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A Nonparametric Group Runs Control Chart for Location Using Sign Statistic

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Abstract

In this paper, we propose a nonparametric group runs control chart using sign statistic for monitoring the shifts in the process location. Performance of the proposed control chart is measured using average run length, standard deviation of run length and percentiles. The study reveals that the proposed nonparametric group runs control chart performs better than the existing nonparametric control charts based on the sign statistic viz sign chart, the synthetic chart, the 2-of-2 chart, exponentially weighted moving average chart and cumulative sum chart. Implementation procedure of the proposed chart is explained using a numerical example.

Keywords: Nonparametric, sign statistic, control chart, group runs, average run length.

1. Introduction

Statistical process control (SPC) refers to the collection of statistical procedures and problem solving tools used to control and monitor the quality of the product. The main aim of SPC is to detect and eliminate or at least reduce unwanted variation in the process output. A control chart is one of the SPC tools used to identify whether a process is under statistical control or not. Parametric control charts are based on the assumption that the process output follows a specific probability distribution. When the process output does not follow the assumption of normality, the performance of the parametric control charts cannot be appropriate. Then there is need of development of the nonparametric control charts or distribution-free control charts. Nonparametric control charts are designed to detect the changes in the process median and mean or changes in the process variability. The chart is said to be the nonparametric control chart if in-control average run length (ARL) does not depend on the underlying process distribution. ARL is the average number of samples that need to be collected before the first out-of-control signal given by a chart.

In the literature related to nonparametric control charts, Bakir and Reynolds (1979) proposed the cumulative sum (CUSUM) control chart to monitor a process centre based on within-group signed-ranks. Chakraborti et al. (2001) presented an extensive overview of the literature on univariate nonparametric control charts. Bakir (2004) suggested a nonparametric Shewhart-type control chart for monitoring the process centre based on the signed-ranks of grouped observations. Chakraborti and Eryilmaz (2007) proposed a nonparametric Shewhart-type control charts based on signed-rank statistic under k-of-k runs rules. Pawar and Shirke (2010) developed a synthetic control chart using signed-rank statistic. Human et al. (2010) suggested the nonparametric control charts based on runs of sign statistic. Mukharjee et al. (2013) proposed distribution-free exceedance CUSUM control charts for location. Cheng and Shiau (2015) proposed a distribution-free multivariate control chart for phase I applications. Liu et al. (2014) proposed dual nonparametric CUSUM control chart based on ranks. Riaz and Abbasi (2016) developed double exponentially weighted moving average (EWMA) control chart for process monitoring. Abid et al. (2016) suggested nonparametric EWMA control chart based on the Wilcoxon signed-rank statistic for monitoring location. Pawar et al. (2018) proposed a nonparametric moving average control charts using the sign and the signed-rank statistics to detect shifts in the location. Pawar et al. (2018) studied the steady-state behavior of the nonparametric synthetic control chart using signed-rank statistic.

In the review of the literature related to nonparametric control charts based on the sign test statistic, Amin et al. (1995) proposed a nonparametric Shewhart-type sign chart and CUSUM control chart based on the sign statistic for monitoring the process location and the process variability. Khilare and Shirke (2010) developed synthetic control chart based on the sign statistic to monitor shifts in the location. The proposed synthetic control chart performs significantly better than the sign chart and Shewhart-type chart. Khilare and Shirke (2012) developed a nonparametric synthetic control chart using sign statistic for a process variability based on the quartiles. Yang et al. (2011) proposed a new nonparametric EWMA control chart based on the sign statistic. Yang and Cheng (2011) developed a new nonparametric CUSUM mean chart using the sign statistic. Graham et al. (2009) developed nonparametric EWMA control using sign statistics. Khilare and Shirke (2015) studied the steady-state behavior of nonparametric control charts using sign statistic. Shirke and Barale (2018) proposed a new nonparametric CUSUM chart for the process dispersion using the sign statistic based on the deciles. Tang et al. (2019) suggested a new nonparametric adaptive EWMA chart based on the sign statistic.

This paper presents the positive-sided and two-sided nonparametric group runs control charts using sign statistic for monitoring shifts in the process median. The main purpose of the proposed chart is to improve the performance of the existing nonparametric control chart based on the sign statistic for a wide class of the process distributions. Rest of the paper is organized as follows: Section 2 presents a sign chart for location. Section 3 gives the conforming run length (CRL) control chart in brief. In Section 4, the nonparametric group runs control chart based on a sign statistic is explained in detail. Section 5 gives a nonparametric CUSUM chart using sign statistic. Section 6 gives the performance study of the proposed nonparametric group runs chart. A numerical example is given in Section 7. In Section 8, Concluding remarks are given.

