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A New Generalization of Power Lindley Distribution and Its Applications

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Abstract

In this paper, a new distribution for modelling the lifetime data has been introduced named as type II generalized Topp Leone-power Lindley (TIIGTL-PLindley) distribution by modifying the cumulative distribution function of power Lindley distribution depending on type II generalized Topp Leone family. Furthermore, main statistical properties of the proposed model such as the reliability function, hazard rate function, probability weighted moments, moments, moment generating function, Renyi entropy, order statistics etc. are determined. For the estimation of unknown parameters, maximum likelihood estimation method is performed. An attempt has been made to fit the TIIGTL-PLindley distribution. In order to illustrate the goodness of fit, flexibility and usefulness, we have included a real life application of this proposed three parameter distribution. Two real data sets are fitted for a range of distributions and compared the performance with the proposed distribution.

Keywords: Goodness of fit, maximum likelihood, reliability function, type II generalized Topp Leone-Lindley distribution.

1. Introduction

1.1. On the extensions of Lindley distribution

Existing distributions have been modified by many statistical researchers in several ways. Among these approaches, generalization through adding parameters in the existing distributions is a widely applied one for developing more flexible distributions, mostly applied in reliability analysis. Many distributions have been recommended in the literature for modelling the lifetime of an item such as exponential, gamma, lognormal, Weibull and Lindley. Among these models Lindley distribution (Lindley 1958) gained popularity in analyzing the lifetime of an item. This distribution plays a major role because it serves as useful reliability model due to its attractive flexibility in modelling the lifetime data. After that properties and many generalizations of Lindley distribution have been discussed by various researchers. Some of them are extended Lindley distribution (Bakouch et al. 2012), two-parameter weighted Lindley distribution (Ghitany et al. 2011), generalized Lindley distribution (Nadarajah et al.2011), transmuted Lindley-geometric distribution (Merovci and Elbatal 2014), generalized Lindley distribution (Oluyede and Yang 2015) etc. Ghitany et al. (2013) suggested an extension of the Lindley distribution by considering the power transformation of Lindley distribution. This is a two parameter distribution with distribution function

$$G_{PL}(x) = 1 - \left(1 + \frac{\theta}{\theta+1} x^{\lambda}\right) e^{-\theta x^{\lambda}} \; ; \; x > 0,\ \lambda, \theta > 0, \tag{1}$$

where λ and θ are the shape and scale parameters, respectively. The corresponding probability density function (PDF) is given by

$$g_{PL}(x) = \lambda \frac{\theta^2}{\theta+1}\left(1 + x^{\lambda}\right) x^{\lambda-1} e^{-\theta x^{\lambda}} \; ; x > 0,\ \lambda, \theta > 0, \tag{2}$$

If $\lambda = 1$, this distribution reduces to the one parameter Lindley distribution.

1.2. On the extension of distributions through Topp Leone generated families

In the recent literature quite a few generalized distributions through generated families of distributions are available and can be helpful for real life data analysis. Among these families Topp Leone (TL) generated family of distributions shows more flexibility to model the lifetime of an item and offer practical outcome through definite conclusions and decisions. Topp Leone generalized family of distributions have been studied by different approaches by various researchers. Sangsanit et al. (2016) introduced TL distribution as a generator for continuous distributions due to its flexibility of providing closed form expressions of probability density and cumulative distribution functions and this TL distribution was introduced by Topp and Leone (1955). They studied important statistical properties and derived related inference. Also type I Topp Leone generalized family has been defined by Al-Shomrani et al. (2016) by using the cumulative distribution function of TL distribution. Recently, type II TL family has been introduced by Elgarhy et.al (2018), they studied some statistical and mathematical properties of the new family and parameters estimated through maximum likelihood estimation method. Yahia and Mohammed (2019) proposed TIITLGIR distribution; they examined some explicit measures for a few of its statistical properties and provided an illustration for showing the flexibility of the proposed distribution for fitting the real data set. Al-Marzoukiet et al. (2020) introduced four parameter TIITLPL distributions and studied its desirable properties, and potentiality of the distribution is checked with the help of some real life data sets. A natural extension of the type II Topp-Leone-G family has been discussed by Bantan et al. (2020) by considering an additional shape parameter, they examined the main properties of their suggested family and investigated quantile function, moments, order statistic, mixture representations, stochastic ordering etc.

