0.2244]	0.9472[0.1577]	0.9456[0.1112]	0.9455[0.0703]
	SCI	0.9623[0.6602]	0.9520[0.3059]	0.9494[0.2305]	0.9512[0.1597]	0.9494[0.1118]	0.9484[0.0704]
	SCJ	0.9527[0.5438]	0.9571[0.2939]	0.9545[0.2248]	0.9523[0.1579]	0.9478[0.1111]	0.9492[0.0701]
0.2	WC	0.9084[0.5606]	0.9359[0.3283]	0.9476[0.2518]	0.9476[0.1763]	0.9488[0.1243]	0.9511[0.0786]
	WCP	0.8975[0.5574]	0.9383[0.3257]	0.9412[0.2519]	0.9468[0.1767]	0.9485[0.1244]	0.9500[0.0784]
	SCI	0.9725[0.8496]	0.9577[0.3502]	0.9538[0.2605]	0.9502[0.1799]	0.9548[0.1254]	0.9468[0.0789]
	SCJ	0.9576[0.6509]	0.9595[0.3323]	0.9558[0.2529]	0.9504[0.1768]	0.9540[0.1243]	0.9501[0.0786]
0.4	WC	0.8579[0.6118]	0.9388[0.3806]	0.9397[0.2927]	0.9477[0.2043]	0.9479[0.1439]	0.9507[0.0906]
	WCP	0.8673[0.6081]	0.9377[0.3813]	0.9391[0.2928]	0.9463[0.2048]	0.9501[0.1439]	0.9481[0.0907]
	SCI	0.9717[1.1465]	0.9537[0.4233]	0.9535[0.3063]	0.9562[0.2095]	0.9516[0.1457]	0.9466[0.0913]
	SCJ	0.9553[0.8456]	0.9561[0.3909]	0.9536[0.2951]	0.9550[0.2053]	0.9493[0.1441]	0.9490[0.0906]
0.6	WC	0.7514[0.6223]	0.9237[0.4718]	0.9338[0.3638]	0.9483[0.2527]	0.9479[0.1774]	0.9463[0.1115]
	WCP	0.7975[0.6439]	0.9255[0.4707]	0.9368[0.3627]	0.9431[0.2525]	0.9495[0.1770]	0.9510[0.1113]
	SCI	0.9764[1.5326]	0.9631[0.5927]	0.9579[0.3965]	0.9457[0.4038]	0.9574[0.1800]	0.9526[0.1119]
	SCJ	0.9549[1.1589]	0.9508[0.5056]	0.9571[0.3708]	0.9574[0.3844]	0.9497[0.1774]	0.9491[0.1117]
0.8	WC	0.4976[0.5036]	0.8480[0.6057]	0.9113[0.5182]	0.9352[0.3647]	0.9415[0.2536]	0.9494[0.1578]
	WCP	0.6941[0.6498]	0.8583[0.6074]	0.9153[0.5143]	0.9345[0.3637]	0.9446[0.2527]	0.9485[0.1578]
	SCI	0.9649[1.9401]	0.9732[1.1412]	0.9688[0.7203]	0.9595[0.3977]	0.9518[0.2622]	0.9527[0.1602]
	SCJ	0.9531[1.6636]	0.9555[0.8832]	0.9529[0.5848]	0.9575[0.3713]	0.9527[0.2542]	0.9532[0.1590]

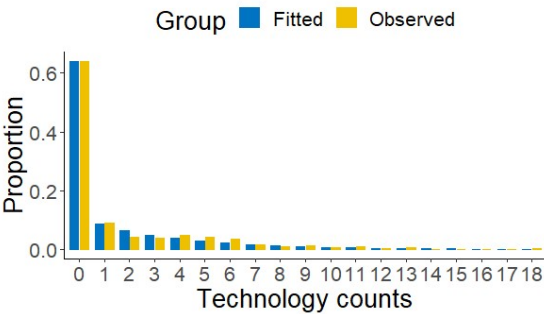


Figure 4 Observed and fitted proportions of the number of technologies adopted

Table 4 Coverage probabilities [interval lengths] for $X \sim \text{ZIG}(\pi, p = 0.2)$