2. A Nonparametric Sign Chart for Location

Let $X_i = (X_{i1}, X_{i2}, ..., X_{in})$ be a random sample of size $n > 1$ taken from a continuous process with median θ at sampling instance $t = 1, 2, ...$ It is assumed that the underlying process distribution is continuous. Let θ_0 be the in-control process median is known or specified to be equal to θ_0. We further assume that if $\theta = \theta_0$, the process is in-control and when $\theta \neq \theta_0$, the process is to be out-of-control. A sample of n observations is taken at a regular time interval from the process and each observation is compared with the target value θ_0. In the comparison procedure, the number of observations below θ_0 and the above θ_0 are recorded for each sample. Define

$$sign(X_{ij}-\theta_0)=\begin{cases} 1, & \text{if } X_{ij}>\theta_0, \\ 0, & \text{if } X_{ij}=\theta_0, \\ -1, & \text{if } X_{ij}<\theta_0, \end{cases} \tag{1}$$

where X_{ij} is the j^{th} observation in the i^{th} sample. Since the distribution of observations is assumed to be continuous, $pr\left(X_{ij}-\theta_0=0\right)=0.$ In practice occasionally zero may occur which can be signed alternatively $+1$ and $-1.$ Let

$$SN_i=\sum_{j=1}^{n} sign\left(X_{ij}-\theta_0\right),\ i=1,2,... \tag{2}$$

where SN_i is the difference between the number of observations above θ_0 and number of observations below θ_0 in the i^{th} sample. A random variable $T_i=\dfrac{SN_i+n}{2}$ gives the number of positive signs in the sample of size n and has a binomial distribution with parameters n and p, where $p=pr\left(X_{ij}>\theta_0\right).$ As long as the median remains at θ_0. Moreover, when the process is in-control $p=p_0=0.5.$ A nonparametric sign chart signals an out-of-control status if $\left|SN_i\right|\geq a_1,$ where $a_1>0$ is a constant. The following section gives conforming run length control chart.

3. Conforming Run Length Control Chart

Bourke (1991) proposed the CRL control chart. According to Bourke (1991) unit/sample CRL is the number of conforming units/ samples between two successive nonconforming units/ samples. According to Wu and Spedding (2000) unit/ sample based CRL is the number of conforming units/ samples between two nonconforming units/ samples including ending nonconforming unit/sample. The charting statistic CRL follows a geometric distribution with cumulative distribution function (cdf)

$$F(CRL)=1-(1-p)^{CRL},\quad CRL=1,2,3,... \tag{3}$$

where p is the probability of the nonconforming unit/sample only. When there is only concern about the detecting of an increase in $p,$ the lower control limit (L_2) of the unit/sample based CRL chart is sufficient and it is given by

$$L_2=\frac{\ln\left(1-\alpha_{CRL}\right)}{\ln\left(1-p_0\right)}, \tag{4}$$

where $\alpha_{CRL}=1-(1-p)^L$ is the type-I error probability of the unit/ sample based CRL chart and p_0 is the in-control fraction nonconforming of the unit/sample based CRL chart. In the following section a nonparametric group runs control chart is explained in detail.

4. A Group Runs Control Chart Based on the Sign Statistic

Gadre and Rattihalli (2004) proposed the group runs control chart to detect shifts in the process mean by combining Shewhart's $\overline{X}$ chart with group based CRL chart. The group runs (GR) chart signals an out-of-control status, if the first value of $CRL\leq L_2$ or two successive values of $CRL\leq L_2$ for the first time. In literature related to the nonparametric control charts, Gadre and Kakade (2014) developed a nonparametric group runs control chart to detect shifts in the process median by combining nonparametric Shewhart's control chart based on the signed-rank statistic and group-based CRL chart. This article proposes a nonparametric group runs (NP-GR) control chart based on the sign statistic to detect shifts in the process location by combining operations of the nonparametric sign chart with the group based CRL chart. Some notations for NP-GR chart**:**

δ : Design shift in the mean,

$ARL(\delta)$: The average number of the sign statistic required to detect a shift in the process median.

δ_1 : Design shift in the median, the magnitude of which is considered large enough to seriously impair the quality of the product,

n : Group size,

a_3 : Upper control limit of the NP-GR chart based on the sign statistic,

L_2 : Lower control limit of the group-based CRL chart,

Y_r : The r^{th} $(r = 1,2,...)$ value of group-based CRL. In other words, it is the number of groups (samples) inspected between $(r-1)^{\text{th}}$ and r^{th} nonconforming group, including the r^{th} nonconforming group,

τ : The minimum required in-control ARL (ARL(0)),

a_1 : Upper control limit of the sign chart,

a_2 : Upper control limit of the synthetic chart based on the sign statistic,

L_1 : Lower control limit of the CRL chart,

$ARL_{GR}(0)$: In-control ARL of the group runs sign chart.