Hassan et al. (2019) proposed type II generalized Topp-Leone-G family (TIIGTL) and examined some of its structural properties such as explicit expressions for the quantile function, generating function, moments, order statistics etc. Assessing the behavior of MLE's, Monte Carlo simulation study is performed and potentiality of their suggested family was proved with the help of real data sets. The cumulative distribution function (CDF) and PDF of the TIIGTL family is given as

$$F(x;\alpha,\beta,\zeta) = 1 - \left\{1 - G(x;\zeta)^{2\beta}\right\}^{\alpha} ; \quad x > 0,\ \lambda, \beta > 0, \tag{3}$$

where α and β are the shape parameters and $G(x;\zeta)$ is the baseline CDF and it depends on ζ. The corresponding PDF can be written as

$$f(x;\alpha,\beta,\zeta) = 2\alpha\beta g(x;\zeta)\left[G(x;\zeta)\right]^{2\beta-1}\left[1 - G(x;\zeta)^{2\beta}\right]^{\alpha-1} ; \quad x > 0,\ \lambda, \beta > 0. \tag{4}$$

In this paper, we introduced and discussed a new four parameter distribution called type II generalized Topp Leone-power Lindley **(**TIIGTL-PLindley**)** distribution**.** This distribution is created based on the type II generalized Topp Leone family (Hassan et.al. 2019) and power Lindley distribution (Ghitani et al. 2013). And compared the performance with some already existing distributions such as exponentiated power Lindley, power Lindley, exponentiated Lindley distributions and their Topp Leone generalized versions.

1.3. Paper Organization

In Section 2, PDF, CDF, reliability function and hazard function of TIIGTL-PLindley distribution has been derived and their shapes have been exhibited for various values of the parameters. Section 3 discusses about probability weighted moments, moments, moment generating function, Renyi entropy, order statistics etc. In Section 4, maximum likelihood estimation method is performed for estimating the unknown parameters. Real life application is provided in Section 5 to illustrate the flexibility, goodness of fit and usefulness of the proposed TIIGTL-P Lindley distribution. Finally, conclusion is given in Section 6.

2. A New Generalization of Power Lindley Distribution

In this section, we have developed a new generalization of Lindley distribution named as TIIGTL-PLindley distribution and presents its probability density function (PDF), reliability function and hazard function. Also their shapes have been exhibited for selected values of the parameters.

The PDF and CDF of the proposed distribution is obtained by compounding the PDF and CDF of TIIGTL distribution with the density function and cumulative distribution function of power Lindley distribution. The CDF and PDF of TIIGTL family are obtained as

$$F_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)=1-\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{\alpha};x>0, \tag{5}$$

with parameters $\alpha,\beta,\theta,\lambda>0.$ Then we obtained the corresponding PDF as

$$\begin{aligned} f_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)=2\alpha\beta\lambda\frac{\theta^{2}}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta-1} \\ \times\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{\alpha-1};x>0. \end{aligned} \tag{6}$$

The survival function (SF) can be written as

$$S_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)=\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{\alpha};\ x>0. \tag{7}$$

The hazard rate function (HRF) is

$$\begin{aligned} H_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)=2\alpha\beta\lambda\frac{\theta^{2}}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta-1} \\ \times\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{-1};\ x>0. \end{aligned} \tag{8}$$

The reverse hazard rate is

$$R_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)=\left\{\frac{\begin{array}{r}2\alpha\beta\lambda\frac{\theta^2}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta-1}\\ \times\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{\alpha-1}\end{array}}{1-\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta}\right\}^{\alpha}}\right\};x>0. \quad (9)$$

Figures 1 to 5 shows the density function, distribution function reliability function, hazard rate function and reverse hazard function of the TIIGTL-P Lindley distribution for selected values of the parameters. Figure 1 shows that PDF of TIIGTL-P Lindley distribution gives decreasing, symmetric and right skewed shapes. Figure 2 shows the CDF of TIIGTL-P Lindley distribution. In Figure 3, SF of the TIIGTL-P Lindley distribution shows decreasing and bathtub shapes. Figure 4 shows HRF of TIIGTL-P Lindley distribution gives increasing and bathtub shapes. Figure 5 shows decreasing, reverse J, and bathtub shapes.

