



Thailand Statistician  
April 2023; 21(2): 244-256  
<http://statassoc.or.th>  
Contributed paper

# Reliability of Coherent Systems with Multiple Types of Dependent Components with an Active Redundancy

Zohreh Mohammadi\*

Department of Statistics, Jahrom University, Jahrom, Fars, Iran.

\* Corresponding author; Email: [z.mohammadi@jahromu.ac.ir](mailto:z.mohammadi@jahromu.ac.ir)

Received: 18 July 2020

Revised: 29 May 2021

Accepted: 30 May, 2021

## Abstract

This study is concerned with the problem of redundancy with an active strategy. We consider a coherent system with multiple types of dependent components and investigate its reliability with redundancies at component and system levels. Previous studies have primarily considered a coherent system with a single type of components, so that the lifetimes of original component and its spare are independent. However, the lifetime of original component and its spare are dependent random variables in most practical systems. In this article some comparisons among various stochastic orderings for different situations have been made and the results are presented in details.

---

**Keywords:** Coherent system, Reliability function, Redundancy allocation, Active strategy, Dependent components

## 1. Introduction

Adding redundant components to a system is one of the most effective methods that has been widely used to enhance the reliability and lifetime of the system, see Kuo and Prasad (2000). Furthermore, either an active or standby redundancy strategy is selected to insert redundant components into the system. In the former, the redundant and original components start functioning simultaneously from the onset, whereas in the latter, the redundant component starts functioning immediately after the corresponding original component in the system fails. This topic has been attracted a great deal of attention in reliability theory and studied for many different system structures, objective functions and distribution assumptions, e.g. see Boland and El-Newehi (1995), Romera et al. (2004), Misra et al. (2011), Zhao et al. (2012), Nanda and Hazra (2013), Laniado and Lillo (2014), Wang et al. (2015) and the references therein.

In reliability engineering, a well known principle states that active redundancy at the component level is more effective than at the system level in usual stochastic ordering. Boland and El-Newehi (1995) showed that this principle does not hold for hazard rate ordering in general. For even series system, it is true when the spares match the original components.

Romera et al. (2004) discussed on the allocation of one or two active redundancies to a  $k$ -out-of- $n$ : $G$  system in order to improve the system in the sense of probability order.

Misra et al. (2011) considered the problem of optimal allocating one or two active spares in series systems in the sense of various stochastic orderings. Zhao et al. (2012) investigated this problem of both active and standby redundancies and discussed the case of allocating  $K$  active redundancies to a series system.

Zhao et al. (2015) considered the problem of stochastic comparison of both standby and active redundancies at component level versus system level. They assumed that the lifetime of original and spare components are independent random variables and follow exponential distribution and presented some results of both series and parallel systems for both of the matching and non-matching spares case, respectively. Eryilmaz (2017) investigated this problem under the cold standby strategy and obtained signature-based expressions for the reliability function of the system after redundancy at component and system levels.

Kavлак (2017) employed signature theory to calculate the reliability and mean residual life function of a coherent system with an active redundancy. She considered a coherent system made of  $n$  independent original components and equipped with  $n$  spare components. She also assumed that the lifetime of original and spare components are dependent random variables. She managed to obtain the reliability of a coherent system with an active redundancy at the component level. However, in Kavлак (2017) a closed form for the reliability at the system level is not given.

Eryilmaz (2016) and Eryilmaz et al. (2018) calculated the reliability function of coherent systems with multiple types of dependent components. We aim to investigate the problem of redundancy allocation for these systems. Previous studies on the problem of redundancy allocation have considered a coherent system with single type of components. We will also present a modified Kavлак’s study.

The rest of this article is organized as follows: Section 2 is devoted to basic definitions, assumptions and the notations that will be used throughout the paper. In Section 3, the closed form is presented for the reliability function of a coherent system consisting of multiple types of dependent components with an active redundancy at component and system levels. The results is used in two special cases and the system reliability with redundancy at component and system levels is compared in terms of stochastic ordering. In Section 4, we investigated the performance of the obtained equations through some examples. Concluding remarks are made in Section 5.

**2. The Structure of the Problem**

Consider a coherent system of  $n$  components, such that  $n_1$  of which are type 1,  $n_2$  of the components are type 2, ...,  $n_m$  of the components are type  $m$  and  $n_1 + n_2 + \dots + n_m = n$ . Assume that the lifetime variable of components of the same type are independent and have identical distribution and full independence is assumed for the lifetime variable of components of different types. Let  $\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)})$  with  $\mathbf{x}^{(i)} = (x_1^{(i)}, \dots, x_{n_i}^{(i)})$  for  $i = 1, \dots, m$ , be state vector. In the case that  $j$ th component of type  $i$  functions  $x_j^{(i)} = 1$ , otherwise  $x_j^{(i)} = 0$ . The structure function  $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ , defined for all possible  $\mathbf{x}$ , takes the value 1 if the system functions and 0 if the system fails for the state vector  $\mathbf{x}$ . Coolen and Coolen-Maturi (2013) introduced the survival signature denoted by  $\Phi(l_1, \dots, l_m)$  for  $l_i = 0, \dots, n_i, i = 1, \dots, m$  and defined as

$$\Phi(l_1, \dots, l_m) = \left[ \prod_{i=1}^m \binom{n_i}{l_i} \right]^{-1} \times \sum_{\mathbf{x} \in S_{l_1, \dots, l_m}} \phi(\mathbf{x}), \tag{1}$$