π		Sample size (n)					
		10	30	50	100	200	500
0	WC	0.9414[0.2700]	0.9499[0.1473]	0.9488[0.1127]	0.9485[0.0791]	0.9483[0.0556]	0.9521[0.0351]
	WCP	0.9444[0.2693]	0.9530[0.1469]	0.9488[0.1129]	0.9520[0.0791]	0.9481[0.0557]	0.9506[0.0351]
	SCI	0.9582[0.2896]	0.9535[0.1499]	0.9532[0.1137]	0.9515[0.0795]	0.9517[0.0558]	0.9498[0.0351]
	SCJ	0.9554[0.2725]	0.9542[0.1476]	0.9508[0.1131]	0.9500[0.0791]	0.9499[0.0557]	0.9515[0.0351]
0.2	WC	0.9419[0.3110]	0.9463[0.1666]	0.9437[0.1265]	0.9507[0.0888]	0.9513[0.0624]	0.9492[0.0393]
	WCP	0.9423[0.3112]	0.9449[0.1655]	0.9498[0.1269]	0.9491[0.0888]	0.9469[0.0623]	0.9515[0.0393]
	SCI	0.9628[0.3513]	0.9559[0.1705]	0.9539[0.1284]	0.9527[0.0892]	0.9518[0.0625]	0.9480[0.0394]
	SCJ	0.9551[0.3163]	0.9529[0.1663]	0.9480[0.1267]	0.9498[0.0889]	0.9513[0.0623]	0.9487[0.0393]
0.4	WC	0.9349[0.3720]	0.9477[0.1963]	0.9515[0.1480]	0.9511[0.1030]	0.9493[0.0721]	0.9489[0.0453]
	WCP	0.9393[0.3821]	0.9483[0.1952]	0.9454[0.1477]	0.9475[0.1031]	0.9498[0.0722]	0.9512[0.0454]
	SCI	0.9633[0.4665]	0.9583[0.2016]	0.9465[0.1511]	0.9515[0.1034]	0.9530[0.0724]	0.9483[0.0454]
	SCJ	0.9569[0.4035]	0.9555[0.1963]	0.9551[0.1485]	0.9512[0.1028]	0.9522[0.0720]	0.9493[0.0453]
0.6	WC	0.8922[0.4573]	0.9493[0.2508]	0.9481[0.1859]	0.9477[0.1276]	0.9484[0.0889]	0.9517[0.0558]
	WCP	0.9164[0.4669]	0.9417[0.2482]	0.9530[0.1861]	0.9493[0.1276]	0.9464[0.0891]	0.9484[0.0558]
	SCI	0.9674[0.7158]	0.9596[0.2627]	0.9561[0.1906]	0.9536[0.1289]	0.9534[0.0895]	0.9478[0.0560]
	SCJ	0.9545[0.5812]	0.9559[0.2513]	0.9589[0.1857]	0.9527[0.1276]	0.9467[0.0889]	0.9518[0.0556]
0.8	WC	0.7037[0.4833]	0.9263[0.3805]	0.9458[0.2824]	0.9479[0.1862]	0.9515[0.1279]	0.9505[0.0790]
	WCP	0.8568[0.5673]	0.9300[0.3834]	0.9402[0.2815]	0.9499[0.1861]	0.9496[0.1267]	0.9508[0.0798]
	SCI	0.9501[1.0552]	0.9622[0.4965]	0.9598[0.3140]	0.9558[0.1921]	0.9510[0.1295]	0.9498[0.0798]
	SCJ	0.9550[0.9650]	0.9593[0.4288]	0.9542[0.2894]	0.9544[0.1854]	0.9526[0.1270]	0.9485[0.0794]

7. Discussion and Conclusions

This paper first discussed existing point estimations for the parameters in ZIG and derived the formulas of the Wald interval using full likelihood for the geometric parameter. However, the MLEs have no closed form, and consequently, the corresponding Wald interval can be evaluated only via computer programming. Also, the estimates of observed and expected FIMs were shown to be equal. Second, the profile likelihood was introduced to eliminate the nuisance parameter, π . The derived Wald interval using profile likelihood and proposed score intervals now have closed forms and can be calculated by hand. This will be very helpful in practice. Finally, the simulation results showed that score intervals depending on both observed and expected FIMs are better than Wald-type intervals in most cases.

Acknowledgments

The research of Dr. Srisuradetchai is currently supported by the Thammasat University Research Unit in Theoretical and Computational Statistics. We gratefully acknowledge financial support from Thammasat University.