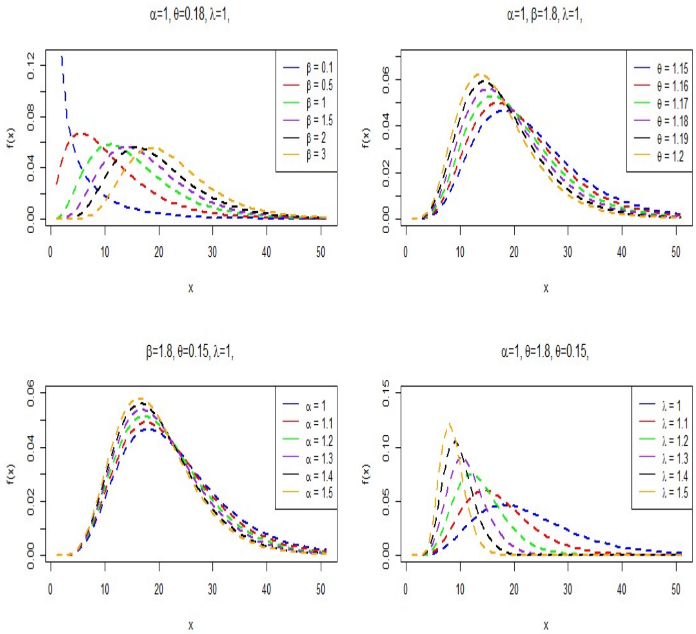

Figure 1 Plots of the PDF of TIIGTL-PLindley distribution for selected values of the parameters α,β,θ and λ

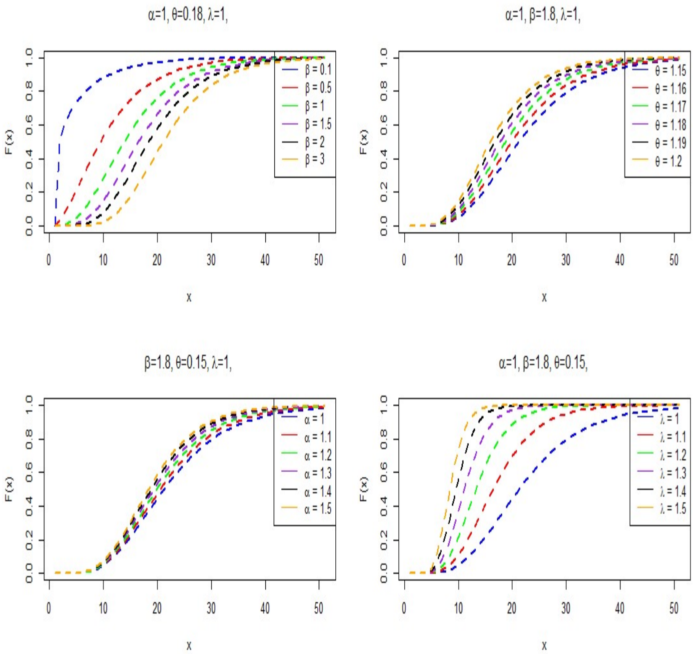

Figure 2 Plots of the CDF of TIIGTL-PLindley distribution for selected values of the parameters α, β, θ and λ