For the implementation of the nonparametric GR chart, the implementation procedure of the NP-GR chart is as follows**:**

1. Inspect n units in a group and calculate the sign statistic SN_i.

2. If $SN_i < a_3$, a group is conforming one and control flow goes back to step1. Otherwise, a group is a nonconforming one and control flow continues to the next step.

3. If $Y_1 \le L_2$ or two successive Y_r's are less than or equal to L_2 for the first time, the process is said to be out-of-control.

4. When the process goes out-of-control, necessary corrective action should be taken to reset and resume it. Once the process restarts, move to Step1.

For the design of the nonparametric GR **(**NP**-**GR**)** chart, the model is

Minimize $ARL_{GR}(\delta_1)$

Subject to the constraint: $ARL_{GR}(0) \ge \tau$.

Let P be the probability of the group being a nonconforming. Then,

$$P = P(\delta) = P(SN_i \ge a_3 / \theta = \theta_0 + \delta). \quad (5)$$

Notice that Y_r $(r = 1,2,3,...)$ is independent and identically distributed (i.i.d.) geometric random variables and its expected value is $1/P$. Therefore, if N is the number of nonconforming groups observed before detecting the process has gone out-of-control then

$$E(N)=\frac{1}{\left(1-(1-P)^{L}\right)^{2}}.$$

If the process signals an out-of-control status for the first time when N^{th} nonconforming group is observed, then ARL of the NP-GR chart is

$$ARL_{GR}(\delta_1)=\frac{1}{P(\delta_1)\left(1-(1-P(\delta_1))^{L}\right)^{2}}. \tag{6}$$

An in-control ARL of a nonparametric GR chart is given by

$$ARL_{GR}(0)=\frac{1}{P(0)\left(1-(1-P(0))^{L}\right)^{2}}. \tag{7}$$

The stepwise procedure to find optimum values of the design parameters (L_2, a_3) of the NP- GR chart is as follows:

1. Specify group size n and $ARL(0)$.
2. Initialize L_2 as 1 and $1 \le a_3 \le n$.
3. Compute $ARL_{GR}(0)$ from the current values of L_2 and a_3 using Equation (7).
4. If $ARL_{GR}(0)$ is approximately equal to the desired $ARL(0)$, then take the current L_2 and a_3 as the final values in the NP- GR chart.

In a nonparametric control chart based on the sign statistic, a specified in-control ARL is depending upon a sample size (n). The in-control ARL of the NP-GR-chart based on the sign statistic is depending upon n, L_2 and a_3.

The problem is to find an optimum pair (L_2, a_3) such that $ARL_{GR}(0)$ is equal to the specified value of the in-control ARL and $ARL_{GR}(\delta)$ is minimum. Thus the optimization problem in which the objective function is to minimize the $ARL_{GR}(\delta)$ subject to equality constraint function for $ARL_{GR}(0)$. Let L_2 be the only independent design variable and a_3 is a function of L_2. Table1 provide the in-control average run length of the upper one-sided nonparametric GR chart.

Table 1 Different pairs of L_2 and a_3 for $n=10$ and specified $ARL(0)=1024$

L_2	a_3	$ARL_{GR}(0)$
2	6	1615.68
2	7	1615.68
3	6	758.62
3	7	758.62
4	6	450.58
4	7	450.58

A specified $ARL(0)$ depends only on sample size (n) and $ARL_{GR}(0)$ on n, L_2 and a_3. For the proposed NP-GR control chart a specified $ARL_{GR}(0)=1024$ when $n=10$. The $ARL_{GR}(0)$ remains same for certain values of n even when a_3 is changed. The following example clears why $ARL_{GR}(0)$ remains same for certain values of n even when a_3 is changed. Let $a_3=6$, then

$(a_3+n)/2=(6+10)/2=8$ and if $a_3=7$, then $(a_3+n)/2=(7+10)/2=8.5$. The $ARL_{GR}(0)$ unchanged for both the choices of a_3 because underlying binomial random variable takes only integer values. Due to discrete nature of sign statistic SN_i, the $ARL_{GR}(0)$ is not approximately equal to the specified $ARL(0)$. In the following section a nonparametric CUSUM chart using sign statistic is explained in detail.

