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On Bivariate Inverse Lindley Distribution Derived From Copula

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Abstract

In the last few decades, copula distribution has become one of the most popular methods to construct bivariate distributions in literature. This paper introduces a new bivariate Inverse Lindley distribution based on the Farlie-Gumbel-Morgenstern (FGM) copula. We study essential mathematical properties and it's application. We have been using the conditional copula distribution method to generate random numbers. Estimation of the parameters for bivariate Inverse Lindley distribution is obtained through maximum likelihood estimation. An example of a real data set is introduced to illustrate the proposed model.

Keywords: Bivariate inverse Lindley distribution, copula, Farlie-Gumbel-Morgenstern copula, maximum likelihood estimate

1. Introduction

It is essential to note that lifetime distributions play a vital role in analyzing the lifetime data in several areas such as medical, engineering, hydrology, etc. Knowing that lifetime distribution's ability to fit the data is also different, the need appears to deal with varying lifetime distributions and compare them based on various criteria to access the most suitable model that fits the data.

In some exceptional cases, we have to deal with two associated lifetime T_1 and T_2 whose marginals are well defined; for example, it may be of interest in studying human organs associated with such as kidney, eyes, lungs, for more detail you can refer Rinne (2008), Bhattacharjee and Misra (2016) and Achcar et al. (2015). The importance of bivariate distributions arises from the ability to cope with different kinds of data which has a parallel clustered system and more than one failure without any order restriction Vincent Raja and Gopalakrishnan (2017). In general, we can notice from the literature that authors derive the bivariate distributions from widespread baseline distribution like Weibull, exponential, Gamma, Pareto, etc.

We can realize that using a standard baseline distribution like Weibull, exponential and Pareto restricted to deal with data that shows non-monotone shapes such as an upside-down bathtub (UBT). In contrast, the significance of inverse distribution raising from the capability to model the upside-down bathtub since its hazard rate displayed the UBT shapes. Accordingly, several authors discussed the inverse distribution and analyzed different data sets showing the (UBT) shapes for the hazard rate for more details see Guo and Gui (2018) and Sharma et al. (2014).

There are various ways of eliciting the bivariate distributions such as Marginal Transformation Method, Copula Method, Conditional Specification Method, and Frailty Approach; further details regarding the methods of establishing bivariate distribution can be found in Vincent Raja and Gopalakrishnan (2017) and Pathak and Vellaisamy (2020).

The copula function is one of the recent topics in statistics; the importance of copula arises as an efficient tool for modeling bivariate and multivariate distributions with broad applications in economics, financial, hydrology and medical data. A copula is a multivariate distribution function whose one-dimension margins are uniform on the interval $[0, 1]$. Here our attention is exclusively on the bivariate copula distribution.

Characterization: $C : [0, 1]^d \to [0, 1]$ is a copula if and only if for every $u, v \in [0, 1]$
$C(u, 0) = 0 = C(0, v)$ [Grounded minimum], $C(u, 1) = u; C(1, v) = v$ [Grounded maximum],
for every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$
$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) > 0$ (2 increasing).

Let X and Y be random variables with joint distribution function F, f density function and marginals $F(X)$ and $F(Y)$ respectively. Sklar (1959) defines copula as a function, which can join or link the joint distribution function to its marginal distribution function via the relation $F(x, y) = P(X \leq x, Y \leq y) = C(F(x), F(y))$ and the associated joint density is $f(x, y) = c(F(x), F(y))f(x)f(y)$, where c is copula density.

In the statistical literature, many authors used copula structure to establish and model bivariate distribution. Kundu and Gupta (2011) construct an absolute continuous bivariate generalized exponential distribution derived on the Clayton copula, also Kundu and Gupta (2017) proposed the bivariate Birnbaum-Saunders distribution from Gaussian copula. Achcar et al. (2015) introduced a Bayesian analysis for a bivariate generalized exponential distribution in the presence of censored data and covariates based on FGM copula functions. Kundu (2015) introduced the bivariate Sinh-normal distribution, based on a bivariate Gaussian copula. Coelho-Barros et al. (2016) used the FGM and Gumbel copula functions to construct bivariate Weibull distributions in the presence of cure fraction and censored data. Elaal and Jarwan (2017) studied a bivariate generalized exponential distribution (BVGE) derived from FGM and Plackett copula functions; they explained two illustrative examples to compare between the estimation methods maximum likelihood, inference functions for marginal and canonical maximum likelihood for the two proposed models using simulated and real data sets. A bivariate modified Weibull distribution embedded by Peres et al. (2018) via a generalized Farlie-Gumbel-Morgenstern copula, taking into consideration the presence of non censored data and censored data, they discussed maximum likelihood and Bayesian approaches for the estimation of the model parameters, Markov chain Monte Carlo (MCMC) methodology used in the Bayesian analysis; estimated the parameters of the posterior distributions by using an example of a real data set to illustrate the proposed methodology. Almetwally et al. (2020) introduced bivariate Weibull derived from FGM copula and discussed essential properties for the bivariate distribution; conducted a simulation study and application of real data for the proposed model.