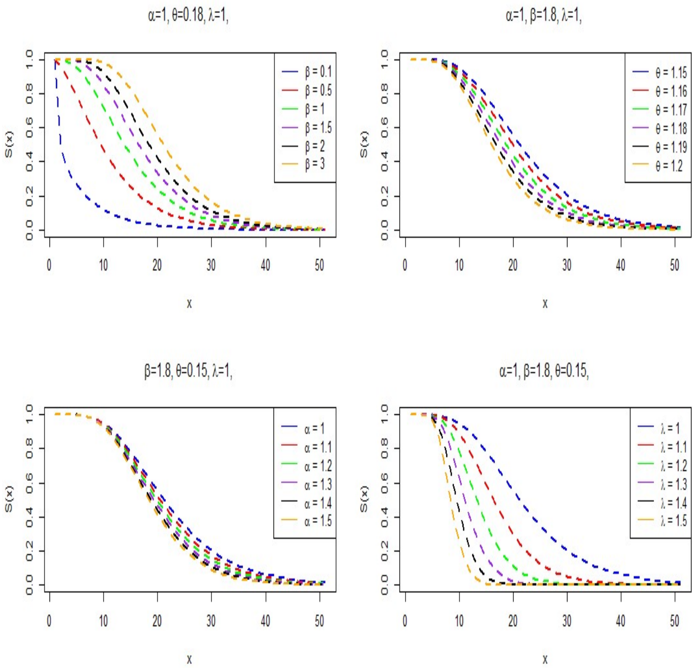

Figure 3 Plots of the SF of TIIGTL-PLindley distribution for selected values of the parameters α, β, θ and λ

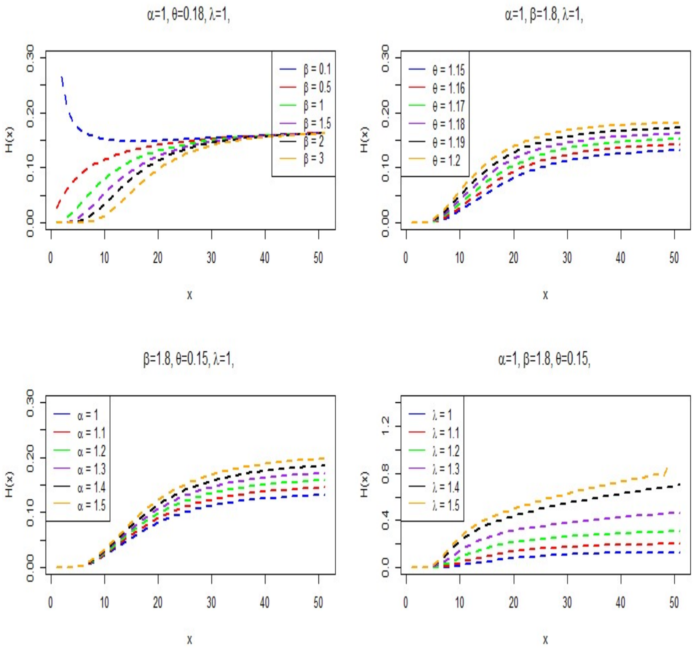

Figure 4 Plots of the HRF of TIIGTL-PLindley distribution for selected values of the parameters α, β, θ and λ

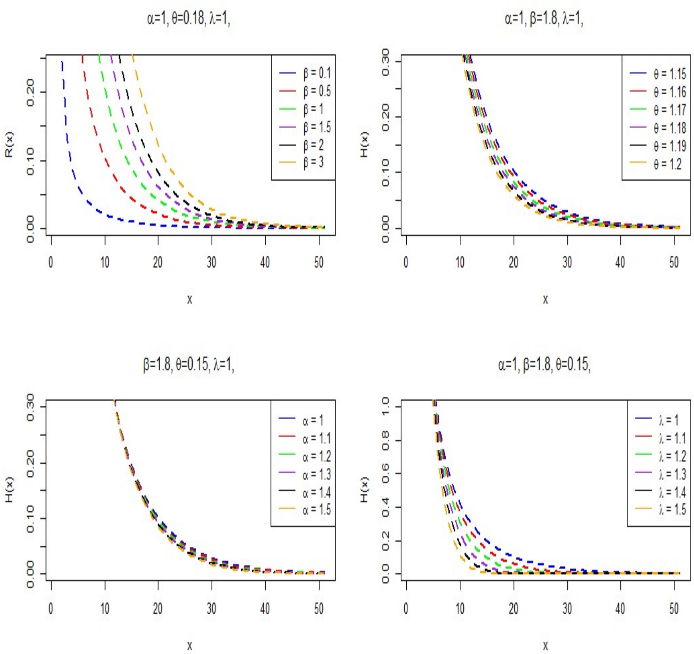

Figure 5 Plots of the RHRF of TIIGTL-PLindley distribution for selected values of the parameters α, β, θ and λ

Remark 1: Expansion of the density function

(i) Using the generalized binomial series expansion $(1-x)^n = \sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k$, we can write the density function of the proposed model as

$$f_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda) = \sum_{k=0}^{\infty} \varsigma_k \lambda \frac{\theta^2}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta(k+1)-1}, \quad (10)$$

where $\varsigma_k = 2\alpha\beta(-1)^k \binom{\alpha-1}{k}$.

