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An Extension of Exponentiated Rayleigh Distribution: Properties and Applications

Ajaz ahmad Bhat* and Sheikh Parvaiz Ahmad
Department of Statistics, University of Kashmir, Srinagar, India.
*Corresponding author; e-mail: bhatajaz.msst@gmail.com

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Abstract

This research article deals with the extension of exponentiated Rayleigh distribution introduced by Surles and Padgett (2001) by considering the power transformation of a random variable. The new distribution is referred to as power exponentiated Rayleigh distribution. We provide a comprehensive description of the statistical properties including ordinary and incomplete moments, mean residual life, mean deviations, information measures and order statistics of the subject distribution. The estimation of the unknown parameters of the model is performed by the method of maximum likelihood estimation. Finally, the usefulness of the proposed model among other models is illustrated by means of real life and simulated data sets using some goodness-of-fit measures.

Keywords: Exponentiated Rayleigh distribution, power transformation, mean deviations, mean residual life, simulation, estimation and data analysis.

1. Introduction

The Rayleigh distribution (Rayleigh 1880) is a well-known probability model of prominent importance and is used to study the problems in the field of communication theory, physical science, medical image analysis and survival analysis. It is a special case from two parameter Weibull distribution when the shape parameter is equal to 2. The important characteristic of Rayleigh distribution is that its hazard rate is an increasing function of time. Due to its significant applicability in the real life problems, numerous researchers have contributed to this model, among them are Siddiqui (1962) studied the genesis and origin of this model. Howlader and Hossain (1995) studied the Bayesian estimation of the Rayleigh distribution based on type-II censored data. Lalitha and Mishra (1996) discussed the modified maximum likelihood estimation for Rayleigh distribution. Abd Elfattah et al. (2006) discussed the efficiency of maximum likelihood estimators under different censored sampling schemes for Rayleigh distribution. Dev and Tanujit (2011) studied the Bayesian estimation of the scale parameter. Ardianti (2018) used classical and Bayesian methods to estimate the parameter of Rayleigh distribution. The probability density function (PDF) and cumulative distribution function (CDF) associated with random variable X from Rayleigh distribution with scale parameter θ is expressed as

$$f(x;\theta)=2\theta x\exp\left(-\theta x^{2}\right) \text{ and } F(x;\theta)=1-\exp\left(-\theta x^{2}\right).$$

In recent years, several extensions of Rayleigh distribution have been made to increase its applicability in medical science, physical analysis and survival analysis. Numerous authors have worked on the generalization of Rayleigh distribution; among them are Voda (2007), Kundu and Raqab (2005), Raqab and Madi (2009), Merovci (2013), Dey et al. (2014), Merovci and Elbatal (2015), Ahmad et al. (2015), Saima et al. (2016), Ateeq et al. (2019), Sofi et al. (2019). Surles and Padgett (2001) proposed the two parameter Burr type X distribution and named it as exponentiated Rayleigh distribution or generalized Rayleigh distribution with PDF and CDF, respectively given as

$$f\left(x;\beta,\theta\right)=2\beta\theta x\exp\left(-\theta x^{2}\right)\left[1-\exp\left(-\theta x^{2}\right)\right]^{\beta-1}, \quad (1)$$

$$F\left(x;\beta,\theta\right)=\left[1-\exp\left(-\theta x^{2}\right)\right]^{\beta}.$$

The main aim of this research paper is to enhance the flexibility of the model by inducing one extra shape parameter for improving its goodness-of-fit to real data. The advantage of the proposed model over exponentiated Rayleigh distribution is that the latter cannot model lifetime phenomenon showing various shapes of failure rates. We provide the comprehensive description of the mathematical properties of the powered distribution and estimate its parameters using maximum likelihood method. We will also provide the possible areas of applications.

2. Power Exponentiated Rayleigh Distribution

The section is devoted to construct the PDF and CDF of power exponentiated Rayleigh (PER) distribution by taking the power transformation $V=X^{\frac{1}{\alpha}}$ where X follows exponentiated Rayleigh (ER) distribution with PDF given in (1). The PDF of the PER distribution with parameters α,β and θ and $v>0$ is given as

$$f\left(v;\alpha,\beta,\theta\right)=2\alpha\beta\theta v^{2\alpha-1}\exp\left(-\theta v^{2\alpha}\right)\left(1-\exp\left(-\theta v^{2\alpha}\right)\right)^{\beta-1}, \quad (2)$$

where $\alpha,\beta>0$ are the two shape parameters and $\theta>0$ is the scale parameter. The CDF corresponding to (2) is given as

$$F\left(v;\alpha,\beta,\theta\right)=\left(1-\exp\left(-\theta v^{2\alpha}\right)\right)^{\beta}. \quad (3)$$

The proposed model is very flexible in nature that approaches to different models when its parameters are changed. The flexibility of the proposed model is elucidated in Table 1 where it has 5 sub-models when the value of parameters is chosen carefully.

Table 1 Sub-models of the PER distribution

Distribution	α	β	θ	PDF
Exponentiated Rayleigh (ERD)	1	β	θ	$2\beta\theta v\exp(-\theta v^{2})\left[1-\exp(-\theta v^{2})\right]^{\beta-1}$
Rayleigh (RD)	1	1	θ	$2\theta v\exp(-\theta v^{2})$
Weibull (WD)	$\delta/2$	1	θ	$\delta\theta v^{\delta-1}\exp(-\theta v^{\delta})$
Generalized Exponential (GED)	0.5	β	θ	$\beta\theta\exp\left(-\theta v\right)\left[1-\exp\left(-\theta v\right)\right]^{\beta-1}$
Exponential (ED)	0.5	1	θ	$\theta\exp(-\theta v)$

Figure 1 provides plots of the PDF curves for some values of the shape and scale parameters of the PER distribution.

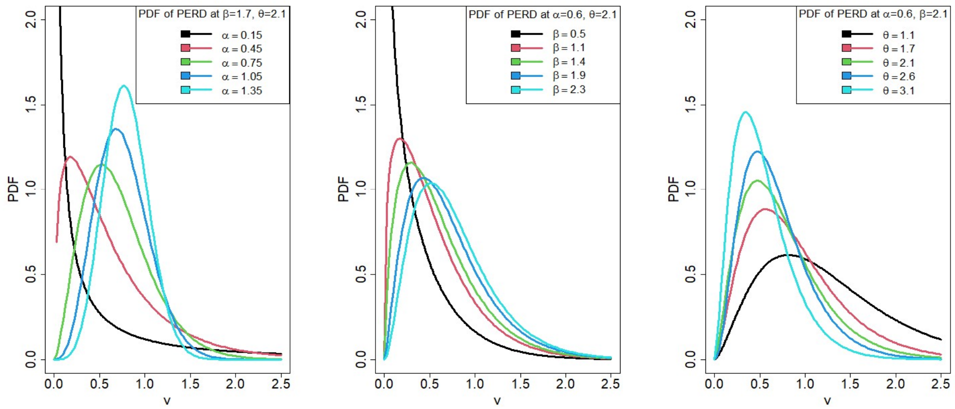

Figure 1 PDF plots of $PERD(\alpha,\beta,\theta)$ for varying values of shape and scale parameters

It is clearly from the plots of the PDF curves that the proposed model is unimodal, decreasing and right skewed in nature.