The article is organized as follows: In Section 2, we recall some notes for the inverse Lindley distribution. In Section 3, we introduce the bivariate inverse Lindley distribution derived from FGM copula (BILD). Several properties have been discussed in Section 4. Estimation of bivariate inverse Lindley distribution has been discussed in Section 5. The simulation study is illustrated in Section 6. Application of real data analysis and a comparison study is presented in Section 7. Finally, some consequences and conclusions are addressed in Section 8.

2. Inverse Lindley Distribution

IA continuous random variable Z is said to follow Inverse Lindley distribution if the probability density function (PDF) is of the form

$$f(z) = \frac{\eta^2 e^{\frac{-\eta}{z}}(1+z)}{(1+\eta)z^3}; \qquad z, \eta > 0, \tag{1}$$

and is denoted by ILD (η).

The cumulative density fuction (CDF) of Z is given by

$$F(z) = e^{-\frac{\eta}{z}} \left(1 + \frac{\eta}{(1+\eta)z}\right); \qquad z, \eta > 0, \tag{2}$$

for $z \geq 0$ and 0 otherwise.

The hazard function is defined as

$$h(z) = \frac{\eta^2(1+z)}{z^2\left[(1+\eta)z\left(e^{\eta/z} - 1\right) - \eta\right]}; z, \eta > 0. \tag{3}$$

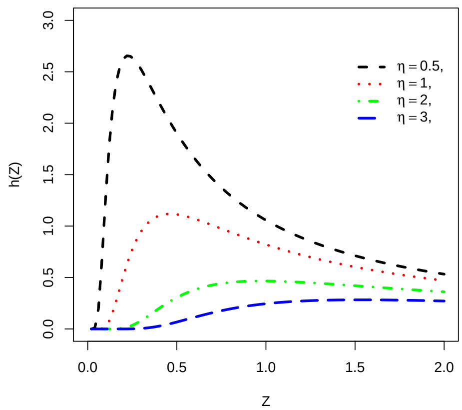

Figure 1 Hazard rate function for IL(η) for different values of the parameter η

3. Model

As previously described in the introduction a copula C is a multivariate cummulative distribution function on $[0,1]^d$ with uniform (0, 1) marginals. There are several kinds of copula discussed in the literature with different properties and featured, here we are using FGM copula to represent the bivariate inverse Lindley distribution. FGM copula one of the most popular family. It had been introduced primarily by Morgenstern (1956); the expression of the joint distribution function considering the FGM copula,

$$C(u,v) = uv[1 + \theta(1-u)(1-v)] \tag{4}$$

where the marginal function $u = F(x)$ and $v = F(y) \in I$, $\theta \in [-1, 1]$ is a dependence parameter and the density function is given by $c(u,v) = [1 + \theta(1-2u)(1-2v)]$.

3.1. Bivariate inverse Lindley distribution

Bivariate inverse Lindley distribution based on FGM copula and by used Skalar's theorem and from Eqn. (4).

$$\begin{aligned} F(x,y) &= C(F(x), F(y)) \\ &= e^{-\frac{\eta}{x}-\frac{\beta}{y}} \left[1 + \frac{\eta}{x+\eta x}\right]\left[1 + \frac{\beta}{y+\beta y}\right] \\ &\quad \times \left[1 + \theta\left(1 - e^{-\frac{\eta}{x}}\left[1 + \frac{\eta}{x+\eta x}\right]\right)\left(1 - e^{-\frac{\beta}{y}}\left[1 + \frac{\beta}{y+\beta y}\right]\right)\right]. \end{aligned}$$

The bivariate inverse Lindley density based on FGM copula and by used Skalar's theorem.

$$\begin{aligned} f(x,y) &= f(x)f(y)c(F(x),(F(y)) \\ &= \frac{\eta^2\beta^2 e^{-\frac{\eta}{x}-\frac{\beta}{y}}(1+x)(1+y)\left[1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right]}{(1+\eta)(1+\beta)x^3y^3}. \end{aligned} \tag{5}$$

3.2. Reliability function

There are several ways to construct the reliability function for the bivariate distribution. We prefer to use the copula approach to express the reliability function for the BILD by using the marginal reliability function $S(x)$ and $S(y)$ where X and Y the random variable and selection dependence structure, i.e., copula, see Nelsen (2006) for details.