where  $S_{l_1, \dots, l_m}$  denote the set of state vectors such that  $\sum_{j=1}^{n_i} x_j^{(i)} = l_i$  for  $i = 1, \dots, m$ .  $\Phi(l_1, \dots, l_m)$  denotes the probability that a system functions given that precisely  $l_i$  components of type  $i$  function  $i \in \{1, \dots, m\}$ . The reliability function of system is represented by

$$\begin{aligned} P(T > t) &= \sum_{l_1=0}^{n_1} \dots \sum_{n_m}^{l_m} \Phi(l_1, \dots, l_m) P\left(\bigcap_{i=1}^m \{C_i(t) = l_i\}\right) \\ &= \sum_{l_1=0}^{n_1} \dots \sum_{n_m}^{l_m} \Phi(l_1, \dots, l_m) \prod_{i=1}^m \left[ \binom{n_i}{l_i} \{F_i(t)\}^{n_i-l_i} \{1 - F_i(t)\}^{l_i} \right] \end{aligned} \tag{2}$$

where  $C_i(t) \in \{0, \dots, n_i\}$  denotes the number of components of type  $i$  in the system that function at time  $t > 0$  and  $F_i$  denotes the distribution function of components of type  $i$ . (See Coolen and Coolen-Maturi (2013) for more details).

Assume that this system is equipped with  $n$  spare components in an active strategy such that the lifetime of original and spare components are shown by  $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(m)}$  and  $\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)}$  where  $\mathbf{X}^{(i)} = (X_1^{(i)}, \dots, X_{n_i}^{(i)})$  and  $\mathbf{Y}^{(i)} = (Y_1^{(i)}, \dots, Y_{n_i}^{(i)})$  for  $i = 1, \dots, m$ , respectively.

Active redundancy can be applied at system or component levels. It is clear that the lifetime of a system with an active redundancy at the component level is

$$T_c = \phi(Z_1^{(1)}, \dots, Z_{n_1}^{(1)}, \dots, Z_1^{(m)}, \dots, Z_{n_m}^{(m)}), \tag{3}$$

where  $Z_j^{(i)} = \max(X_j^{(i)}, Y_j^{(i)})$  for  $i = 1, \dots, m, j = 1, \dots, n_i$  and at the system level is

$$T_s = \max(\phi(\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(m)}), \phi(\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)})). \tag{4}$$

In this article, the reliability function of a coherent system with a given survival signature with an active redundancy at component and system levels are calculated and compared under different conditions. To compare two lifetime variables  $T_c$  and  $T_s$ , we first recall some stochastic order definitions for two non negative random variables  $T_1$  and  $T_2$  with cumulative distribution functions  $F_1$  and  $F_2$ , density functions  $f_1$  and  $f_2$ , and reliability functions  $R_1$  and  $R_2$ , respectively. The random variable  $T_1$  is expected to be less than the random variable  $T_2$

- in the stochastic order,  $T_1 \leq T_2$ , if  $F_1(t) \geq F_2(t)$  for all  $t > 0$ .
- in hazard rate order,  $T_1 \leq_{hr} T_2$ , if  $\frac{R_1(t)}{R_2(t)}$  is a decreasing function of  $t$ .
- in likelihood ratio order,  $T_1 \leq_{lr} T_2$ , if  $\frac{f_1(t)}{f_2(t)}$  is a decreasing function of  $t$ .

### 3. The Reliability of Coherent Systems Consisting of Single Type of Components

In this section, the problem of redundancy is detailed for a coherent system with multiple type of components under the different scenarios.

First, the problem is investigated under general dependence. Consider a coherent system of  $n$  components, such that  $n_1$  of which are type 1,  $n_2$  of the components are type 2,  $\dots$ ,  $n_m$  of the components are type  $m$  and  $n_1 + n_2 + \dots + n_m = n$ . Assume that the lifetime variable of components of the same type are exchangeable and dependent and the lifetime variable of components of different type are dependent.

Eryilmaz (2016) and Eryilmaz et al. (2018) presented a closed form for the reliability of a coherent system with multiple types of dependent components under the above condition.

Now, assume that this system is equipped with  $n$  spare components in an active strategy with a similar condition. In addition, assume that the lifetime of spare and original components of different types are dependent, i.e.

$$\begin{aligned} &P(X_{i_1}^{(1)} \leq x_1, \dots, X_{i_{n_1}}^{(1)} \leq x_{n_1}, \dots, X_{i_{n-n_m+1}}^{(m)} \leq x_{n-n_m+1}, \dots, X_{i_n}^{(m)} \leq x_n, \\ &Y_{i_1}^{(1)} \leq y_1, \dots, Y_{i_{n_1}}^{(1)} \leq y_{n_1}, \dots, Y_{i_{n-n_m+1}}^{(m)} \leq y_{n-n_m+1}, \dots, Y_{i_n}^{(m)} \leq y_n) = \\ &P(X_{\pi_1(i_1)}^{(1)} \leq x_1, \dots, X_{\pi_1(i_{n_1})}^{(1)} \leq x_{n_1}, \dots, X_{\pi_m(i_{n-n_m+1})}^{(m)} \leq x_{n-n_m+1}, \dots, X_{\pi_m(i_n)}^{(m)} \leq x_n, \\ &Y_{\sigma_1(i_1)}^{(1)} \leq y_1, \dots, Y_{\sigma_1(i_{n_1})}^{(1)} \leq y_{n_1}, \dots, Y_{\sigma_m(i_{n-n_m+1})}^{(m)} \leq y_{n-n_m+1}, \dots, Y_{\sigma_m(i_n)}^{(m)} \leq y_n) \end{aligned} \tag{5}$$

for any permutations  $(\pi_1(i_1), \dots, \pi_1(i_{n_1}))$  and  $(\sigma_1(i_1), \dots, \sigma_1(i_{n_1}))$  of  $\{i_1, \dots, i_{n_1}\}, \dots$ ,  $(\pi_m(i_{n-n_m+1}), \dots, \pi_m(i_n))$  and  $(\sigma_m(i_{n-n_m+1}), \dots, \sigma_m(i_n))$  of  $\{i_{n-n_m+1}, \dots, i_n\}$  for all  $i_{n_1} \leq n_1, \dots, i_n \leq n_m$ .