(ii) Also, PDF of TIIGTL-PLindley distribution can be expressed as

$$f_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda) = \sum_{k=0}^{\infty} W_k h_{2\beta(k+1)}, \quad (11)$$

where $W_k = \frac{\varsigma_k}{2\beta(k+1)}$ and $h_b(x) = b\, g_{PL}(x) G_{PL}(x)^{a-1}$, $g_{PL}(x)$ and $G_{PL}(x)$ are the PDF and CDF of power Lindley distribution given in (1) and (2).

(iii) CDF can be expressed as

$$F_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda) = \sum_{z=0}^{\infty} S_z \left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta z}, \quad (12)$$

where $S_z = \sum_{i=0}^{h}(-1)^{i+z}\binom{h}{i}\binom{\alpha_i}{z}$.

3. Properties of TIIGTL-PLindley Distribution

In this section, we obtain the probability weighted moments, ordinary moments, incomplete moments, Renyi entropy and the PDF of the r^{th} order statistic.

3.1. Probability weighted moments

For a random variable X, the probability weighted moment is denoted by $\tau_{r,h}$ and is defined by

$$\tau_{r,h} = E[X^r F(x)^h] = \int_{-\infty}^{+\infty} x^r f(x) F(x)^h \, dx. \quad (13)$$

Substituting Equations (10) and (12) in Equation (13)

$$\tau_{r,h} = \int_{-\infty}^{+\infty} \sum_{k,z=0}^{\infty} S_z \varsigma_k x^r \lambda \frac{\theta^2}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta(k+1+z)-1} dx.$$

Then, $\tau_{r,h} = \sum_{k,z=0}^{\infty} S_z \varsigma_k \tau_{r,2\beta(k+z+1)-1}$ where

$$\tau_{r,2\beta(k+z+1)-1} = \int_{-\infty}^{\infty} x^r \lambda \frac{\theta^2}{\theta+1}\left(1+x^{\lambda}\right)x^{\lambda-1}e^{-\theta x^{\lambda}}\left[1-\left(1+\frac{\theta}{\theta+1}x^{\lambda}\right)e^{-\theta x^{\lambda}}\right]^{2\beta(k+1+z)-1} dx.$$

3.2. Moments

The r^{th} ordinary moment of X follows from Equation (10) as follows

$$\mu_r' = \int_{-\infty}^{+\infty} \sum_{k,z=0}^{\infty} \varsigma_k x^r \lambda \frac{\theta^2}{\theta+1}\left(1+x^\lambda\right)x^{\lambda-1}e^{-\theta x^\lambda}\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta(k+1)-1} dx, \tag{14}$$

where $\varsigma_k = 2\alpha\beta(-1)^k \binom{\alpha-1}{k}$ and another formula of r^{th} moment of X is given by

$$\mu_r' = \sum_{k,z=0}^{\infty} \varsigma_k \tau_{r,2\beta(k+1)-1}, \tag{15}$$

where $\tau_{r,2\beta(k+1)-1}$ is the probability weighted moment. And the moment generating function of TIIGTL-PLindley distribution can be written as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r = \sum_{k,r=0}^{\infty} \frac{t^r}{r!}\varsigma_k \tau_{r,2\beta(k+1)-1}. \tag{16}$$

3.3. Renyi entropy

It is a measure of variation or uncertainty of a random variable X. The Renyi entropy of X with PDF $f(x)$ is defined by

$$I_\varepsilon(X) = \frac{1}{1-\varepsilon}\log\int_{-\infty}^{+\infty} f(x)^\varepsilon dx, \ \varepsilon > 0, \varepsilon \neq 1.$$