The joint reliability function for the copula is as following

$$S(x,y) = \hat{C}(S(x),S(y)),$$

where the marginal reliability function $u = S(x)$ and $v = S(y)$. The reliability function $S(x,y)$ of BILD

$$\begin{aligned} S(x,y) &= 1-e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)-e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right) \\ &\quad +e^{-\frac{\eta}{x}-\frac{\beta}{y}}\left(1+\frac{\eta}{x+\eta x}\right)\left(1+\frac{\beta}{y+\beta y}\right)\left[1+e^{-\frac{\eta}{x}-\frac{\beta}{y}}\theta\left(1+\frac{\eta}{x+\eta x}\right)\left(1+\frac{\beta}{y+\beta y}\right)\right] \end{aligned}$$

in the next Figures 2 - 5 three dimension graphs have been reported for the CDF, PDF, reliability and hazard functions of BILD, for the numerous values parameter η = 0.4(pink), 0.8(red), 1.3(blue), 1.6(green), numerous values of β = 0.5(pink), 0.2(red), 1.2(blue), 0.4(green) and numerous values of copula parameter θ = 0.1(pink), 0.7(red), -0.1(blue), -0.5(green).

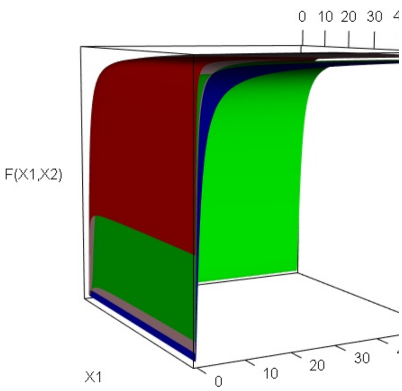

Figure 2 CDF FGM-BILD

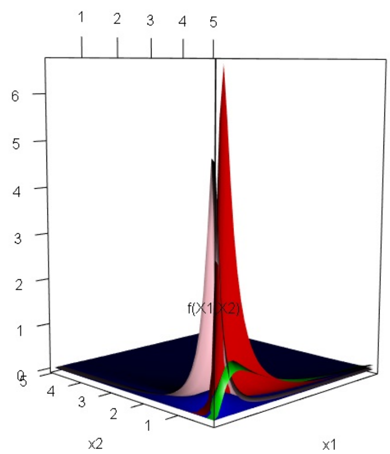

Figure 3 PDF FGM-BILD

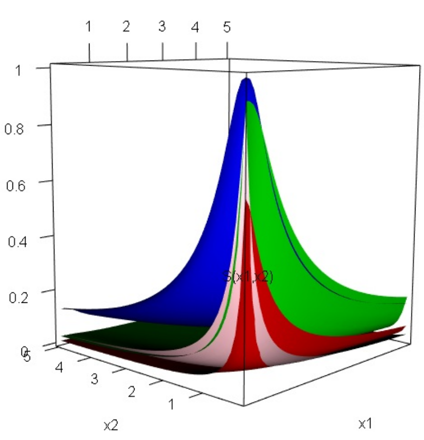

Figure 4 Reliability FGM-BILD

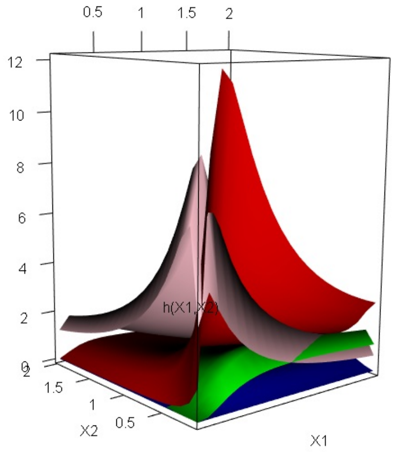

Figure 5 Hazard FGM-BILD

4. Materials and Methods

4.1. Some statistical properties

Some statistical properties have been reported for the bivariate inverse Lindley distribution derived from FGM copula with $\eta > 0$, $\beta > 0$, $\theta \in [-1, 1]$ as shown in the following: Let $(X, Y) \sim$ BILD(η, β, θ), $X \sim$ IL(η) and $Y \sim$ IL(β), then the following result can be easily derived:

(i) the marginal distribution of X and Y can be found by solving the integral of the BILD density from Eqn. (5)

$$\begin{aligned} f(x, \eta) &= \int_0^\infty f(x, y)\, dy = \frac{\eta^2 e^{\frac{-\eta}{x}}(1+x)}{(1+\eta)x^3}; x > 0, \eta > 0 \\ f(y, \beta) &= \int_0^\infty f(x, y)\, dx = \frac{\beta^2 e^{\frac{-\beta}{y}}(1+y)}{(1+\beta)y^3}; y > 0, \beta > 0 \end{aligned}$$