In the following, the lifetime of this system with an active redundancy at the component and system levels is presented.

**Corollary 1** Assume that the lifetime of spare and original components are dependent and the condition (5) satisfies, then the reliability function of a system with an active redundancy at the component level is

$$\begin{aligned}
 P(T_c > t) &= \sum_{l_1=0}^{n_1} \dots \sum_{l_m=0}^{n_m} \Phi(l_1, \dots, l_m) \binom{n_1}{l_1} \dots \binom{n_m}{l_m} \\
 &\times \sum_{k_1=0}^{l_1} \dots \sum_{k_m=0}^{l_m} (-1)^{k_1+\dots+k_m} \binom{l_1}{k_1} \dots \binom{l_m}{k_m} \beta_{n_1-l_1+k_1, \dots, n_m-l_m+k_m} \quad (6)
 \end{aligned}$$

where  $\beta_{r_1, \dots, r_m} = P(X_1^{(i)} \leq t, \dots, X_{r_i}^{(i)} \leq t, Y_1^{(i)} \leq t, \dots, Y_{r_i}^{(i)} \leq t; i = 1, \dots, m)$  for  $r_i = 1, \dots, n_i$ .

**Proof:** The proof is inspired by Theorem 1 Eryilmaz et al. (2018). Consider the Eqn. (2) and define

$$W_j^{(i)} = \begin{cases} 0 & \text{if } Z_j^{(i)} > t \\ 1 & \text{if } Z_j^{(i)} \leq t. \end{cases}$$

So,  $P(C_1(t) = l_1, \dots, C_m(t) = l_m)$  in Eqn. (2) can be written as

$$P\left(\sum_{j=1}^{n_1} W_j^{(1)} = n_1 - l_1, \dots, \sum_{j=1}^{n_m} W_j^{(m)} = n_m - l_m\right).$$

The proof is completed using the Theorem 1 Eryilmaz et al. (2018).

In order to calculate the reliability function of a coherent system with an active redundancy at the system level, this system must be considered to be made of  $2m$  different types of dependent components. Considering aforementioned assumptions, the survival signature theory could be employed in order to derive the reliability function. Based on Theorem 1 Eryilmaz et al. (2018), the following corollary could be achieved.

**Corollary 2** The reliability function of a system with an active redundancy at the system level is

$$\begin{aligned}
 P(T_s > t) &= \sum_{i_1=0}^{n_1} \dots \sum_{i_m=0}^{n_m} \sum_{j_1=0}^{n_1} \dots \sum_{j_m=0}^{n_m} \Phi_*(i_1, \dots, i_m, j_1, \dots, j_m) \\
 &\times P(\min(X_{1:i_1}^{(1)}, \dots, X_{1:i_m}^{(m)}, Y_{1:j_1}^{(1)}, \dots, Y_{1:j_m}^{(m)}) > t), \quad (7)
 \end{aligned}$$

where  $X_{1:r}^{(i)} = \min(X_1^{(i)}, \dots, X_r^{(i)})$ ,  $Y_{1:r}^{(i)} = \min(Y_1^{(i)}, \dots, Y_r^{(i)})$

$$\begin{aligned}
 \Phi_*(i_1, \dots, i_m, j_1, \dots, j_m) &= \sum_{l_1=0}^{i_1} \dots \sum_{l_m=0}^{i_m} \sum_{s_1=0}^{j_1} \dots \sum_{s_m=0}^{j_m} (-1)^{\sum_{k=1}^m (i_k - l_k + j_k - s_k)} \\
 &\times \binom{n_1}{l_1} \dots \binom{n_m}{l_m} \binom{n_1}{s_1} \dots \binom{n_m}{s_m} \binom{n_1 - l_1}{i_1 - l_1} \dots \binom{n_m - l_m}{i_m - l_m} \\
 &\times \binom{n_1 - s_1}{j_1 - s_1} \dots \binom{n_m - s_m}{j_m - s_m} \Phi(l_1, \dots, l_m, s_1, \dots, s_m)
 \end{aligned}$$

and  $\Phi(l_1, \dots, l_m, s_1, \dots, s_m)$  represents the survival signature.

Now, assume that the lifetime variables of all components are independent. In addition, suppose that  $n_i$  original and spare components from  $i$ th type have the same distribution  $F_i$  and  $G_i$ , respectively for  $i = 1, \dots, L$ . Further, consider that spare components are independent from original components.