While considering the generalized binomial theory in the PDF in Equation (10), then the PDF $f_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)^\varepsilon$ can be expressed as

$$f_{TIIGTL-PL}(x;\alpha,\beta,\theta,\lambda)^\varepsilon = \sum_{k=0}^{\infty} t_k \left[\lambda \frac{\theta^2}{\theta+1}\left(1+x^\lambda\right)x^{\lambda-1}e^{-\theta x^\lambda}\right]^\varepsilon \left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta(k+\varepsilon)-\varepsilon},$$

where $t_k = (2\alpha\beta)^\varepsilon(-1)^k \binom{\varepsilon(\alpha-1)}{k}$. Then, the Renyi entropy of TIIGTL-PLindley distribution is given by

$$I_\varepsilon(X) = \frac{1}{1-\varepsilon}\log\sum_{k=0}^{\infty} t_k \left[\lambda \frac{\theta^2}{\theta+1}\left(1+x^\lambda\right)x^{\lambda-1}e^{-\theta x^\lambda}\right]^\varepsilon \left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta(k+\varepsilon)-\varepsilon}. \tag{17}$$

3.4. Order statistics

Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with their corresponding distribution function $F(.)$. Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding ordered random sample from a population of size n. The PDF of the r^{th} order statistics is defined as

$$f_{r:n}(x) = \frac{f(x)}{B(r,n-r+1)}\sum_{\gamma=0}^{n-r}(-1)^\gamma \binom{n-r}{\gamma} F(x)^{\gamma+r-1}, \tag{18}$$

where $B(.,.)$ stands for the beta function. Inserting (10) and (12) in (18), replacing h with $\gamma - r + 1$, then the PDF of the order statistics for TIIGTL-PLindley distribution is defined by

$$f_{r:n}(x;\alpha,\beta,\theta,\lambda) = \frac{g(x;\theta,\lambda)}{B(r,n-r+1)}\sum_{\gamma=0}^{n-r}\sum_{k,z=0}^{\infty} \varsigma_k P_{z,y} G(x;\theta)^{2\beta(k+z+1)-1}$$

$$f_{r:n}(x;\alpha,\beta,\theta,\lambda)=\frac{\lambda\frac{\theta^2}{\theta+1}\left(1+x^\lambda\right)x^{\lambda-1}e^{-\theta x^\lambda}}{B(r,n-r+1)}\sum_{\gamma=0}^{n-r}\sum_{k,z=0}^{\infty}\varsigma_k P_{z,y}\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta(k+z+1)-1}, \quad (19)$$

where $P_{z,y}=(-1)^\gamma\binom{n-r}{\gamma}S_z$. Further the k^{th} moment of r^{th} order statistics for TIIGTL-PLindley distribution is defined by

$$E(X_{r:n}^k)=\int_{-\infty}^{+\infty}x^k f_{r:n}(x;\alpha,\beta,\theta,\lambda)dx=\frac{1}{B(r,n-r+1)}\sum_{\gamma=0}^{n-r}\sum_{k,z=0}^{\infty}\varsigma_k P_{z,y}\,\varsigma_k\tau_{r,2\beta(k+z+1)-1}. \quad (20)$$

4. Parameter Estimation of TIIGTL-PLindley Distribution

Let $X_1,X_2,...,X_n$ be the observed values from the TIIGTL-PLindley distribution with set of parameters $\phi=(\alpha,\beta,\theta,\lambda)'$, then the likelihood function of ϕ is given by

$$\ln L(\phi)=\left\{\begin{array}{l} n\log 2+n\log\alpha+n\log\beta+\sum_{i=1}^{n}\log\left[\lambda\frac{\theta^2}{\theta+1}\left(1+x^\lambda\right)x^{\lambda-1}e^{-\theta x^\lambda}\right]\\ +\left(2\beta-1\right)\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]\\ +\left(\alpha-1\right)\sum_{i=1}^{n}\log\left\{\left[1-\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta}\right]\right\}\end{array}\right\}. \quad (21)$$