3. Reliability Analysis

In this section, the expression for the reliability function R(v), hazard rate function h(v), reverse hazard rate function r(v), cumulative hazard rate function H(v), mills ratio (M.R) are acquired and are respectively given as

$$R(v)=1-\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta},\quad h(v)=\frac{2\alpha\beta\theta v^{2\alpha-1}\exp\left(-\theta v^{2\alpha}\right)\left(1-\exp\left(-\theta v^{2\alpha}\right)\right)^{\beta-1}}{1-\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta}},$$

$$r(v)=\frac{2\alpha\beta\theta v^{2\alpha-1}\exp\left(-\theta v^{2\alpha}\right)}{\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]},\quad H(v)=-\ln\left[1-\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta}\right],$$

$$M.R.=\frac{\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta}}{1-\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta}}.$$

From Figure 2, we conclude that the hazard rate of PER distribution can be decreasing, bathtub, Increasing, constant and j-shaped. The advantage of PER distribution over ER distribution is that the latter cannot model lifetime phenomenon exhibiting various shapes of failure rates such as J-shaped and bathtub.

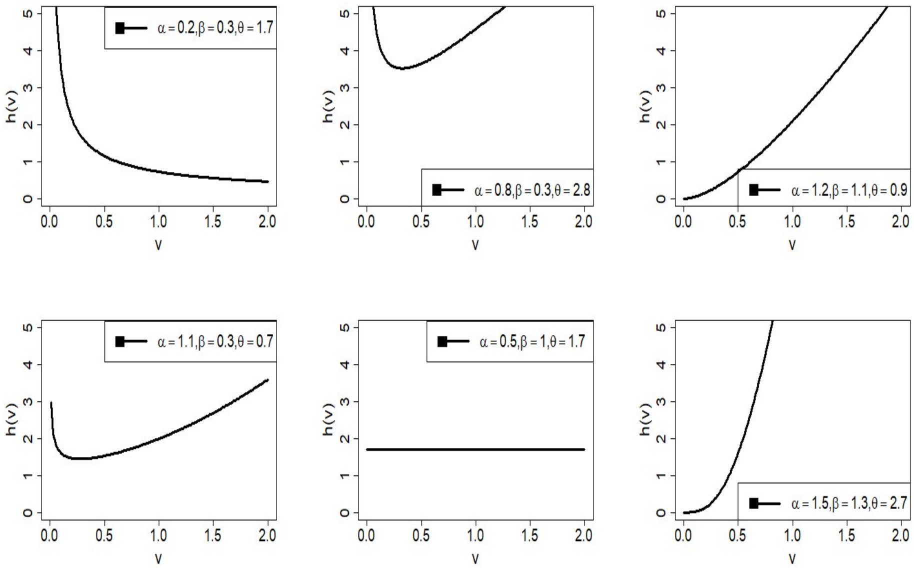

Figure 2 Hazard Rate Function plots of $PERD(\alpha,\beta,\theta)$ for varying values of shape and scale parameters

Useful Expansion

The mixture representation of PDF and CDF of the PER distribution is presented by using the series expansion

$$(1-d)^{c-1}=\sum_{k=0}^{\infty}a_k d^k, |d|\langle 1 \text{and} k\rangle 0, \quad (1-d)^{c}=\sum_{j=0}^{\infty}b_j d^j, |d|\langle 1 \text{and}\ j\rangle 0,$$

where

$$a_k=\frac{(-1)^k\,\Gamma(c)}{\Gamma(c-k)\Gamma(k+1)} \quad \text{and} \quad b_j=\frac{(-1)^j\,\Gamma(c+1)}{\Gamma(c-j+1)\Gamma(j+1)}. \tag{4}$$

Thus, the PDF of PER distribution in (2) can be expressed in the mixture form as

$$f(v)=2\alpha\beta\theta v^{2\alpha-1}\sum_{k=0}^{\infty}a_k \exp\left[-\theta v^{2\alpha}(k+1)\right]. \tag{5}$$

The CDF in (3) can be expressed in the mixture form as

$$F(v)=\sum_{j=0}^{\infty}b_j \exp\left(-j\theta v^{2\alpha}\right). \tag{6}$$

4. Some Properties of PER distribution

This section presents some of the mathematical and statistical properties of the PER distribution including quantile function, moments, mean deviations, mean residual life, mean waiting time and generating functions.

4.1. Quantile function and simulation

The quantile function for any distribution defined by Hyndman and Fan (1996) can be written in the form of

$$Q(u) = V_q = F^{-1}(u).$$

where $Q(u)$ is the quantile function of a random variable V for the given distribution function $F(v)$ and u is the uniform random variable defined on the unit interval $0 < u < 1$. Inverting the distribution function of the PER distribution given in (3) as defined above will give the quantile function as follows

$$Q(u) = \left[-\frac{1}{\theta} \log\left(1 - u^{\frac{1}{\beta}} \right) \right]^{\frac{1}{2\alpha}}. \tag{7}$$

This above equation is very helpful to obtain moments like skewness and kurtosis as well as the median and can be used for generation of random variables of the PER distribution. The median of the distribution is obtained by setting $u = 0.5$ in (7) as

$$Median = \left[-\frac{1}{\theta} \log\left(1 - (0.5)^{\frac{1}{\beta}} \right) \right]^{\frac{1}{2\alpha}}.$$

Accordingly, the random samples can be simulated from PER distribution by setting $Q(u) = V$ and this procedure is called inverse transformation method of simulation. Thus

$$V = \left[-\frac{1}{\theta} \log\left(1 - u^{\frac{1}{\beta}} \right) \right]^{\frac{1}{2\alpha}}.$$

Bowley's measure of skewness based on quartiles is given as

$$B = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)}.$$

Similarly, the Moors measure of kurtosis based on octiles is given as

$$M = \frac{Q\left(3/8\right) - Q\left(1/8\right) + Q\left(7/8\right) - Q\left(5/8\right)}{Q\left(6/8\right) - Q\left(2/8\right)}.$$

where $Q(.)$ is given in (7).