By Eqn. (2) , the reliability function of the system under active redundancy at component level is

$$P(T_c > t) = \sum_{l_1=1}^{n_1} \dots \sum_{l_m=1}^{n_m} \Phi(l_1, \dots, l_m) \prod_{i=1}^m \binom{n_i}{l_i} \{F_i(t)G_i(t)\}^{n_i-l_i} \{1 - F_i(t)G_i(t)\}^{l_i} \quad (8)$$

and the reliability function of the system under active redundancy at system level is

$$\begin{aligned} P(T_s > t) &= 1 - \{1 - P(\phi(\mathbf{X}) > t)\}\{1 - P(\phi(\mathbf{Y}) > t)\} \\ &= 1 - \{1 - \sum_{l_1=1}^{n_1} \dots \sum_{l_m=1}^{n_m} \Phi(l_1, \dots, l_m) \prod_{i=1}^m \binom{n_i}{l_i} \{F_i(t)\}^{n_i-l_i} \{1 - F_i(t)\}^{l_i}\} \\ &\quad \times \{1 - \sum_{l_1=1}^{n_1} \dots \sum_{l_m=1}^{n_m} \Phi(l_1, \dots, l_m) \prod_{i=1}^m \binom{n_i}{l_i} \{G_i(t)\}^{n_i-l_i} \{1 - G_i(t)\}^{l_i}\} \end{aligned} \quad (9)$$

where  $\Phi(l_1, \dots, l_m)$  is the survival signature and calculated by Eqn. (1).

In sum, stochastic ordering between the reliability of a coherent system with multiple type of components with redundancy at component and system levels is very complicated. The following subsections present comparisons in details.

**3.1.  $(k_1, k_2, \dots, k_m)$ -out-of- $n$**

In this section, a special case of coherent systems is considered. Ordinary  $k$ -out-of- $n$  systems operate if at least  $k$  components work. A number of studies have considered the effectiveness of adding redundancy to  $k$ -out-of- $n$  system. In general, a system may have multiple types of components with different functions, each type of which may be required for the operation of the whole system. These systems have  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  structure. The  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  system operates if at least  $k_1$  components of type 1,  $k_2$  components of type 2,  $\dots$ ,  $k_m$  components of type  $m$  work. Eryilmaz (2019) studied this system. The lifetime of the  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  system is presented as

$$T = \min(X_{n_1-k_1+1:n_1}^{(1)}, X_{n_2-k_2+1:n_2}^{(2)}, \dots, X_{n_m-k_m+1:n_m}^{(m)}), \quad (10)$$

where  $X_{r:n_i}^{(i)}$  is the  $r$ th smallest order statistics among  $X_1^{(i)}, X_2^{(i)}, \dots, X_{n_i}^{(i)}$  for  $i = 1, \dots, m$ .

Thus, the lifetime variable of  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  system under active redundancy at the component level is

$$T_c = \min(Z_{n_1-k_1+1:n_1}^{(1)}, Z_{n_2-k_2+1:n_2}^{(2)}, \dots, Z_{n_m-k_m+1:n_m}^{(m)}) \quad (11)$$

and by Eqn. (8) the reliability function of system is represented as

$$\begin{aligned} P(T_c > t) &= \prod_{i=1}^m P(Z_{n_i-k_i+1:n_i}^{(i)} > t) \\ &= \prod_{i=1}^m \sum_{j=0}^{n_i-k_i} \binom{n_i}{j} \{F_i(t)G_i(t)\}^j \{1 - F_i(t)G_i(t)\}^{n_i-j}. \end{aligned}$$

The lifetime variable of  $(k_1, k_2, \dots, k_L)$ -out-of- $n$  system under active redundancy at the system level is

$$T_s = \max\{\min(X_{n_1-k_1+1:n_1}^{(1)}, \dots, X_{n_m-k_m+1:n_m}^{(m)}), \min(Y_{n_1-k_1+1:n_1}^{(1)}, \dots, Y_{n_m-k_m+1:n_m}^{(m)})\} \quad (12)$$

and by Eqn. (9) the reliability function of system is

$$P(T_s > t) = 1 - \left\{ 1 - \prod_{i=1}^m \sum_{j=0}^{n_i - k_i} \binom{n_i}{j} \{F_i(t)\}^j \{1 - F_i(t)\}^{n_i - j} \right\} \\ \times \left\{ 1 - \prod_{i=1}^m \sum_{j=0}^{n_i - k_i} \binom{n_i}{j} \{G_i(t)\}^j \{1 - G_i(t)\}^{n_i - j} \right\}.$$

For convenience,  $\prod_i$  is used instead of  $\prod_{i=1}^m$  in the rest of this article.

**Theorem 1** Under the matching spare, (i.e.  $F_i(t) = G_i(t)$ ), if let  $k_i = 1$  for  $i = 1, \dots, m$ , then the lifetime variable of a system with an active redundancy at the component level is bigger than the one at the system level in terms of hazard rate ordering,  $T_c \geq_{hr} T_s$ .

**Proof:** It is enough to show that  $\varphi_1(t) = \frac{P(T_s > t)}{P(T_c > t)}$  is a decreasing function of  $t$  where

$$P(T_c > t) = \prod_{i=1}^m (1 - F_i^{2n_i}(t))$$

and  $P(T_s > t) = 1 - \left\{ 1 - \prod_{i=1}^m (1 - F_i^{n_i}(t)) \right\}^2$ .

Let  $x_i = 1 - F_i^{n_i}(t)$  for  $i = 1, \dots, m$  so,  $\varphi_1(t)$  can be represented by

$$\varphi_2(x_1, \dots, x_m) = \frac{2 \prod_i x_i - \prod_i x_i^2}{\prod_i (2x_i - x_i^2)} = \frac{2 - \prod_i x_i}{\prod_i (2 - x_i)}.$$

It can be easily seen that each  $x_i$  is a decreasing function of  $t$ . The proof is completed if  $\varphi_2$  is an increasing function in each  $x_i$ .

$$\frac{\partial \varphi_2}{\partial x_i} = \frac{\{ \prod_{j \neq i} (2 - x_j) \} \{ 2 - 2 \prod_{j \neq i} x_j \}}{\prod_i (2 - x_i)^2},$$

since  $0 < x_i < 1$  so,  $\frac{\partial \varphi_2}{\partial x_i} \geq 0$ .