The elements of the score function $U(\phi)=(U_\alpha,U_\beta,U_\theta,U_\lambda)$ are given by

$$U_\alpha=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta}\right\}$$

$$U_\beta=\left\{\begin{array}{l}\frac{n}{\beta}+2\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]\\ \quad -2\left(\alpha-1\right)\sum_{i=1}^{n}\frac{\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta}\times\log\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]}{1-\left[1-\left(1+\frac{\theta}{\theta+1}x^\lambda\right)e^{-\theta x^\lambda}\right]^{2\beta}}\end{array}\right\}. \quad (22)$$

$$U_\theta = \left\{ \begin{array}{l} \sum_{i=1}^{n} \frac{\left(\theta + 2 - x\theta - x\theta^2\right)}{\theta(\theta+1)} \\ -2(\alpha-1)\beta \sum_{i=1}^{n} \left\{ \frac{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta-1} \left[x^\lambda e^{-\theta x^\lambda} \left(\begin{array}{l} \theta^2 (x^\lambda + 1) \\ +\theta(x^\lambda + 2) \end{array} \right) \right]}{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta}} \right\} \\ -(2\beta-1) \sum_{i=1}^{n} \left\{ \frac{\left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta-1} \left[x^\lambda e^{-\theta x^\lambda} \left(\theta^2 (x^\lambda + 1) + \theta(x^\lambda + 2) \right) \right]}{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta}} \right\} \\ -2(\alpha-1)\beta \sum_{i=1}^{n} \left\{ \frac{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta-1} \left[x^\lambda e^{-\theta x^\lambda} \left(\theta^2 (x^\lambda + 1) + \theta(x^\lambda + 2) \right) \right]}{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta}} \right\} \end{array} \right\}. \qquad (23)$$

$$U_\lambda = \left\{ \begin{array}{l} \sum_{i=1}^{n} \log x \left(1 + \theta x^\lambda\right) + (2\beta - 1) \sum_{i=1}^{n} \left\{ \frac{\frac{\theta^2}{\theta+1} x^\lambda \left(x^\lambda + 1\right) \log x \, e^{-\theta x^\lambda}}{\left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]} \right\} \\ +2(\alpha-1)\beta \sum_{i=1}^{n} \left\{ \frac{\left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta-1} \left[\frac{\theta^2}{\theta+1} \left(x^\lambda + 1\right) x^\lambda \log x \, e^{-\theta x^\lambda} \right]}{1 - \left[1 - \left(1 + \frac{\theta}{\theta+1} x^\lambda\right) e^{-\theta x^\lambda}\right]^{2\beta}} \right\} \end{array} \right\}. \qquad (24)$$

Maximum likelihood estimators can be obtained by solving Equations (21)-(24) simultaneously by setting $U_\alpha, U_\beta, U_\theta$ and U_λ equal to zero. Since it is difficult to solve these equations analytically, it can be solved numerically using Newton-Raphson method in R software.

5. Applications

In this section, we considered two practical datasets for proving the effectiveness of TIIGTL-PLindley distribution. First data set has been obtained from Dumonceaux and Antle (1973). The data set consists of 20 observations concerning the maximum flood level data, which is given in Table 1 and Figure 6.

Table 1 Observations concerning the maximum flood level data

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265

We present a comparison study to evaluate the goodness of fit of the proposed distribution with the existing Lindley distribution and its generalizations. While considering the results in Table 2, it clearly shows that TIIGTL-PLindley is strong competitor for the existing generalized Lindley distributions and Lindley distribution.

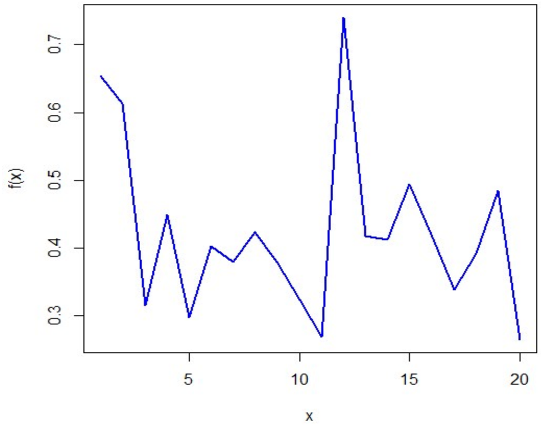

Figure 6 Maximum flood level data

Table 2 Values of parameter estimates and statistical criteria concerning maximum flood level data