4.2. Ordinary moments

Moments are the constants of a population and help to determine the various properties of the probability model such as mean, variance, skewness and kurtosis. Suppose V is a continuous random variable, then the s^{th} ordinary moment about origin (non-central moment) of V is defined as

$$\mu_s' = E\left(V^s\right) = \int_0^\infty v^s f(v)\, dv, \tag{8}$$

where $f(v)$ is the PDF of the PER distribution and is stated from (5) as

$$f(v) = 2\alpha\beta\theta v^{2\alpha-1} \sum_{k=0}^{\infty} a_k \exp\left[-\theta v^{2\alpha} (k+1) \right]. \tag{9}$$

where a_k is defined in (4). Inserting (9) in (8), we obtain

$$\mu_s' = 2\alpha\beta\theta\sum_{k=0}^{\infty} a_k \int_0^{\infty} v^{s+2\alpha-1}\exp\left(-\theta v^{2\alpha}(k+1)\right)dv. \tag{10}$$

Using integration by substitution method in Equation (10) leads to the following operations: Let

$$\theta v^{2\alpha}(k+1) = u \Rightarrow v = \left[\frac{u}{\theta(k+1)}\right]^{\frac{1}{2\alpha}},$$

which implies that $dv = \dfrac{u^{\frac{1}{2\alpha}-1}}{2\alpha\left[\theta(k+1)\right]^{\frac{1}{2\alpha}}}du.$ Substituting for v, u and dv in (10) and simplifying; we have

$$\mu_s' = \frac{\beta\Gamma\left(\frac{s}{2\alpha}+1\right)}{\theta^{\frac{s}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k}{(k+1)^{\frac{s}{2\alpha}+1}},$$

where $\Gamma(.)$ is the gamma function. In particular, the mean and variance of PER distribution are respectively, obtained as

$$\mu_1' = \frac{\beta\Gamma\left(\frac{1}{2\alpha}+1\right)}{\theta^{\frac{1}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k}{(k+1)^{\frac{1}{2\alpha}+1}},$$

$$\mu_2 = \sigma^2 = \frac{\beta}{\theta^{\frac{1}{\alpha}}}\left[\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{k=0}^{\infty}\frac{a_k}{(k+1)^{\frac{1}{\alpha}+1}} - \beta\left(\Gamma\left(\frac{1}{2\alpha}+1\right)\sum_{k=0}^{\infty}\frac{a_k}{(k+1)^{\frac{1}{2\alpha}+1}}\right)^2\right].$$

The classical measures of skewness and kurtosis based on non-central moments can be calculated by using the following expressions

$$\gamma_1 = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{(\mu_2)^{3/2}} \text{ and } \gamma_2 = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{(\mu_2)^2} - 3.$$

4.3. Incomplete moments

The s^{th} incomplete moment, denoted by $\varphi_s(v)$, of V is mathematically defined as

$$\varphi_s(v) = \int_0^v v^s f(v)dv. \tag{11}$$

Inserting (5) in (11) and using lower incomplete gamma function, we obtain

$$\varphi_s(v) = \frac{\beta}{\theta^{\frac{s}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{s}{2\alpha}+1\right),\theta(k+1)v^{2\alpha}\right]}{(k+1)^{\frac{s}{2\alpha}+1}}, \tag{12}$$

where $\gamma(a,b)$ is the lower incomplete gamma function. By setting $s=1$, the first incomplete moment of V can be obtained as

$$\varphi_1(v)=\frac{\beta}{\theta^{\frac{1}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)v^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}. \tag{13}$$

The first incomplete moment can be used to construct the Lorenz and Bonferroni inequality curves as well as the mean deviation from mean and mean deviation from median. The Lorenz curve (LC) for PER distribution calculated as

$$LC=\frac{\varphi_1(v)}{\mu}=\frac{\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)v^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}}{\sum_{k=0}^{\infty}\frac{a_k\Gamma\left(\frac{1}{2\alpha}+1\right)}{(k+1)^{\frac{1}{2\alpha}+1}}}.$$

Similarly, the expression for Bonferroni curve (BC) of PER distribution is obtained as

$$BC=\frac{LC}{F(v;\alpha,\beta,\theta)}=\frac{\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)v^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}}{\sum_{j=0}^{\infty}b_j exp\left(-j\theta v^{2\alpha}\right)\sum_{k=0}^{\infty}\frac{a_k\Gamma\left(\frac{1}{2\alpha}+1\right)}{(k+1)^{\frac{1}{2\alpha}+1}}}.$$

4.4. Mean deviations

The mean deviation about mean, denoted by $D(\mu)$ is defined by

$$D(\mu)=2\mu F(\mu)-2\varphi_1(\mu).$$

where μ is the mean of the distribution, $F(\mu)$ is the distribution function and $\varphi_1(\mu)$ is the first incomplete moment defined in (13).

Using the value of μ, $F(\mu)$ and $\varphi_1(\mu)$ in the above expression, we obtain the required expression of mean deviation about mean for PER distribution as

$$D(\mu)=\frac{2\beta}{\theta^{\frac{1}{2\alpha}}}\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_k b_j\Gamma\left(\frac{1}{2\alpha}+1\right)}{(k+1)^{\frac{1}{2\alpha}+1}}exp\left(-j\theta\mu^{2\alpha}\right)-\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)\mu^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}\right].$$

The mean deviation about median, denoted by $D(M)$ is defined by

$$D(M)=\mu-2\varphi_1(M).$$

Using the value of μ and $\varphi_1(M)$ in the above expression, we obtain the required result as

$$D(M)=\frac{\beta}{\theta^{\frac{1}{2\alpha}}}\left[\sum_{k=0}^{\infty}\frac{a_k\Gamma\left(\frac{1}{2\alpha}+1\right)}{(k+1)^{\frac{1}{2\alpha}+1}}-2\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)M^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}\right].$$

4.5. Conditional moments

For the prediction in lifetime models, the conditional moments are very important. Suppose V is a random variable from PER distribution, then the s^{th} conditional moment is given by

$$E\left(V^s \left| V \right\rangle w\right)=\frac{1}{R(w)}\int_w^{\infty}v^s f(v)dv,$$

where $R(w)$ is the reliability function. Thus, the required expression for the conditional moments of the PER distribution is obtained as

$$E\left(V^s \left| V \right\rangle w\right)=\frac{\beta}{\theta^{\frac{s}{2\alpha}}\left[1-\sum_{j=0}^{\infty}b_j exp\left(-j\theta w^{2\alpha}\right)\right]}\sum_{k=0}^{\infty}\frac{a_k\Gamma\left[\left(\frac{s}{2\alpha}+1\right),\theta(k+1)w^{2\alpha}\right]}{(k+1)^{\frac{s}{2\alpha}+1}}.$$