5. A Nonparametric CUSUM Control Chart

Monitoring small shifts in process mean is equivalent to monitoring small changes in process proportion p. Let $\Delta=|p_0-p_1|$, $\Delta>0$ and we are interested in to detect a shift of size p_1 quickly. If the process is in-control then $p=p_0$ or if the process is out-of-control that is mean θ has shifted then $p=p_1$. First we define two CUSUM monitoring statistics for the i^{th} subgroup sample as follows,

$$C_i^+=\max\left(0,SN_i-(np_0+k_1)+C_{i-1}^+\right), \qquad (8)$$
$$C_i^-=\max\left(0,(np_0-k_1)-SN_i+C_{i-1}^-\right),$$

where SN_i is defined in Equation (2) count the number of observations above θ_0 and number of observations below θ_0 in the i^{th} sample and k_1 is the reference value with $k_1=\dfrac{n\Delta}{2}$. The starting values of the above defined CUSUM statistics are chosen as zero, that is, $C_0^+=0$ and $C_0^-=0$. Let h be the parameter of a nonparametric CUSUM control chart using sign statistic. If $C_0^+>h$ or $C_0^->h$, then the process signals out-of-control status. Moreover, if $C_0^+>h$, then the upward shift is detected in a process location and if $C_0^->h$, then the downward shift is detected in a process location. The two design parameters k_1 and h of a nonparametric CUSUM control chart using sign statistic are chosen in such a way that they would satisfy the specified ARL.

We first obtain ARL of the upper-sided CUSUM chart using Markov chain approach. Brook Evans (1972) has computed the ARL of a CUSUM chart by employing Markov chain approach. Yang and Cheng (2011) computed ARL of a nonparametric CUSUM chart based on the sign statistic to detect shifts in the process location. We follow the procedure given by Yang and Cheng (2011) to compute the ARL of a nonparametric CUSUM chart using sign statistic. To obtain the ARL of a nonparametric CUSUM chart using sign statistic, we divide the region $(0,h)$ into $M-1$ sub-intervals of equal width of $2w$, where $w=h/(2(M-1))$. The first subinterval is $(-\infty,0]$, the k^{th} subinterval is (m_k-w,m_k+w), where m_k be the midpoints of the k^{th} subinterval with $m_1=0$, $m_k=(2k-3)h/(2(M-1))$ for $k=2,3,4,...,M$ and $(M+1)^{\text{th}}$ interval as (h,∞). These all $(M+1)$ subintervals can be viewed as states of the Markov chain. The $(M+1)^{\text{th}}$ state is the absorbing state and remaining $1,2,...,M$ states are transient states of the Markov chain $(C_i^+, i=0,1,2,...)$.

Let P_{kj}^+ be the transition probability that statistic C_i^+ reaches state j at time i, given that C_{i-1}^+ was in the state k at time $(i-1)$. The transition probabilities can be calculated as follows:

$$
\begin{aligned}
P_{k1}^{+} &= P\left(C_i^{+} \le 0 / C_{i-1}^{+} = m_k\right), \\
P_{k1}^{+} &= P\left(SN_i - \left(np_0 + k_1\right) + C_{i-1}^{+} \le 0 / C_{i-1}^{+} = m_k\right), \\
P_{k1}^{+} &= P\left(SN_i \le np_0 + k_1 - m_k\right), \\
P_{k1}^{+} &= \sum_{s=0}^{\left[np_0 + k_1 - m_k\right]} {}^{n}c_s p^s \left(1-p\right)^{n-s},
\end{aligned} \tag{9}
$$

$k = 1,2,3,...,M$ and $i = 1,2,3,...$

$$
\begin{aligned}
P_{kj}^{+} &= P\left(m_j - w \le C_i^{+} \le m_j + w / C_{i-1}^{+} = m_k\right), \\
P_{kj}^{+} &= P\left(m_j - w \le SN_i - \left(np_0 + k_1\right) + C_{i-1}^{+} \le m_j + w / C_{i-1}^{+} = m_k\right), \\
P_{kj}^{+} &= P\left(m_j - w + np_0 + k_1 - m_k \le SN_i \le m_j + w + np_0 + k_1 - m_k\right), \\
P_{kj}^{+} &= P\left(SN_i \le m_j - m_k + w + np_0 + k_1\right) - P\left(SN_i \le m_j - m_k - w + np_0 + k_1\right), \\
P_{kj}^{+} &= \sum_{s=0}^{\left[\left(m_j - m_k + w + np_0 + k_1\right)^{-}\right]} {}^{n}c_s p^s \left(1-p\right)^{n-s} - \sum_{s=0}^{\left[\left(m_j - m_k - w + np_0 + k_1\right)^{-}\right]} {}^{n}c_s p^s \left(1-p\right)^{n-s},
\end{aligned} \tag{10}
$$