References

- Adarabioyo MI, Ipinoyomi RA. Comparing zero-inflated Poisson, zero-inflated negative binomial and zero inflated geometric in count data with excess zero. *Asian J Prob Stat*. 2019; 4(2): 1-10.
- Aryal TR. Inflated geometric distribution to study the distribution of rural out-migrants. *Journal of*

- the Institute of Engineering. 2011; 8(1-2): 266-268.
- Baofu P. The future of post-human formal science: a preface to a new theory of abstraction and application. Newcastle upon Tyne: Cambridge Scholars Press; 2010.
- Cole SR, Chu H, Greenland S. Maximum likelihood, profile likelihood, and penalized likelihood: a primer. *Am J Epidemiol*. 2014; 179(2): 252-260.
- Fahrmeir L, Kneib T, Lang S, Marx B. Regression: Models, Methods and Applications. Berlin: Springer; 2013.
- Hall DB. Zero-inflated Poisson and binomial regression with random effects: a case study. *Biometrics*. 2000; 56(4): 1030-1039.
- Held L, Bové DS. Applied statistical inference: likelihood and Bayes. Heidelberg: Springer-Verlag Berlin; 2014.
- Henningsen A, Toomet O. MaxLik: A package for maximum likelihood estimation in R. *Comput Stat*. 2011; 26(3): 443-458.
- Hussein MJM, Hamodi HA. Comparison count regression models for the number of infected of pneumonia. *Glob J Pure Appl Math*. 2017; 13(9): 5359-5366.
- Isgin T, Bilgic A, Forster DL, Batte M. Using count data models to determine the factors affecting farmers' quantity decisions of precision farming technology adoption. *Comput Electron Agr*. 2008; 62(2): 231-242.
- Ismail MIAB, Sukono, Bon ATB, Hidayat Y, Lesmana E, Supian S. Geometric-Gamma collective modified value-at-risk model in life insurance risk. *Int J Eng Technol*. 2018; 7(3.20): 372-376.
- Iwunor CCO. Estimating of parameters of the inflated geometric distribution for rural out-migration. *Genus*. 1995; 51 (3/4): 253-260.
- Johnson NL, Kemp AW, Kotz S. Univariate discrete distributions. New Jersey: John Wiley; 2005.
- Joshi RD. A generalized inflated geometric distribution. Master [Thesis]. Huntington: Marshall University; 2015.
- Kolev N, Minkova L, Neytchev P. Inflated-parameter family of generalized power series distributions and applications in analysis of over dispersed insurance data. *ARCH Research Clearing House*. 2001; 2: 295-320.
- Lambert D. Zero-inflated Poisson regression, with an application to defects in manufacturing. *Technometrics*. 1992; 34(1): 1-14.
- Mallick A, Joshi R. Parameter estimation and application of generalized inflated geometric distribution. *J Stat Theory Appl*. 2018; 17(3): 491-519.
- Pandya M, Pandya HD, Pandya SL. Bayesian inference on mixture of geometric with degenerate distribution: zero inflated geometric distribution. *Int J Res Rev Appl Sci*. 2012; 13(1):53-65.
- Pawitan Y. In all likelihood: statistical modelling and inference using likelihood. Oxford: Oxford University Press; 2013.
- R Core Team. R: a language and environment for statistical computing. Vienna, Austria: R Foundation for Statistical Computing; 2020.
- Sastry DVS, Bhati D, Rattihalli RN, Gómez-Déniz E. On zero-distorted generalized geometric distribution. *Commun Stat Theory Methods*. 2016; 45(18): 5427-5442.
- Sharma HL. A probability distribution for rural out migration at micro level. *Rural Demogr*. 1985; 12(1-2): 63-69.
- Tripodi R, Pelillo M. A game-theoretic approach to word sense disambiguation. *Comput Linguist*. 2017; 43(1): 31-70.
- Venzon DJ, Moolgavkar SH. A method for computing profile-likelihood-based confidence intervals. *J R Stat Soc Ser C Appl Stat*. 1988; 37(1): 87-94.

- Wackerly DD, Mendenhall W, Scheaffer RL. Mathematical statistics with applications. Belmont: Cengage Learning; 2008.
- Yee TW. Vector generalized linear and additive models with an implementation in R. New York: Springer-Verlag; 2015.
- Yu SZ. Hidden semi-Markov models. Artificial Intelligence. 2010; 174(2): 215-243.
- Zipkin EF, Leirness JB, Kinlan BP, O'Connell AF, Silverman ED. Fitting statistical distributions to sea duck count data: Implications for survey design and abundance estimation. Stat Methodol. 2014; 17: 67-81.