Similarly, the s^{th} reversed conditional moment is given by

$$E\left(V^s \mid V\le w\right)=\frac{1}{F(w)}\int_0^{w}v^s f(v)dv,$$

where $F(w)$ is the distribution function. Thus, the s^{th} reversed conditional moment of PER distribution is obtained as

$$E\left(V^s \mid V\le w\right)=\frac{\beta}{\theta^{\frac{s}{2\alpha}}\sum_{j=0}^{\infty}b_j exp\left(-j\theta w^{2\alpha}\right)}\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{s}{2\alpha}+1\right),\theta(k+1)w^{2\alpha}\right]}{(k+1)^{\frac{s}{2\alpha}+1}}.$$

4.6. Probability weighted moments (PWM)

The general formula for probability weighted moments (PWM) is defined by

$$PWM=\int_0^{\infty}v^s f(v)\left[F(v)\right]^t dv.$$

Using (2) and (3), we have

$$PWM=2\alpha\beta\theta\int_0^{\infty}v^{s+2\alpha-1}exp\left(-\theta v^{2\alpha}\right)\left(1-exp\left(-\theta v^{2\alpha}\right)\right)^{\beta t+\beta-1}dv.$$

Solving the integral, we obtain the required expression as

$$PWM=\frac{\beta}{\theta^{\frac{s}{2\alpha}}}\sum_{k=0}^{\infty}\frac{w_k\Gamma\left(\frac{s}{2\alpha}+1\right)}{(k+1)^{\frac{s}{2\alpha}+1}}.$$

4.7. Residual and reversed residual life functions

This sub-section presents some of the statistical properties related to residual and reversed residual life functions including mean related to residual and reversed residual life functions of the PER distribution.

4.7.1. Residual life function

The conditional random variable $R_{(t)} = \left(V - t \middle| V \rangle t\right)\ ; t \geq 0$ is used to explain the residual life of a lifetime component and interpreted as the period from time t until the time of failure. The survival function of residual lifetime $R_{(t)}$ of PER distribution is given by

$$S_{R_{(t)}}(v) = \frac{S(v+t)}{S(t)} = \frac{1-\left[1-\exp\left(-\theta(v+t)^{2\alpha}\right)\right]^{\beta}}{1-\left[1-\exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}, v > 0.$$

The corresponding PDF of $R_{(t)}$ will be

$$f_{R_{(t)}}(v) = \frac{2\alpha\beta\theta(v+t)^{2\alpha-1}\exp\left[-\theta(v+t)^{2\alpha}\right]\left[1-\exp\left(-\theta(v+t)^{2\alpha}\right)\right]^{\beta-1}}{1-\left[1-\exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}.$$

Consequently, the hazard rate function of $R_{(t)}$ is given by

$$h_{R_{(t)}}(v) = \frac{2\alpha\beta\theta(v+t)^{2\alpha-1}\exp\left[-\theta(v+t)^{2\alpha}\right]\left[1-\exp\left(-\theta(v+t)^{2\alpha}\right)\right]^{\beta-1}}{1-\left[1-\exp\left(-\theta(v+t)^{2\alpha}\right)\right]^{\beta}}.$$

4.7.2. Mean residual life (MRL) function

The MRL function is defined as the expected life of an item to survive after the age t. It provides information about the whole interval in which the item will survive is to be believed. The MRL function denoted by $\pi_1(t) = E\left(V - t \middle| V \rangle t\right)$, is given by

$$\pi_1(t) = \left[R(t)\right]^{-1}\int_t^{\infty} vf(v)\,dv - t = \frac{1}{1-\left[1-exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}\left[2\alpha\beta\theta\sum_{k=0}^{\infty} a_k \int_t^{\infty} v^{2\alpha}\exp\left[-\theta v^{2\alpha}(k+1)\right]dv\right] - t.$$

Solving the integral yields, the result as

$$\pi_1(t) = \frac{1}{1-\left[1-exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}\left[\frac{\beta}{\theta^{\frac{1}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k}{(k+1)^{\frac{1}{2\alpha}+1}}\Gamma\left[\left(\frac{1}{2\alpha}+1\right),(k+1)\theta t^{2\alpha}\right]\right] - t.$$

4.7.3. Reversed residual life function

The conditional random variable $\overline{R}_{(t)} = \left(t - V \mid V \leq t\right); t \geq 0$ is used to explain the reversed residual life of a lifetime component and is interpreted as the time elapsed from the failure of an item given that its life $\leq t$. The survival function of reversed residual lifetime $\overline{R}_{(t)}$ of PER distribution is given as

$$S_{\bar{R}_{(t)}}(v)=\frac{F(t-v)}{F(t)}=\frac{\left[1-\exp\left(-\theta(t-v)^{2\alpha}\right)\right]^{\beta}}{\left[1-\exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}.$$

The corresponding PDF of $\bar{R}_{(t)}$ will be

$$f_{\bar{R}_{(t)}}(v)=\frac{2\alpha\beta\theta(t-v)^{2\alpha-1}\exp\left[-\theta(t-v)^{2\alpha}\right]\left[1-\exp\left(-\theta(t-v)^{2\alpha}\right)\right]^{\beta-1}}{\left[1-\exp\left(-\theta t^{2\alpha}\right)\right]^{\beta}}.$$

Accordingly, the associated failure rate of $\bar{R}_{(t)}$; $t\geq 0$ is given as

$$h_{\bar{R}_{(t)}}(v)=\frac{2\alpha\beta\theta(t-v)^{2\alpha-1}\, exp\left[-\theta(t-v)^{2\alpha}\right]\left[1-exp\left(-\theta(t-v)^{2\alpha}\right)\right]^{\beta-1}}{\left[1-exp\left(-\theta(t-v)^{2\alpha}\right)\right]^{\beta}}.$$

4.7.4. Mean reversed residual life (MRRL) function

The MRRL function, also known as mean past lifetime, is very useful to presage the actual time of failure of an already failed component. The MRRL function denoted by $\pi_2(t)=E(t-V\mid V\leq t)$, is given by

$$\pi_2(t)=t-\left[F(t)\right]^{-1}\int_0^t vf(v)\,dv,$$

Inserting (2) and (3) and solving the integral yields the result as

$$\pi_2(t)=t-\left[1-exp\left(-\theta t^{2\alpha}\right)\right]^{-\beta}\left[\frac{\beta}{\theta^{\frac{1}{2\alpha}}}\sum_{k=0}^{\infty}\frac{a_k\gamma\left[\left(\frac{1}{2\alpha}+1\right),\theta(k+1)t^{2\alpha}\right]}{(k+1)^{\frac{1}{2\alpha}+1}}\right].$$

4.8. Generating functions

For PER distribution, by the definition of moment generating function of V, we have

$$M_V(t)=E\left(e^{tV}\right)=\int_0^{\infty}e^{tV}f(v)\,dv.$$

Using power series expansion,

$$e^{tV}=\sum_{s=0}^{\infty}\frac{t^s}{\Gamma(s+1)}v^s \quad\text{and}\quad M_V(t)=\sum_{s=0}^{\infty}\frac{t^s}{\Gamma(s+1)}\int_0^{\infty}v^s f(v)\,dv.$$