### 3.2. The reliability of coherent systems consisting of single type of components

Consider a coherent system with a single type of components. If the lifetime of components in each group (original and spare) are exchangeably dependent the reliability function of a system with an active redundancy at the component level is stated by Eqn. (6)

$$P(T_c > t) = \sum_{l=0}^n \Phi(l) \binom{n}{l} \sum_{k=0}^l (-1)^k \binom{l}{k} \beta_{n-l+k}, \tag{13}$$

where  $\beta_0 = 1$  and  $\beta_r = P(X_1 \leq t, \dots, X_r \leq t, Y_1 \leq t, \dots, Y_r \leq t)$  for  $r = 1, \dots, n$ . By Eqn. (7), the reliability function of a system with an active redundancy at the system level is

$$P(T_s > t) = \sum_{i=0}^n \sum_{j=0}^n \Phi_*(i, j) P(\min(X_{1:i}, Y_{1:j}) > t) \tag{14}$$

where  $X_{1:i} = \min(X_1, \dots, X_i), Y_{1:i} = \min(Y_1, \dots, Y_j),$

$$\begin{aligned} \Phi_*(m_1, m_2) &= \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} (-1)^{m_1-l_1+m_2-l_2} \binom{n}{l_1} \binom{n}{l_2} \\ &\times \binom{n-l_1}{m_1-l_1} \binom{n-l_2}{m_2-l_2} \Phi(l_1, l_2). \end{aligned}$$

$\Phi(l_1, l_2) = \frac{r_{n,n}(l_1, l_2)}{\binom{n}{l_1} \binom{n}{l_2}}$  represents the survival signature and  $r_{n,n}(l_1, l_2)$  denotes the number of path sets of the system including exactly  $l_1$  original components and  $l_2$  spare components.

**Theorem 2** *If  $(X, Y)$  is positively quadrant dependent (PQD), i.e., for all  $x > 0, y > 0 F_{X,Y}(x, y) \geq F_X(x)F_Y(y)$ , then*

$$P(T_s > t) \leq P(T_{s,ind} > t) \tag{15}$$

that is,  $T_s \leq_{st} T_{s,ind}$ , where  $T_{s,ind}$  is the lifetime of the system when the original component and its spare are independent. *If  $(X, Y)$  is negatively quadrant dependent (NQD), i.e.,  $F_{X,Y}(x, y) \leq F_X(x)F_Y(y)$  for all  $x > 0, y > 0$ , then*

$$P(T_s > t) \geq P(T_{s,ind} > t) \tag{16}$$

that is,  $T_s \geq_{st} T_{s,ind}$ .

**Proof:** Huang et al. (2013) proved that if  $(X, Y)$  is PQD, the bivariate order statistics  $(X_{r:n}, Y_{s:n}), 1 < r, s < n$  would be PQD. By contrast, where  $(X, Y)$  is NQD,  $(X_{r:n}, Y_{s:n}), 1 < r, s < n$  is NQD. The result can be easily derived using corollary 3 Eryilmaz (2015) that calculates the lifetime of a system with an active redundancy at the system level as follows:

$$\begin{aligned} P(T_s > t) &= \sum_{i=1}^n p_{n-i+1}(n)P(X_{n-i+1:n} > t) + \sum_{j=1}^n q_{n-j+1}(n)P(Y_{n-j+1:n} > t) \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n p_{n-i+1}(n)q_{n-j+1}(n)P(X_{n-i+1:n} > t, Y_{n-j+1:n} > t) \end{aligned} \tag{17}$$

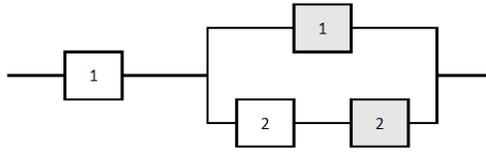
where  $p_i(n) = P(T = X_{i:n})$  and  $q_i(n) = P(T = Y_{i:n})$  for  $i = 1, \dots, n$  represent the signatures associated with the lifetimes  $T$ .

Accordingly, for the reliability of a system with an active redundancy at the component level, a result in contrast with that stated in Theorem 2 would be obtained. This has been confirmed by Kavlak (2017).

#### 4. Case Study

In this section, the performance of the obtained equations in the previous sections is investigated.

**Example 1** Consider the system in Figure 1 which is made of 4 components with  $m = 2$  types with  $n_1 = n_2 = 2$ . Types 1 and 2 components are presented respectively by blank and black boxes. Assume that this system is equipped which 4 spare components in an active redundancy strategy such that the lifetime of spare and original components are dependent at the condition (5) satisfies.



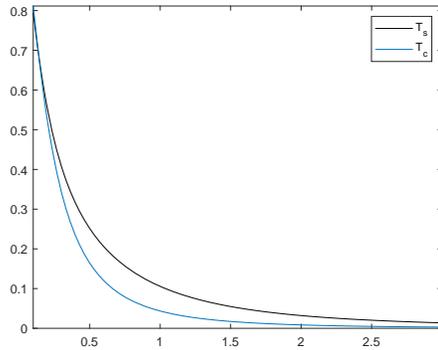
**Figure 1** Systems with two types of components

If the joint distribution of spare and original components is modeled by the multivariate pareto distribution as

$$P(X_1^{(i)} > x_1^{(i)}, X_2^{(i)} > x_2^{(i)}, Y_1^{(i)} > y_1^{(i)}, Y_2^{(i)} > y_2^{(i)}, i = 1, 2) = (1 + \theta_1 \sum_{j=1}^2 x_j^{(1)} + \theta_2 \sum_{j=1}^2 x_j^{(2)} + \gamma_1 \sum_{j=1}^2 y_j^{(1)} + \gamma_2 \sum_{j=1}^2 y_j^{(2)})^{-\alpha} \tag{18}$$

for  $\alpha > 0, \theta_i, \gamma_i > 0$  and  $x_j^{(i)}, y_j^{(i)} > 0, i, j = 1, 2$ . Then, the reliability function of the system with an active redundancy at the component and system level can be calculated by the Eqns. (6) and (7), respectively.