$k = 1,2,3,...,M$ and $j = 2,3,...,M$ and $i = 1,2,3,...,$ where $SN_i \to B\left(n,p\right)$ and $\left(\beta\right)^{-}$ represent the largest integer not greater than β. Let $b = \left(b_1, b_2, ..., b_M\right)'$ be a vector of the initial probabilities that the process started in state $1,2,3,...,M$. Since we assume that $C_i^{+} = C_i^{-} = 0,$ we get $b_1 = 1$ and $b_k = 0$ for $k \ne 1$. Let $P^{+} = (P_{kj}^{+})$ be the transition probability matrix of order $(M+1)\times(M+1)$ and $R^{+} = (P_{kj}^{+})$ be a $M \times M$ sub-matrix of P^{+} by deleting last row and last column. Then the ARL of the upper-sided CUSUM chart can be obtained as $ARL^{+} = b'(1-R^{+})^{-1}1',$ where $1' = \left(1,1,...,1\right)$ is the $1 \times N$ vector with elements 1. The in-control ARL of a chart can be obtained by putting $p = p_0$, so $ARL^{+} = ARL_0^{+}$ be the in-control ARL and if $p = p_1$ then $ARL^{+} = ARL_1^{+}$ be the out-of-control ARL. In similar way one can compute ARL of a lower-sided nonparametric CUSUM chart, which is denoted by ARL^{-}. Then the ARL for two-sided nonparametric CUSUM chart can be obtained as follows:

$$
ARL = \frac{ARL^{+} \times ARL^{-}}{ARL^{+} + ARL^{-}}. \tag{11}
$$

The following section gives the performance study of the proposed control chart.

6. Performance Study of the Nonparametric GR Chart

The performance of the NP-GR chart is evaluated using popular measure ARL, standard deviation of run length (SDRL), 25^{th} percentile (Q1), 50^{th} percentile (Q2= median run length) and 75^{th} percentile (Q3). Performance of the various control charts is investigated under normal and non-normal distributions. When the underlying process distribution is normal, in-control value of process center is $\theta_0 = 0$ and variance $\sigma_0^2 = 1$. For out-of-control process, the parameters of normal distribution are θ and σ^2, where $\theta = \theta_0 + \delta\sigma_0$. The performance of two control charts is compared by making the ARL(0) of the two control charts same. Therefore, we computed adjusted ARL of a chart B with respect to the chart A as

$$\left[\text{adjusted}\, ARL(\delta)\right]_B = \frac{\left[ARL(\delta)\right]_B}{\left[ARL(0)\right]_B} \times \left[ARL(0)\right]_A .$$

6.1. Performance study of the positive-sided GR chart

The positive-sided NP-GR chart signals an out-of-control status if a charting statistic $SN_i \geq a_3$. The charting statistic SN_i is a sequence of Bernoulli trials at each sampling subgroup i, each resulting in a either a signal "1" or a no-signal "0". The resulting Bernoulli sequence has independent elements with probability of success (signal) or group being a nonconforming one is given by

$$P^+ = P^+(\delta) = pr(SN_i \geq a_3), \quad i = 1,2,3,... \text{ or } P^+ = pr\left[T_i \geq \frac{a_3 + n}{2}\right].$$

We have compared the performance of the proposed NP- GR chart with the sign chart, nonparametric synthetic and CUSUM control charts using sign statistic. Tables 2 to 6 provide ARL and SDRL values of the $N(0,1)$, double exponential$\left(0,1/\sqrt{2}\right)$, uniform$\left(-\sqrt{3},\sqrt{3}\right)$, Cauchy$(0, 0.2605)$ and gamma gamma$(4,1/2)$ for $n = 10$ and ARL(0) = 1024.

Table 2 The ARL and SDRL profile of the positive-sided sign chart, the synthetic chart, GR chart and CUSUM chart under normal distribution
($a_1 = 10, a_2 = 9, L_1 = 9, a_3 = 6, L_2 = 3, k_1 = 1.5, h = 4.4$)

Shift (δ)	Sign Chart		Synthetic Chart		GR Chart		CUSUM Chart	
	ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL
0	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5
0.25	169.0	168.5	65.2	64.7	46.7	46.2	44.1	43.6
0.5	40.0	39.5	10.2	9.7	6.9	6.4	8.2	7.7
1	5.6	5.1	2.0	1.4	1.7	1.1	2.8	2.2
2	1.3	0.6	1.0	0.0	1.4	0.7	1.9	1.3

Table 3 The ARL and SDRL profile of the positive-sided sign chart, the synthetic chart, GR chart and CUSUM chart under Laplace distribution
($a_1 = 10, a_2 = 9, L_1 = 9, a_3 = 6, L_2 = 3, k_1 = 1.5, h = 4.4$)