Inserting (5) in the above expression, we obtain the required result as

$$M_V(t)=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^s}{\Gamma(s+1)}\frac{\beta\Gamma\left(\frac{s}{2\alpha}+1\right)}{\theta^{\frac{s}{2\alpha}}}\frac{a_k}{(k+1)^{\frac{s}{2\alpha}+1}}.$$

Using the well- known relation $\phi_V(t)=M_V(it)$, we obtain the characteristic function of PER distribution as

$$\phi_V(t)=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(it)^s}{\Gamma(s+1)}\frac{\beta\Gamma\left(\frac{s}{2\alpha}+1\right)}{\theta^{\frac{s}{2\alpha}}}\frac{a_k}{(k+1)^{\frac{s}{2\alpha}+1}}.$$

5. Measures of Information

This section presents some of the well-known measures of the entropy. The loss of information or existence of randomness in a random variable is measured by entropy. Two measures of entropy including Renyi entropy and beta entropy are discussed in this section.

5.1. Renyi ntropy of PERD

The extended form of Shannon entropy is Renyi entropy as $\delta\to 1$. Suppose V be a random variable with PDF $f(v)$, then Renyi entropy (1960) is defined by

$$I_\delta(v)=(1-\delta)^{-1}\,log\int_0^\infty f^\delta(v)\,dv;\;\; \delta>0\,\text{and}\,\delta\neq 1.$$

Replacing $f(v)$ with (2), we have

$$I_\delta(v)=(1-\delta)^{-1}\log\int_0^\infty(2\alpha\beta\theta)^\delta\, v^{\delta(2\alpha-1)}\exp\left(-\delta\theta v^{2\alpha}\right)\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\delta(\beta-1)}dv.$$

Using the expansion of $\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\delta(\beta-1)}$ as given by

$$\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\delta(\beta-1)}=\sum_{k=0}^{\infty}e_k\exp\left(-k\theta v^{2\alpha}\right),$$

where $e_k=\dfrac{(-1)^k\,\Gamma\left[\delta(\beta-1)+1\right]}{\Gamma[k+1]\Gamma\left[\delta(\beta-1)-k+1\right]}$. Hence, simple mathematical calculation reduces the Renyi entropy as follows

$$I_\delta(v)=(1-\delta)^{-1}\log\left[(2\alpha)^{\delta-1}\,\beta^\delta\theta^{\frac{(\delta-1)}{2\alpha}}\sum_{k=0}^{\infty}\frac{e_k\Gamma\left[\frac{\delta(2\alpha-1)+1}{2\alpha}\right]}{(k+\delta)^{\frac{\delta(2\alpha-1)+1}{2\alpha}}}\right].$$

5.2. Harvda and Charvat entropy

The Harvda and Charvat entropy (1967) of a random variable V is defined by

$$I_\delta(v)=(\delta-1)^{-1}\left[1-\int_0^\infty f^\delta(v)\,dv\right];\;\; \delta>0\;\;\text{and}\;\;\delta\neq 1,$$

$$I_\delta(v)=(\delta-1)^{-1}\left[1-\int_0^\infty(2\alpha\beta\theta)^\delta\, v^{\delta(2\alpha-1)}\exp\left(-\delta\theta v^{2\alpha}\right)\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\delta(\beta-1)}dv\right].$$

Thus, the Harvda and Charvat entropy for PER distribution is obtained as

$$I_\delta(v)=(\delta-1)^{-1}\left[1-(2\alpha)^{\delta-1}\beta^\delta\theta^{\frac{(\delta-1)}{2\alpha}}\sum_{k=0}^{\infty}\frac{e_k\Gamma\left[\frac{\delta(2\alpha-1)+1}{2\alpha}\right]}{(k+\delta)^{\frac{\delta(2\alpha-1)+1}{2\alpha}}}\right].$$

6. General Order Statistics

Theorem 1. *The PDF of the general order statistics of PER distribution is given by*

$$f_{v_{(s)}}(v)=\frac{2\alpha\beta\theta v^{2\alpha-1}}{\beta(s,n-s+1)}\sum_{j,k=0}^{\infty}(-1)^{j+k}\binom{n-s}{j}\binom{\beta j+\beta s-1}{k}\exp\left[-(k+1)\theta v^{2\alpha}\right].$$

Proof: Suppose $V_{(1)},V_{(2)},\ldots,V_{(n)}$ be the order statistics of a random sample follows PER distribution with PDF $f_V(v)$ and CDF $F_V(v)$. Then, the PDF of s^{th} order statistics, denoted by $V_{(S)}$ is given by

$$f_{v_{(s)}}(v)=\frac{n!}{(s-1)!(n-s)!}f_V(v)\left[F_V(v)\right]^{s-1}\left[1-F_V(v)\right]^{n-s}. \tag{14}$$

Prior to incorporate (2) and (3) in (14), we use binomial expansion of $\left[1-F_V(v)\right]^{n-s}$ as

$$\left[1-F_V(v)\right]^{n-s}=\sum_{j=0}^{\infty}(-1)^j\binom{n-s}{j}\left[F(v)\right]^j.$$

Thus, we obtain

$$f_{v_{(s)}}(v)=\frac{f(v)}{\beta(s,n-s+1)}\sum_{j=0}^{\infty}(-1)^j\binom{n-s}{j}\left[F(v)\right]^{j+s-1}. \tag{15}$$

Incorporating (2) and (3) in (15), we get

$$f_{v_{(s)}}(v)=\frac{2\alpha\beta\theta v^{2\alpha-1}}{\beta(s,n-s+1)}\sum_{j=0}^{\infty}(-1)^j\binom{n-s}{j}\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta j+\beta s-1}\exp\left(-\theta v^{2\alpha}\right).$$

Again, using Binomial expansion of $\left[1-exp\left(-\theta v^{2\alpha}\right)\right]^{\beta j+\beta s-1}$ as

$$\left[1-\exp\left(-\theta v^{2\alpha}\right)\right]^{\beta j+\beta s-1}=\sum_{k=0}^{\infty}(-1)^k\binom{\beta j+\beta s-1}{k}\exp\left(-k\theta v^{2\alpha}\right).$$

We obtain the general order statistics of PER distribution as

$$f_{v_{(s)}}(v)=\frac{2\alpha\beta\theta v^{2\alpha-1}}{\beta(s,n-s+1)}\sum_{j,k=0}^{\infty}(-1)^{j+k}\binom{n-s}{j}\binom{\beta j+\beta s-1}{k}\exp\left[-(k+1)\theta v^{2\alpha}\right].$$