In general, the relationship between these reliability functions depend on the values of  $\theta_1, \theta_2, \gamma_1, \gamma_2$  and  $\alpha$ . So, Comparison is performed for some selected of these parameters. For  $(\theta_1, \theta_2, \gamma_1, \gamma_2) = (1, 2, 1, 2, 3)$ , the reliability function of the system with an active redundancy at the component and system level are plotted in Figure 2 and it can be seen that the reliability function of the system with an active redundancy at the component level is smaller than the reliability function of the system with an active redundancy at the system level.



**Figure 2** The reliability function of the system with an active redundancy at the component ( $T_c$ ) and system( $T_s$ ) level, for  $(\theta_1, \theta_2, \gamma_1, \gamma_2) = (1, 2, 1, 2, 3)$

**Example 2** Consider a coherent system made of 3 components with the different structure functions which are listed in the Table 1. Suppose that the joint distribution of the components is modeled by multivariate Pareto distribution, i.e.

$$P(X_1 > x_1, \dots, X_3 > x_3, Y_1 > y_1, \dots, Y_3 > y_3) = (1 + \theta_1 \sum_{i=1}^3 x_i + \theta_2 \sum_{i=1}^3 y_i)^{-\alpha} \tag{19}$$

for  $x_i > 0, y_i > 0, i = 1, 2, 3$  and  $\alpha, \theta_1, \theta_2 > 0$ . Several properties of this distribution has been discussed by Lai et al. (2001).

For a coherent system with given structure function, the reliability function of a system with an active redundancy at the component (system) level can be simply calculated using Eqn. (13) and Table 1 (Eqn. (14) and Table 2).

For instance, a coherent system with the structure function  $\min(X_1, \max(X_2, X_3))$ , they are given by

$$\begin{aligned}
 P(T_c > t) = & 2(1 + 2t\theta_1)^{-\alpha} + 2(1 + 2t\theta_2)^{-\alpha} + 4(1 + t\theta_1 + t\theta_2)^{-\alpha} \\
 & - 7(1 + 2t\theta_1 + t\theta_2)^{-\alpha} - 7(1 + t\theta_1 + 2t\theta_2)^{-\alpha} + 8(1 + 2t\theta_1 + 2t\theta_2)^{-\alpha} \\
 & + 3(1 + 3t\theta_1 + t\theta_2)^{-\alpha} + 3(1 + t\theta_1 + 3t\theta_2)^{-\alpha} - 3(1 + 3t\theta_1 + 2t\theta_2)^{-\alpha} \\
 & - 3(1 + 2t\theta_1 + 3t\theta_2)^{-\alpha} + (1 + 3t\theta_1 + 3t\theta_2)^{-\alpha} - (1 + 3t\theta_1)^{-\alpha} - (1 + 3t\theta_2)^{-\alpha} \quad (20)
 \end{aligned}$$

and

$$\begin{aligned}
 P(T_s > t) = & 2(1 + 2t\theta_1)^{-\alpha} + 2(1 + 2t\theta_2)^{-\alpha} - (1 + 3t\theta_1)^{-\alpha} \\
 & - (1 + 3t\theta_2)^{-\alpha} - 4(1 + 2t\theta_1 + 2t\theta_2)^{-\alpha} + 2(1 + 3t\theta_1 + 2t\theta_2)^{-\alpha} \\
 & + 2(1 + 2t\theta_1 + 3t\theta_2)^{-\alpha} - (1 + 3t\theta_1 + 3t\theta_2)^{-\alpha} \quad (21)
 \end{aligned}$$

It is obvious that  $F_{X,Y}(x, y) \geq F_X(x)F_Y(y)$ , So,  $(X, Y)$  are positively quadrant dependent (PQD) and thus,  $T_s \leq_{st} T_{s,ind}$  and  $T_c \leq_{st} T_c$ .

**Table 1** Survival signature,  $\Phi(l)$ , of the system with given structure function

Structure function		l			
		0	1	2	3
1	Series	0	0	0	1
2	$\min(X_1, \max(X_2, X_3))$	0	0	$\frac{2}{3}$	1
3	2-out-of-3	0	0	1	1
4	$\max(X_1, \min(X_2, X_3))$	0	$\frac{1}{3}$	$\frac{2}{3}$	1

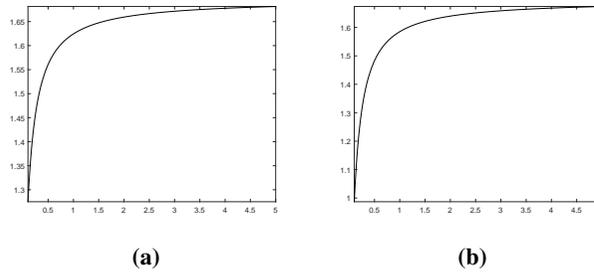
**Table 2** Minimal survival signature,  $\Phi_*(i, j)$ , of the system with given structure function. Model1: Series, Model2:  $\min(X_1, \max(X_2, X_3))$ , Model3: 2-out-of-3, Model4:  $\max(X_1, \max(X_2, X_3))$

i	j	Model1	Model2	Model3	Model4	i	j	Model1	Model2	Model3	Model4
0	0	0	0	0	0	0	1	0	0	0	1
0	2	0	2	3	1	0	3	1	-1	-2	-1
1	0	0	0	0	1	1	1	0	0	0	-1
1	2	0	0	0	-1	1	3	0	0	0	1
2	0	0	2	3	1	2	1	0	0	0	-1
2	2	0	-4	-9	-1	2	3	0	2	6	1
3	0	1	-1	-2	-1	3	1	0	0	0	1
3	2	0	2	6	1	3	3	-1	-1	-4	-1