Shift (δ)	Sign Chart		Synthetic Chart		GR Chart		CUSUM Chart	
	ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL
0	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5
0.25	75.5	75.0	21.8	21.3	14.8	14.3	15.3	14.7
0.5	17.0	16.5	4.4	3.8	3.1	2.6	4.5	4.0
1	3.7	3.1	1.6	0.9	1.5	0.9	2.4	1.8
2	1.3	0.7	1.1	0.2	1.4	0.7	1.9	1.3

Table 4 The ARL and SDRL profile of the positive-sided sign chart, the synthetic chart, GR chart and CUSUM chart under Cauchy distribution
($a_1 = 10, a_2 = 9, L_1 = 9, a_3 = 6, L_2 = 3, k_1 = 1.5, h = 4.4$)

Shift (δ)	Sign Chart		Synthetic Chart		GR Chart		CUSUM Chart	
	ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL
0	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5
0.25	19.4	18.9	4.9	4.4	3.4	2.9	4.9	4.4
0.5	5.3	4.8	1.9	1.3	1.7	1.1	2.7	2.1
1	2.3	1.7	1.3	0.6	1.4	0.7	2.1	1.5
2	1.5	0.9	1.1	0.3	1.4	0.7	2	1.4

Table 5 The ARL and SDRL profile of the positive-sided sign chart, the synthetic chart, GR chart and CUSUM chart under uniform distribution
($a_1 = 10, a_2 = 9, L_1 = 9, a_3 = 6, L_2 = 3, k_1 = 1.5, h = 4.4$)

Shift (δ)	Sign Chart		Synthetic Chart		GR Chart		CUSUM Chart	
	ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL
0	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5
0.25	265.9	265.4	126.4	125.9	95.9	95.4	90.4	89.9
0.5	81.1	80.6	23.9	23.4	16.2	15.7	16.5	16.0
1	10.7	10.2	3.0	2.5	2.3	1.7	3.6	3.1
2	1.0	0.0	1.0	0.1	1.3	0.7	1.9	1.

Table 6 The ARL and SDRL profile of the positive-sided sign chart, the synthetic chart, GR chart and CUSUM chart under gamma distribution
($a_1 = 10, a_2 = 9, L_1 = 9, a_3 = 6, L_2 = 3, k_1 = 1.5, h = 4.4$)

Shift (δ)	Sign Chart		Synthetic Chart		GR Chart		CUSUM Chart	
	ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL	adj.ARL	SDRL
0	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5	1024.0	1023.5
0.25	143.1	142.6	51.5	51.0	36.3	35.8	34.4	33.9
0.5	26.6	26.1	6.6	6.1	4.5	4.0	6.0	5.5
1	2.5	1.9	1.3	0.6	1.4	0.8	2.2	1.6
2	1.0	0.0	1.0	0.1	1.3	0.7	1.9	1.3

From Tables 2 to 6, it is observed that:

- The ARL and SDRL values of the proposed NP-GR chart are significantly smaller than the synthetic and sign charts for all shifts. That is the proposed NP-GR chart performs significantly better than the synthetic and sign charts under all considered process distributions.
- Performance of the proposed NP-GR chart is also significantly better than the CUSUM chart under heavy-tailed distributions.

 Under normal, uniform and gamma distributions ARL and SDRL values of the NP-GR chart are smaller than the CUSUM chart except a shift of magnitude equal to 0.25.

 Under the Cauchy distribution, the performance of the proposed NP-GR chart is significantly better than the all other distributions under study.

 Therefore, the proposed NP-GR chart is a more efficient than the synthetic and CUSUM charts.

The ARL and SDRL does not give the complete information about the performance study of a control chart due to a highly positively skewed distribution of run length. Quantiles give a more information about the performance study of the control chart. Therefore, we computed 25th percentile, 50th percentile, 75th percentile and inter quartile range (IQR). Tables 7 to 11 give three quartiles and IQR of the positive-sided synthetic, GR and CUSUM charts.