7. Estimation and Simulation

7.1. Maximum likelihood estimation

Let $V_1,V_2,\ldots,V_n$ be a random sample drawn from the PER distribution with parameter vector $\varphi=(\alpha,\beta,\theta)^T$. Then, the log-likelihood function of n observations for φ is given by

$$\ell(\varphi)=m\log(2\alpha\beta\theta)+(2\alpha-1)\sum_{j=1}^{m}\log(v_i)-\theta\sum_{j=1}^{m}v_i^{2\alpha}+(\beta-1)\sum_{j=1}^{m}\log\left[1-exp\left(-\theta v_i^{2\alpha}\right)\right].$$

The components of the score vector, $\mathcal{U}(\varphi) = \frac{\partial \ell}{\partial \varphi} = \left(\mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_\theta\right)^T$, are given by

$$\mathcal{U}_\alpha = \frac{m}{\alpha} + 2\sum_{j=1}^{m} \log(v_i) - \theta \sum_{j=1}^{m} v_i^{2\alpha} \log(v_i) + (\beta - 1)\theta \sum_{j=1}^{m} \frac{\exp\left(-\theta v_i^{2\alpha}\right) v_i^{2\alpha} \log(v_i)}{\left[1 - exp\left(-\theta v_i^{2\alpha}\right)\right]}.$$

$$\mathcal{U}_\beta = \frac{m}{\beta} + \sum_{j=1}^{m} \log\left[1 - \exp\left(-\theta v_i^{2\alpha}\right)\right].$$

and

$$\mathcal{U}_\theta = \frac{m}{\theta} - \sum_{j=1}^{m} v_i^{2\alpha} + (\beta - 1)\sum_{j=1}^{m} \frac{\exp\left(-\theta v_i^{2\alpha}\right) v_i^{2\alpha}}{\left[1 - \exp\left(-\theta v_i^{2\alpha}\right)\right]}.$$

The ML estimates $\hat{\varphi} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}\right)^T$ of the parameters $\varphi = (\alpha, \beta, \theta)^T$ are investigated by equating the above non-linear system of equations $\mathcal{U}_\alpha = \mathcal{U}_\beta = \mathcal{U}_\theta = 0$ and solving them simultaneously. For doing this, the statistical software R can be used to obtain the desired results.

Since, all the second order derivatives exist. Therefore, for interval estimation of the parameters, we obtain the 3×3 observed information matrix $\mathcal{J}(\varphi) = \frac{\partial^2 \ell}{\partial x \partial y}; (\text{for}\, x, y = \alpha, \beta, \theta)$, whose elements can be computed numerically.

7.2. Simulation

Here, we perform the Monte Carlo simulation study to investigate the performance of ML estimators of the unknown parameters for the PER distribution. The random numbers are generated from PER distribution using inverse transformation method of simulation. The values of parameters are chosen to be $\alpha = 1.5, \beta = 2.5$ and $\theta = 1.3$. The R-software is used to generate data sets of different samples sizes 20, 40, 75 and 150. The summary of the results is presented in Tables 2 to 5.

Table 2 ML estimates and comparison of distributions using simulated data set of size 20

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha} = 2.435$	1.769	7.769	10.756	9.269	8.352
	$\hat{\beta} = 0.217$					
	$\hat{\theta} = 1.882$					
ERD	$\hat{\beta} = 0.558$	6.475	10.475	12.466	11.181	10.864
	$\hat{\theta} = 2.105$					
RD	$\hat{\theta} = 2.980$	11.118	13.118	14.114	13.340	13.312
WD	$\hat{\delta} = 1.526$	9.080	13.080	15.071	13.786	13.469
	$\hat{\theta} = 2.483$					
GED	$\hat{\beta} = 1.382$	11.798	15.798	17.789	16.504	16.187
	$\hat{\theta} = 2.379$					
ED	$\hat{\theta} = 1.968$	12.919	14.919	15.915	15.141	15.113

Table 3 ML estimates and comparison of distributions using simulated data set of size 40

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha} = 2.549$ $\hat{\beta} = 0.267$ $\hat{\theta} = 2.121$	3.352	9.352	14.419	10.018	11.184
ERD	$\hat{\beta} = 0.855$ $\hat{\theta} = 2.558$	9.245	13.245	16.623	13.569	14.466
RD	$\hat{\theta} = 2.827$	9.892	11.892	13.581	11.997	12.503
WD	$\hat{\delta} = 1.937$ $\hat{\theta} = 2.768$	9.833	13.833	17.211	14.157	15.054
GED	$\hat{\beta} = 2.478$ $\hat{\theta} = 3.125$	15.892	19.892	23.270	20.216	21.113
ED	$\hat{\theta} = 1.895$	28.859	30.859	32.548	30.964	31.470

Table 4 ML estimates and comparison of distributions using simulated data set of size 75

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha} = 2.429$ $\hat{\beta} = 0.193$ $\hat{\theta} = 1.682$	8.519	14.519	21.471	14.857	17.295
ERD	$\hat{\beta} = 0.523$ $\hat{\theta} = 2.025$	22.199	26.199	30.834	26.366	28.049
RD	$\hat{\theta} = 3.125$	49.489	51.489	53.806	51.544	52.414
WD	$\hat{\delta} = 1.346$ $\hat{\theta} = 2.450$	31.416	35.316	40.051	35.583	37.267
GED	$\hat{\beta} = 1.287$ $\hat{\theta} = 2.435$	36.571	40.571	45.206	40.738	42.422
ED	$\hat{\theta} = 2.094$	39.162	41.162	43.479	41.217	42.087

8. Data Analysis

This section is devoted to illustrate the practical applications of the proposed PER distribution. In order to assess the flexibility of the new model, we analyze three real life data sets taken from literature and the numerical results of PER distribution are compared with its sub-models, namely ER distribution, Rayleigh distribution (RD), Weibull distribution (WD), exponential distribution (ED) and generalized exponential distribution (GED). The model selection is carried out by using different model selection criterions including the negative log-likelihood, Akaike information criteria (AIC) (Akaike 1974), Schwarz Information Criteria (SIC) (Schwarz 1978), Corrected Akaike information criteria (AICC) (Bazdogan 1987) and Hannan-Quinn information criteria (HQIC) (Hanna and Quinn 1979). Also, Kolmogorov-Simonov test statistics along with corresponding p-value has been calculated.