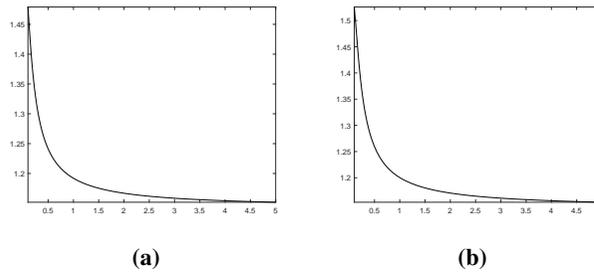
To carry out a stochastic comparison between two lifetime variables  $T_s$  and  $T_c$  in the example 2, the graph of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  are plotted for some values of parameters  $\theta_1, \theta_2$  and  $\alpha$  where  $f_{T_c}(t)$  and  $f_{T_s}(t)$  show the density function of  $T_c$  and  $T_s$ , respectively. As it is seen in figures 3, 5 and 7, these functions were plotted for  $(\theta_1, \theta_2, \alpha) = (2, 3, 2)$ . For a system with series and  $\min(X_1, \max(X_2, X_3))$  structure, they are increasing functions of  $t$ . Therefore,  $T_c$  dominates  $T_s$  in terms of hazard rate and likelihood ratio ordering, i.e.  $T_s \leq_{hr} T_c$  and  $T_s \leq_{lr} T_c$ . For a system with  $\max(X_1, \min(X_2, X_3))$ , they are not monotone. Consequently, the hazard rate and likelihood ratio ordering does not hold.

Figures 4, 6 and 8 plot the functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for  $(\theta_1, \theta_2, \alpha) = (2, 5, 7)$ . For a system with series and  $\min(X_1, \max(X_2, X_3))$  structure, they are decreasing functions of  $t$  then  $T_c \leq_{hr} T_s$  and  $T_c \leq_{lr} T_s$ . But, For a system with  $\max(X_1, \min(X_2, X_3))$  structure, they are increasing functions of  $t$  then  $T_s \leq_{hr} T_c$  and  $T_s \leq_{lr} T_c$ .

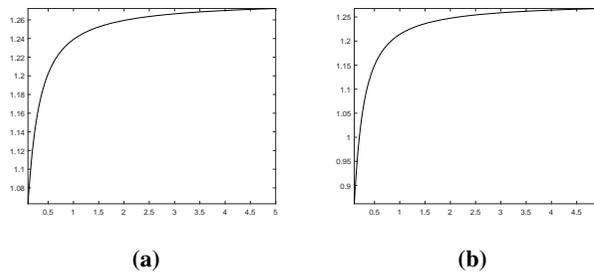
In general, the relation between the lifetime variable of a system with an active redundancy at the component and system level heavily depends on the structure of a system and the values of the parameters. Therefore, an engineer should consider the distribution and the structure of the system for selecting the most effective redundancy type.



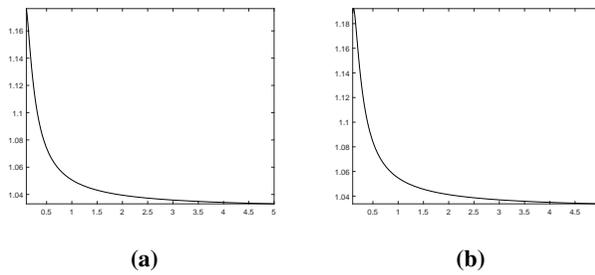
**Figure 3** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a series system and  $(\theta_1, \theta_2, \alpha) = (2, 3, 2)$ , in (a) and (b), respectively



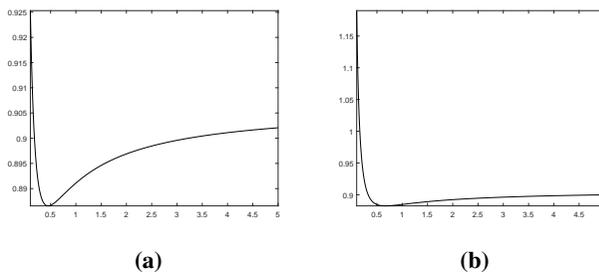
**Figure 4** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a series system and  $(\theta_1, \theta_2, \alpha) = (2, 5, 7)$ , in (a) and (b), respectively



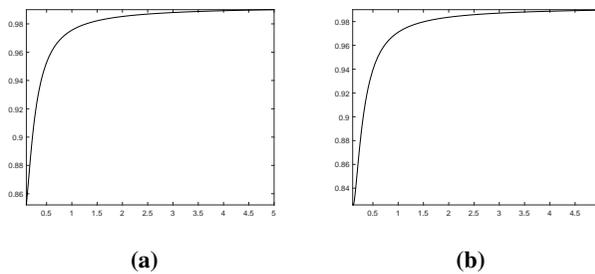
**Figure 5** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a system with structure function  $\min(X_1, \max(X_2, X_3))$  and  $(\theta_1, \theta_2, \alpha) = (2, 3, 2)$ , in (a) and (b), respectively