Table 7 Three quartiles and IQR values of the positive-sided synthetic, GR and CUSUM charts under normal distribution

Shift (δ)	Synthetic Chart				GR Chart				CUSUM Chart			
	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0	294	709	1419	1125	294	709	1419	1125	294	709	1419	1125
0.25	19	45	90	71	13	32	64	51	13	30	60	47
0.5	3	7	13	10	2	4	9	7	2	5	11	9
1	0	1	2	2	0	1	2	2	1	2	3	2
2	0	0	0	0	0	1	1	1	0	1	2	2

Table 8 Three quartiles and IQR values of the positive-sided synthetic, GR and CUSUM charts under Laplace distribution

Shift (δ)	Synthetic Chart				GR Chart				CUSUM Chart			
	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0	294	709	1419	1125	294	709	1419	1125	294	709	1419	1125
0.25	6	15	30	24	4	10	20	16	4	10	21	17
0.5	1	3	5	4	1	2	4	3	1	3	6	5
1	0	1	1	1	0	1	1	1	1	1	3	2
2	0	0	1	1	0	1	1	1	0	1	2	2

Table 9 Three quartiles and IQR values of the positive-sided synthetic, GR and CUSUM charts under Cauchy distribution

Shift (δ)	Synthetic Chart				GR Chart				CUSUM Chart			
	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0	294	709	1419	1125	294	709	1419	1125	294	709	1419	1125
0.25	1	3	6	5	1	2	4	3	1	3	6	5
0.5	0	1	2	2	0	1	2	2	1	1	3	2
1	0	0	1	1	0	1	1	1	0	1	2	1
2	0	0	1	1	0	1	1	1	0	1	2	1

Table 10 Three quartiles and IQR values of the positive-sided synthetic, GR and CUSUM charts under uniform distribution

Shift	Synthetic Chart				GR Chart				CUSUM Chart			
(δ)	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0	294	709	1419	1125	294	709	1419	1125	294	709	1419	1125
0.25	36	66	125	89	27	66	132	105	26	62	125	101
0.5	7	11	22	15	5	11	22	17	5	11	22	17
1	1	1	4	3	1	1	2	1	1	2	4	3
2	0	0	2	2	0	0	1	1	0	1	2	1

Table 11 Three quartiles and IQR values of the positive-sided synthetic, GR and CUSUM charts under gamma distribution

Shift	Synthetic Chart				GR Chart				CUSUM Chart			
(δ)	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0	294	709	1419	1125	294	709	1419	1125	294	709	1419	1125
0.25	15	35	71	56	10	25	50	40	10	24	47	37
0.5	2	4	8	6	1	3	6	5	2	4	8	6
1	0	0	1	1	0	1	1	1	0	1	2	2
2	0	0	0	0	0	1	1	1	0	1	2	2

Tables 7 to 11 reveals that:

- The quartile and IQR values of the proposed NP-GR chart are significantly smaller than the synthetic control chart for all distributions under study. Therefore, the performance of the NP-GR chart is superior than the synthetic control chart.
- As compare to the CUSUM chart, the NP-GR chart is superior under heavy-tailed distributions,
- Under normal, uniform and gamma distributions the NP-GR chart is more efficient than the CUSUM chart except a shift of magnitude equal to 0.25.

6.2. Performance study of the two-sided nonparametric GR chart

If interest of practitioners in up-ward shift or down-ward shift from the target value of median, the two-sided control chart may be useful. For the proposed NP-GR chart the signaling indicator is defined as

$$\xi_i = \begin{cases} 1, & \text{if } SN_i \le -a_3 \text{ or } SN_i \ge a_3, \quad i = 1,2,3,\ldots, \\ 0, & \text{otherwise,} \end{cases}$$

with the corresponding signaling probability P is

$$P = P(\delta) = pr[\xi_i = 1] = pr[SN_i \le -a_3 \ or \ SN_i \ge a_3].$$

Table 12 ARL values of two-sided NP-GR, NP-2/2 and NP-EWMA charts under normal, t, Laplace and logistic distributions with ARL(0) = 500

Shift	$N(0,1)$			$t(4)$			Laplace			Logistic		
(δ)	GR	2/2	EWMA	GR	2/2	EWMA	GR	2/2	EWMA	GR	2/2	EWMA
0.00	500	500	500	500	500	500	500	500	500	500	500	500
0.50	5.62	15.45	9.01	6.97	18.29	6.94	2.54	7.88	6.56	4.13	12.05	8.00
1.00	1.41	4.17	4.78	1.61	4.93	4.21	1.25	3.53	4.29	1.32	3.81	4.53
1.50	1.13	3.06	3.65	1.20	3.33	3.37	1.13	3.04	3.57	1.13	3.04	3.59
2.00	1.11	2.96	3.15	1.12	3.03	3.16	1.11	2.96	3.22	1.11	2.96	3.18
2.50	1.10	2.53	3.01	1.11	2.98	3.05	1.10	2.39	3.07	1.10	2.43	3.04