Table 5 ML estimates and comparison of distributions using simulated data set of size 150

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha}=2.419$ $\hat{\beta}=0.267$ $\hat{\theta}=2.124$	11.074	17.074	26.106	17.238	20.743
ERD	$\hat{\beta}=0.803$ $\hat{\theta}=2.591$	31.508	35.508	41.529	33.590	37.954
RD	$\hat{\theta}=3.125$	36.426	38.426	41.437	38.453	39.649
WD	$\hat{\delta}=1.876$ $\hat{\theta}=2.850$	35.525	39.525	45,546	39.607	41.971
GED	$\hat{\beta}=2.164$ $\hat{\theta}=2.987$	62.759	66.759	72.780	66.841	69.205
ED	$\hat{\theta}=1.943$	100.74	102.74	105.76	102.77	103.97

The descriptive synopsis of the three real life data sets is presented in Table 2. The ML estimates with their corresponding standard errors in parenthesis for the three data sets are presented in Tables 3, 5 and 7 respectively whilst the Tables 4, 6 and 8 lists the numerical values of the negative log-likelihood, AIC, SIC, AICC, HQIC, Kolmogorov-Smirnov distance along with corresponding p-value, respectively. These numerical results are acquired using R program. Based on the model selection criterions, we infer that the proposed model fits better than its sub-models to these data sets. Figure (4) displays the fitted density and distribution plots of the PER distribution to the three data sets. It is clear from these plots that PER distribution provide close fit to the three real life data sets.

Data set 1: The data set due to Smith and Naylor (1987) consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. This data set was also analysed by Oguntunde et al. (2015) to demonstrate the applicability of Weibull-Exponential distribution. The data are:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67,1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

Data set 2: This data set as a second real life application consists of 66 observations represents the breaking stress of carbon fibres of 50 mm gauge length is taken from Nichols and Padgett (2006) and recently Alzaatreh et al. (2014) analysed this data for Gamma Normal distribution. The data set is given as

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

Data set 3: This data set taken from the literature of Badar and Priest (1982) represent the strength data measured in GPa, of 69 single carbon fibres tested under tension at gauge lengths of 20 mm. The data is presented as follows:

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Table 6 ML Estimates and the statistics -2ℓ, AIC, SIC, AICC and HQIC using data set 1

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha}=2.419$	30.682	36.682	43.111	37.089	39.211
	$\hat{\beta}=0.267$					
	$\hat{\theta}=2.124$					
ERD	$\hat{\beta}=0.803$	48.442	52.442	56.728	52.642	54.127
	$\hat{\theta}=2.591$					
RD	$\hat{\theta}=3.125$	99.582	101.582	103.725	101.648	102.424
WD	$\hat{\delta}=1.876$	37.741	41.741	46.027	41.941	43.426
	$\hat{\theta}=2.850$					
GED	$\hat{\beta}=2.164$	68.889	72.889	77.175	73.089	74.575
	$\hat{\theta}=2.987$					
ED	$\hat{\theta}=1.943$	177.661	179.661	181.804	179.727	180.504

Table 7 ML Estimates and the statistics -2ℓ, AIC, SIC, AICC and HQIC using data set 2

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha}=1.498$	172.902	178.902	185.471	179.289	181.498
	$\hat{\beta}=1.270$					
	$\hat{\theta}=0.042$					
ERD	$\hat{\beta}=1.637$	180.875	184.875	189.254	185.066	186.605
	$\hat{\theta}=0.159$					
RD	$\hat{\theta}=0.119$	196.417	198.417	200.607	198.480	199.282
WD	$\hat{\delta}=2.531$	180.824	184.824	189.203	185.015	186.554
	$\hat{\theta}=0.064$					
GED	$\hat{\beta}=5.784$	194.439	198.439	202.818	198.630	200.169
	$\hat{\theta}=0.844$					
ED	$\hat{\theta}=0.362$	265.989	267.989	270.179	268.052	268.854

Table 8 ML Estimates and the statistics -2ℓ, AIC, SIC, AICC and HQIC using data set 3

Model	ML Estimates	-2ℓ	AIC	SIC	AICC	HQIC
PERD	$\hat{\alpha}=1.498$	104.459	110.459	117.161	110.828	113.118
	$\hat{\beta}=1.270$					
	$\hat{\theta}=0.042$					
ERD	$\hat{\beta}=1.637$	113.622	117.622	122.090	117.804	119.395
	$\hat{\theta}=0.159$					
RD	$\hat{\theta}=0.119$	174.493	176.493	178.727	176.553	177.379
WD	$\hat{\delta}=2.531$	127.910	131.910	136.378	132.092	133.683
	$\hat{\theta}=0.064$					
GED	$\hat{\beta}=5.784$	137.023	141.023	145.491	141.205	142.796
	$\hat{\theta}=0.844$					
ED	$\hat{\theta}=0.362$	261.735	263.735	265.969	263.795	264.621

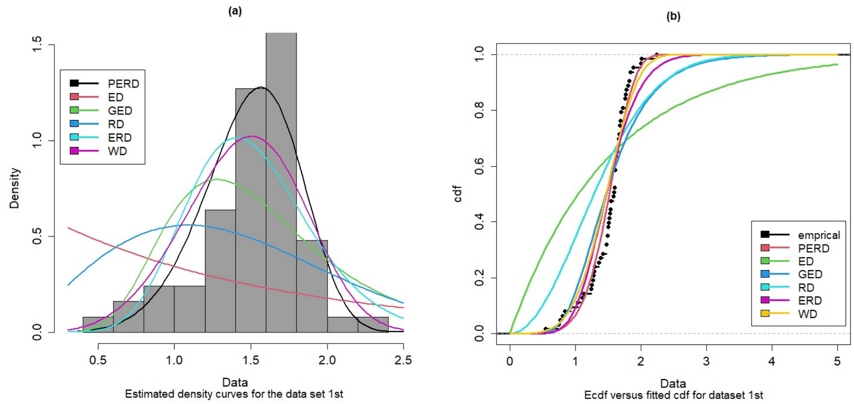

Figure 5 Estimated PDF and CDF plots of PER distribution and other competitive models for the data set 1

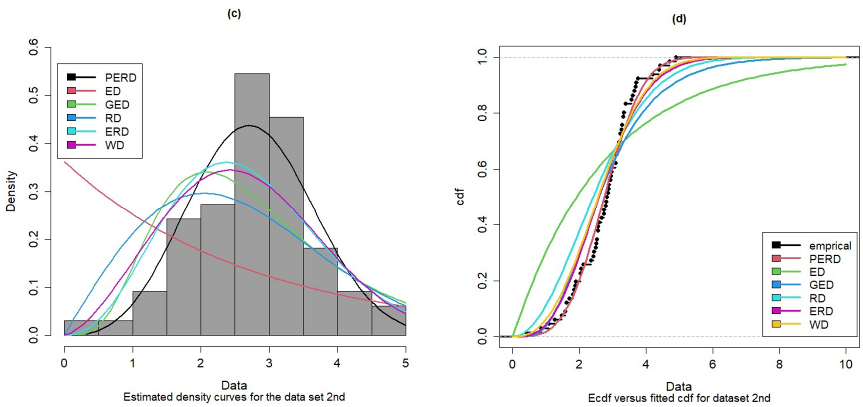

Figure 6 Estimated PDF and CDF plots of PER distribution and other competitive models for the data set 2