**Figure 6** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a system with structure function  $\min(X_1, \max(X_2, X_3))$  and  $(\theta_1, \theta_2, \alpha) = (2, 5, 7)$ , in (a) and (b), respectively



**Figure 7** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a system with structure function  $\max(X_1, \min(X_2, X_3))$  and  $(\theta_1, \theta_2, \alpha) = (2, 3, 2)$ , in (a) and (b), respectively



**Figure 8** The graphs of functions  $\frac{P(T_c > t)}{P(T_s > t)}$  and  $\frac{f_{T_c}(t)}{f_{T_s}(t)}$  for a system with structure function  $\max(X_1, \min(X_2, X_3))$  and  $(\theta_1, \theta_2, \alpha) = (2, 5, 7)$ , in (a) and (b), respectively

**5. Conclusion**

In this article, coherent systems with two types of active redundancy were considered at the component and system level. We presented expressions for the reliability of a coherent system that consists of multiple types of dependent components with an active redundancy for both cases. The reliability function of the system were evaluated for the very general case, where the original and spare components are dependent. We illustrated our results for system with  $m = 1$  and  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  system. First, it was shown that in a system with  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  structure, the lifetime variable of a system with an active redundancy at the component level is bigger than the lifetime variable of that system with an active redundancy at system level in terms of hazard rate ordering ( $T_c \geq_{hr} T_s$ ) when  $k_i = 1$  and the original and spare components of type  $i$  have the same distribution

for  $i = 1, \dots, m$ .

In addition, it was shown in a system with  $m = 1$ , If  $(X, Y)$  is positively quadrant dependent (PQD), then the reliability function of system with an active redundancy at system level, when the components are dependent, is less than the reliability function when the components are independent ( $T_s \leq T_{s,ind}$ ), and if  $(X, Y)$  is negatively quadrant dependent (NQD), then  $T_s \geq T_{s,ind}$ .

Finally, some example with a multivariate Pareto distribution were considered and its reliability function with redundancy at both cases were provided. It was shown that in general, the relation between the lifetime variable of a system with an active redundancy at the component and system level heavily depends on the structure of a system and the values of the parameters.

### Acknowledgements

The author thanks the referees for their helpful comments and suggestions, which were very useful in improving the manuscript.

### References

- Boland PJ, El-Newehi E. Component redundancy vs system redundancy in the hazard rate ordering. *IEEE Trans Reliab.* 1995; 44(4): 614-619.
- Coolen FP, Coolen-Maturi T. Generalizing the signature to systems with multiple types of components. In: *Complex systems and dependability*. Berlin: Springer; 2013.
- Eryilmaz S. Systems composed of two types of nonidentical and dependent components. *Naval Res Log.* 2015; 62(5): 388-394.
- Eryilmaz S. Reliability of systems with multiple types of dependent components. *IEEE Trans Reliab.* 2016; 65(2): 1022-1029.
- Eryilmaz S. The effectiveness of adding cold standby redundancy to a coherent system at system and component levels. *Reliab Eng Syst Saf.* 2017; 165: 331-335.
- Eryilmaz S.  $(k_1, k_2, \dots, k_m)$ -out-of- $n$  system and its reliability. *J Comput Appl Math.* 2019; 346: 591-598.
- Eryilmaz S, Coolen FP, Coolen-Maturi T. Mean residual life of coherent systems consisting of multiple types of dependent components. *Naval Res Log.* 2018; 65(1): 86-97.
- Huang J, Dou X, Kuriki S, Lin GD. Dependence structure of bivariate order statistics with applications to Bayramoglus distributions. *J Multivar Anal.* 2013; 114: 201-208.
- Kavlak KB. Reliability and mean residual life functions of coherent systems in an active redundancy. *Naval Res Log.* 2017; 64(1): 19-28.
- Kuo W, Prasad VR. An annotated overview of system-reliability optimization. *IEEE Trans Reliab.* 2000; 49(2): 176-187.
- Lai C, Xie M, Bairamov I. Dependence and ageing properties of bivariate Lomax distribution. *System and Bayesian Reliability: Essays in Honor of Professor Richard E Barlow on His 70th Birthday*. World Scientific; 2001. p. 243-256.
- Laniado, H. Lillo RE. Allocation policies of redundancies in two-parallel-series and two-series-parallel systems. *IEEE Trans Reliab.* 2014; 63(1): 223-229.
- Louie H, Slougher JM. Probabilistic modeling and statistical characteristics of aggregate wind power. In: *Large scale renewable power Generation*. Singapore: Springer; 2014.
- Misra N, Misra AK, Dhariyal ID. Active redundancy allocations in series systems. *Probab Eng Inf Sci.* 2011; 25(2): 219-235.
- Nanda AK, Hazra NK. Some results on active redundancy at component level versus system level. *Oper Res Lett.* 2013; 41(3): 241-245.
- Romera R, Valds JE, Zequeira RI. Active-redundancy allocation in systems. *IEEE Trans Reliab.* 2004; 53(3): 313-318.
- Zhao P, Chan PS, Ng HKT. Optimal allocation of redundancies in series systems. *Eur J Oper Res.* 2012; 220(3): 673-683.

Zhao P, Zhang Y, Li L. Redundancy allocation at component level versus system level. *Eur J Oper Res.* 2015; 241(2): 402-411.

Wang W, Xiong J, Xie, M. Cold-standby redundancy allocation problem with degrading components. *Int J Gen Syst.* 2015; 44(7-8): 876-888.