We also compared the performance of the proposed NP-GR chart with NP-EWMA chart based on the sign statistic to detect a shift in the process median proposed by Graham et al. (2009). Table 12 gives the ARL values of the NP-GR, NP-2-of-2 and NP-EWMA charts under normal, t, double exponential and logistic distributions. The ARL values of the NP-EWMA sign chart with smoothing parameter $\lambda = 0.05$ and $L = 2.612$ for various shifts in the location are given in Graham et al. (2009). From Table 12, it is observed that the ARL values of the proposed NP-GR chart have significantly better performance over the NP- 2-of-2 and NP-EWMA charts using sign statistic for normal, Laplace and logistic distributions. The proposed NP-GR chart under t distribution is more efficient than the NP-EWMA chart except a shift of magnitude equal to 0.5 and NP-2-of-2 chart. The ARL values of the NP-GR chart under Laplace distribution are significantly less as compared to the other distributions under study for all shifts in the process median.

7. Numerical Example

The operations of the proposed control chart can be illustrated using data related to the diameter of casting taken from Montgomery (2009) (Exercise example no.-6.69, page no.-286). The data set contains 20 samples each of size five. The median of the data set is to be $\mu_0 = 11.7531$. To have an in-control ARL equal to 32, the parameters of the upper-sided synthetic and group runs control charts are $a_2 = a_3 = 3$ and $L_1 = 1, L_2 = 3,$ respectively. A sample is conforming one when $SN_i < a_3$. Table 8 depicts the values of the sign statistic for 20 samples. Figure1 gives the upper-sided synthetic and group runs control charts using sign statistic. The synthetic control chart signals an out-of-control status, if $CRL \le L_1$ and GR chart signals an out-of-control status, if $CRL_1 \le L_2$ or two successive CRL's are less than or equal to L_2 for the first time. Figure1 shows that a sign statistic of sample two is plotted above a_3. That is, sample two is nonconforming and CRL_1 at this time epoch is 2 which is less than L_2; hence, GR chart signals an out-of-control status at time epoch 2. However, synthetic control chart does not signals an out-of-control status at time epoch 2. The GR chart produces an out-of-control signal at all-time epochs except a time epoch 17. The synthetic control chart signals an out-of-control status at time point 3 for the first time. The synthetic control chart also produces an out-of-control signal at time points 4, 10, 18, 19 and 20. The GR chart produces early out-of-control signal than the synthetic control chart. Therefore, the GR chart based on sign statistic performs better than the synthetic control chart.

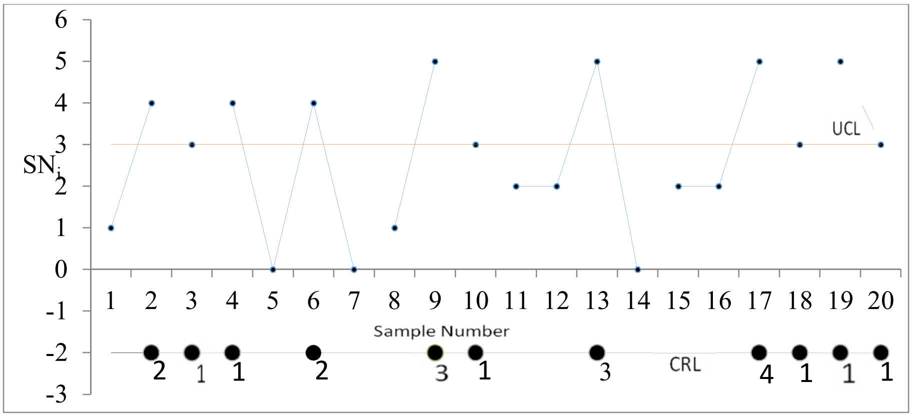

Figure 1 Nonparametric GR control chart using sign statistic

Table 13 Sample number and sign statistic (SN_i)

Sample Number	SN_i
1	1
2	4
3	3
4	4
5	0
6	4
7	0
8	1
9	5
10	3
11	2
12	2
13	5
14	0
15	2
16	2
17	5
18	3
19	5
20	3

8. Conclusions

In this paper, we proposed the positive-sided and two-sided NP-GR charts based on the sign statistic to monitor shifts in the process median. The performance in term of ARL, SDRL, quartiles and IQR of the proposed NP-GR chart under various distributions is studied. The proposed NP-GR chart is more efficient than the NP-synthetic, NP-CUSUM and NP-EWMA charts based on the sign statistic. The study also shows that the proposed NP-GR chart performs better under heavy-tailed distributions. A numerical example also indicates that the NP-GR chart signals faster than the synthetic chart. The proposed NP-GR chart has a higher power of detecting an out-of-control signal.

Implementation procedure of the NP-GR chart is easy and simple for practitioners. We recommend for use of the proposed NP-GR chart to practitioners.

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