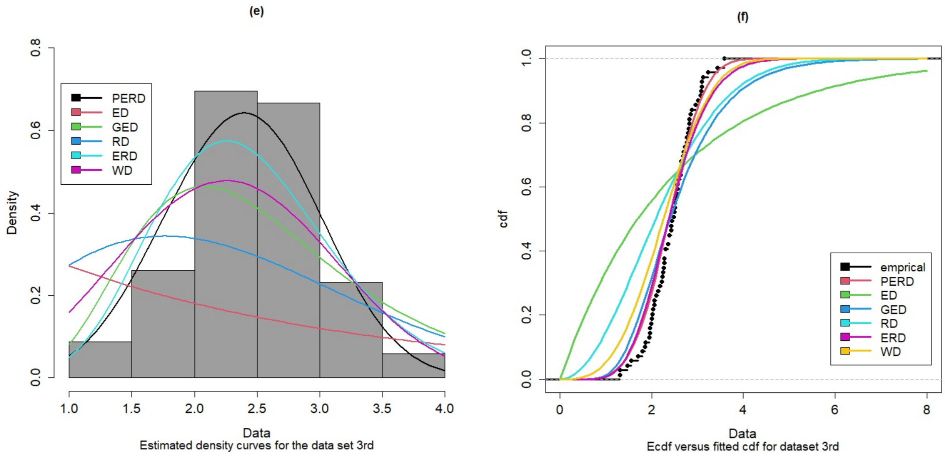

Figure 7 Estimated PDF and CDF plots of PER distribution and other competitive models for the data set 3

9. Concluding Remarks

In this article, we propose a new model called the power exponentiated Rayleigh distribution which extends the exponentiated Rayleigh distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for the moments; mean residual life, moment generating function, characteristic function and Order statistics. The new model is capable of modeling data sets with non-monotone hazard rate. A simulation study is carried out to investigate the behavior of ML estimates for finite sample size. The estimation of parameters is approached by the method of maximum likelihood estimation. The applications of the power exponentiated Rayleigh distribution to real data are provided which show that the new distribution can be used quite effectively to provide better fits than the other competing distributions. We prospect that the proposed model will draw wider applications in statistics.

References

Abd Elfattah AM, Hassan AS, Ziedan DM. Efficiency of maximum likelihood estimators under different censored sampling schemes for Rayleigh distribution. Interstat. 2006; 1: 1-16.

Ahmad A, Ahmad SP, Ahmed A. Characterization and estimation of transmuted Rayleigh distribution. J Stat Appl Prob. 2015; 4(2): 315-321.

Akaike H. A new look at the statistical model identification. IEEE Automat. Contr. 1974; 19: 716-723.

Alzaatreh A, Famoye F, Lee C. The gamma-normal distribution: properties and applications. Comput Stat Data Anal. 2014; 69: 67-80.

Ardianti F, Sutarman. Estimating parameter of Rayleigh distribution by using maximum likelihood method and Bayes method. In IOP Conference Series: Materials Science and Engineering, IOP Publishing. 2018; 300(1): 012-036.

Ateeq K, Qasim BT, Alvi RA. (2019). An extension of Rayleigh distribution and applications. Cogent Math Stat. 2019; 6(1): 1-16.

Badar MG, Priest AM. Statistical aspects of fibre and bundle strength in hybrid composites. In: Hayashi T, Kawata K, Umekawa S. editors. Progress in Science and Engineering Composites. Tokyo: ICCM-IV; 1982. p. 1129-1136.

Bazdogan H. Model selection and Akaike's information criterion: The general theory and its analytical extensions. Psychometrika. 1987; 52: 345-370.

Dey S, Tanujit D. Rayleigh distribution revisited via extension of Jeffrey's prior information and a new loss function. Revstat-Stat. J. 2011; 9(3): 213-226.

Dey S, Dey T, Kundu D. Two-parameter Rayleigh distribution: Different methods of estimation. Am. J Math Manag Sci. 2014; 33: 55-74.

Hannan EJ and Quinn BG. The determination of the order of an autoregression. J Royal Stat Soc Series B. 1979; 41: 190-195.

Harvda J, Charvat F. Quantification method in classification processes: concept of structural entropy. Kybernetika. 1967; 3: 30-35.

Howlader HA, Hossain A. On Bayesian estimation and prediction from Rayleigh distribution based on type II censored data. Commun Stat Theory Methods. 1995; 24(9): 2249-2259.

Hyndman RJ, Fan Y. Sample quantiles in statistical packages. Am. Stat. 1996; 50(4): 361-365.

Kundu D, Raqab MZ. Generalized Rayleigh distribution: different methods of estimation. Comput Stat Data Anal. 2005; 49: 187-200.

Lalitha S, Mishra A. Modified maximum likelihood estimation for Rayleigh distribution. Commun Stat Theory Methods. 1996; 25: 389-401.

Merovci F. Transmuted Rayleigh distribution. Austrian J Stat. 2013; 42(1): 21-31.

Merovci F, Elbatal I. Weibull-Rayleigh distribution: theory and applications. Appl Math Inform Sci. 2015; 9(5): 1-11.

Mudasir S, Jan U, Ahmad SP. Weighted Rayleigh distribution revisited via informative and non-informative priors. Pak J Stat. 2019; 35(4): 321-348.

Nichols MD, Padgett WJ. A bootstrap control for Weibull percentiles. Qual Reliab Engng Int. 2006; 22: 141-151.

Oguntunde PE, Balogun OS, Okagbue H I, Bishop SA. The Weibull-Exponential distribution: its properties and applications. J Appl Sci. 2015; 15(11): 1305-1311.

Raqab MZ, Madi MT. Bayesian analysis for the exponentiated Rayleigh distribution. Int J Stat. 2009; LXVII: 269-288.

Rayleigh J. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. Philos Mag 1880; 10: 73-78.

Renyi A. On measures of entropy and information. Berkeley Symposium on Mathematical Statistics and Probability. 1960; 1(1): 547-561.

Saima N, Ahmad SP, Ahmed A. Bayesian analysis for generalized Rayleigh Distribution. J Math Theory Model. 2016; 6(5): 2224-5804.

Schwarz G. Estimating the dimensions of a model. Ann Stat. 1978; 6(2): 461-464.

Siddiqui MM. Some problems connected with Rayleigh distributions. J Res Natl Bur Stand. 1962; 60D: 167-174.

Smith RL, Naylor JC. A comparison of maximum likelihood and Bayesian estimators for the three parameter Weibull distribution. Appl Stat. 1987; 36: 358-369.

Surles JG, Padgett WJ. Inference for reliability and stress- strength for a scaled Burr type X distribution. Lifetime Data Anal. 2001; 7: 187-200.

Voda VG. A new generalization of Rayleigh distribution. Reliab: Theory Appl. 2007; 2(2): 47-56.