Appendix

Consider the following: $\hat{I}_{11} - \hat{J}_{11}$

$$\begin{aligned}
 &= \frac{n_0(1-\hat{\pi}_{ML})^2}{[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2} - \frac{n(1-\hat{\pi}_{ML})^2}{\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}} + \frac{n_1}{\hat{p}_{ML}^2} + \frac{\sum x_i}{(1-\hat{p}_{ML})^2} - \frac{n(1-\hat{\pi}_{ML})(1-\hat{p}_{ML} + \hat{p}_{ML}^2)}{\hat{p}_{ML}^2(1-\hat{p}_{ML})}, \\
 &= (1-\hat{\pi}_{ML})^2 \left(\frac{n_0}{[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2} - \frac{n}{\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}} \right) + \\
 &\quad \frac{n_1(1-\hat{p}_{ML})^2 + \hat{p}_{ML}^2 \sum x_i - n(1-\hat{\pi}_{ML})(1-\hat{p}_{ML})(1-\hat{p}_{ML} + \hat{p}_{ML}^2)}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2}, \\
 &= (1-\hat{\pi}_{ML})^2 (\hat{I}_{12} - \hat{J}_{12}) + \frac{n_1(1-\hat{p}_{ML})^2 + \hat{p}_{ML}^2 \sum x_i - n(1-\hat{\pi}_{ML})(1-\hat{p}_{ML})(1-\hat{p}_{ML} + \hat{p}_{ML}^2)}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2}, \\
 &= 0 + \frac{n_1(1-\hat{p}_{ML})^2 + n_1\hat{p}_{ML} - n(1-\hat{\pi}_{ML})(1-\hat{p}_{ML})(1-\hat{p}_{ML} + \hat{p}_{ML}^2)}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2}, \\
 &= \frac{(1-\hat{p}_{ML} + \hat{p}_{ML}^2)[n_1 - n(1-\hat{\pi}_{ML})(1-\hat{p}_{ML})]}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2}, \\
 &= \frac{(1-\hat{p}_{ML} + \hat{p}_{ML}^2)n_1[1 - (1-\hat{\pi}_{ML})(1-\hat{p}_{ML})(1+n_0/n_1)]}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2}, \\
 &= \frac{(1-\hat{p}_{ML} + \hat{p}_{ML}^2)n_1(\hat{I}_{12} - \hat{J}_{12})(\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML})^2/n_0}{\hat{p}_{ML}^2(1-\hat{p}_{ML})^2} = 0,
 \end{aligned}$$

and consider $I_{22} - J_{22}$

$$= \frac{n_0(1-\hat{p}_{ML})^2(1-\hat{\pi}_{ML})^2 + n_1[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2 - n(1-\hat{p}_{ML})(1-\hat{\pi}_{ML})[\hat{\pi}_{ML} + (1-\pi)\hat{p}_{ML}]}{[\hat{\pi}_{ML} + (1-\pi)\hat{p}_{ML}]^2(1-\hat{\pi}_{ML})^2}.$$

Substituting n_0 above by n_0 from (6) for n_0 , the numerator of the above equation will be

$$\begin{aligned}
 &n_1(1-\hat{p}_{ML})(1-\hat{\pi}_{ML}) + n_1[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]^2 - n(1-\hat{p}_{ML})(1-\hat{\pi}_{ML})[\hat{\pi}_{ML} + (1-\pi)\hat{p}_{ML}], \\
 &= [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]\{(n_1 - n)(1-\hat{p}_{ML})(1-\hat{\pi}_{ML}) + n_1[\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]\}, \\
 &= [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}]\{(n_1 - n)(1-\hat{p}_{ML})(1-\hat{\pi}_{ML}) + n_0(1-\hat{\pi}_{ML})(1-\hat{p}_{ML})\}, \\
 &= [\hat{\pi}_{ML} + (1-\hat{\pi}_{ML})\hat{p}_{ML}](1-\hat{p}_{ML})(1-\hat{\pi}_{ML})\{n_1 - n + n_0\} = 0.
 \end{aligned}$$