Distribution	Parameter estimate values				Statistical values				
	α	β	λ	θ	–Log	AIC	BIC	KS	p-value
TIIGTL-PL	1.922	1.645	1.436	0.092	46.5583	101.117	107.872	0.0825	0.8732
TL-EPL	9.1321	7.0096	0.097	300.538	89.6808	187.362	194.117	0.0950	0.8630
TL-PL	2,755.4	4.3244	0.139	-	89.5073	185.015	190.081	0.0955	0.8592
TL-GL	1025.14	0.1403	-	0.0109	94.0351	194.070	199.137	0.1525	0.3097
TL-L	0.6692	0.2074	-	-	98.9111	98.911	201.822	205.20	0.1678
EPL	-	8.9099	0.129	6196.21	89.4702	89.470	184.940	190.007	0.0959
PL	-	0.5867	0.799	-	95.9425	95.943	195.885	199.263	0.1346
GL	-	0.3588	-	0.7460	97.9109	97.911	199.822	203.199	0.166
L	-	0.4242	-	-	98.7913	98.791	199.583	201.272	0.2157

Table 3 Active repair time data

0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.2, 22.0, 24.5

In the second example, we analyze a real data set which was taken from the paper Jorgensen (1982) shows the active repair times (h) for an airborne communication transceiver. The data is given in Table 3 and exhibits in Figure 7. While analyzing the result obtained for various distributions based on this data set, outcome shows that the performance of the proposed TIIGTL-PLindley distribution is superior compared to the existing models.

Table 4 Parameter estimates and statistical criteria of the active repair time data

Distribution	Parameter estimate values				Statistical values				
	α	β	λ	θ	–Log	AIC	BIC	KS	p-value
TIIGTL-PL	0.480	153.311	-	17.956	5.831	19.663	23.646	0.173	0.941
TL-EPL	4.580	7.732	0.664	21.080	16.147	40.294	47.05	0.117	0.947
TL-PL	547.989	6.770	0.647	-	16.299	38.598	43.664	0.125	0.915
TL-GL	109.269	6.008	-	0.641	16.173	38.345	43.412	0.117	0.947
TL-L	53.071	6.142	-	-	16.146	36.292	39.670	0.123	0.924
EPL	-	13.27	0.6129	847.276	16.306	38.612	43.678	0.128	0.897
PL	-	15.288	3.521	-	13.249	30.497	33.875	0.199	0.406
GL	-	11.682	-	54.989	16.155	36.310	39.687	0.122	0.926
L	-	2.960	-	-	2.179	6.358	8.047	0.453	0.001

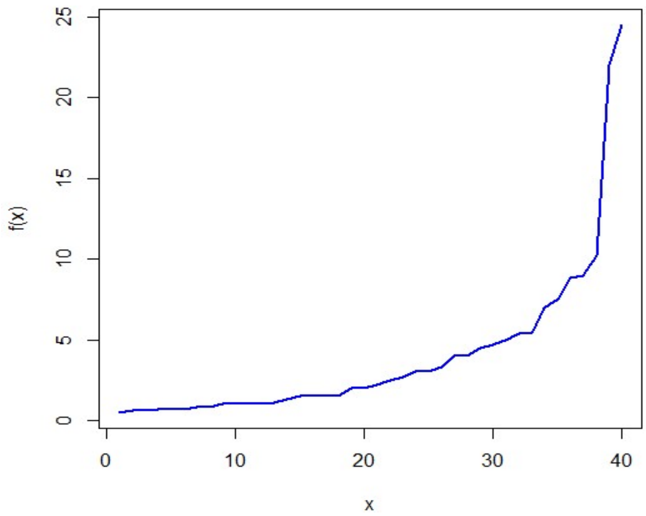

Figure 7 Active repair time data

6. Conclusion

A new generalization of Lindley distribution has been proposed in this paper. The new distribution is obtained by considering a type II Topp Leone family known as TIIGTL family. Main statistical properties of the proposed model such as the reliability function, hazard rate function, probability weighted moments, moments, moment generating function, Renyi entropy, order statistics etc. are determined. MLE is used for carrying out the estimation of parameters of the new distribution. The potentiality of the new distribution is evaluated using two real life data sets.

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