(ii) the conditional density of X given Y is

$$f(x|y) = \frac{\eta^2 e^{-\frac{\eta}{x}}(1+x)\left[1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right]}{(1+\eta)x^3}$$

(iii) the conditional distribution of X given Y is

$$F(x|y) = e^{-\frac{\eta}{x}}\left[1+\frac{\eta}{x+\eta x}\right]\left[1+\theta\left(1-e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2\,e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right]$$

(iv) the conditional reliability function of X given Y is

$$S(x|y) = 1-e^{-\frac{\eta}{x}}\left[1+\frac{\eta}{x+\eta x}\right]\left[1+\theta\left(1-e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2\,e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right]$$

In the next result, we obtain the expression of moment generating and product moments for the BILD.

Theorem 1 *If* $(X,Y) \sim$ *BILD*(η,β,θ), *then the moment generating function can be expressed as follows,*

$$\begin{aligned} M_{(x,y)}\left(t_1,t_2\right) &= \sum_{m=0}^{\infty}\left(\frac{t_1^m}{m!}\right)\sum_{n=0}^{\infty}\left(\frac{t_2^n}{n!}\right)\frac{\eta^m\beta^n m\Gamma(1-m)\Gamma(-n)}{16(1+\eta)^2(1+\beta)^2}\left[2^m\mathrm{B}\left(2^n\mathrm{A}-4(1+\beta)(1+\beta-n)\right)\theta\right. \\ &\quad \left.+4(1+\eta)(1+\eta-m)\left(-2^n\mathrm{A}\,\theta+4(1+\beta)(1+\beta-n)(1+\theta)\right)\right] \end{aligned}$$

$\mathrm{A}=\left(4(1+\beta)^2-(5+4\beta)n+n^2\right),\ \mathrm{B}=\left(4(1+\eta)^2-(5+4\eta)m+m^2\right).$

Proof: From the bivariate density in Eqn. (5), calculate the following integral

$$\mathbf{M}_{(x,y)}\left(t_1,t_2\right)=E\left(e^{t_1x}e^{t_2y}\right)=\int_0^{\infty}\int_0^{\infty}e^{t_1x}e^{t_2y}f\left(x,y\right)dxdy$$

after simplification, we got the result in Theorem 1.

Theorem 2 *If* $(X,Y) \sim$ *BILD*(η,β,θ), *then the* r^{th} *and* s^{th} *moments around zero is given by*

$$\begin{aligned} \mu'_{rs} &= \frac{\eta^r\beta^s s\Gamma(1-r)\Gamma(-s)}{16(1+\eta)^2(1+\beta)^2}\left[2^rB\left(2^sA-4(1+\beta)(1+\beta-s)\right)\theta\right. \\ &\quad \left.+4(1+\eta)(1+\eta-r)\left(-2^sA\,\theta+4(1+\beta)(1+\beta-s)(1+\theta)\right)\right] \end{aligned}$$

$$A=\left(4(1+\beta)^2-(5+4\beta)s+s^2\right),\ B=\left(4(1+\eta)^2-(5+4\eta)r+r^2\right).$$

Proof: To prove Theorem 2 and by using Eqn. (5), start with

$$\mu'_{rs} = E\left(X^rY^s\right)=\int_0^{\infty}\int_0^{\infty}x^ry^sf\left(x,y\right)dxdy$$

after simplification, we got the result.

4.2. Bivariate failure rate

The bivariate failure (hazard) rate function for BILD derived from FGM copula is defined as (see Basu 1971).

$$h(x,y) = \frac{f(x,y)}{s(x,y)}$$

$$= \frac{\eta^2(1+\eta)\beta^2(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}(1+x)(1+y)\left(1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right)}{x\left(\eta+x+\eta x-(1+\eta)e^{\eta/x}x\right)y\left(\beta+y+\beta y-(1+\beta)e^{\beta/y}y\right)\left((1+\eta)(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}xy+\theta(\eta+x+\eta x)(\beta+y+\beta y)\right)}.$$

4.3. Hazard gradient function

Johnson and Kotz (1975) defined the hazard components function as

$$\begin{aligned} \zeta_1(x,y) &= -\frac{\partial}{\partial x}\ln S(x,y) \\ \zeta_2(x,y) &= -\frac{\partial}{\partial y}\ln S(x,y). \end{aligned}$$

Here, we found the hazard gradient function for the BILD, knowing that $\zeta_1(x,y)$ is the failure rate of X with given information $Y>y$,

$$\begin{aligned}\zeta_1(x,y) &= \zeta_1(x|Y>y)\\ &= -\frac{\eta^2(1+x)\left((1+\eta)(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}xy+\theta\left(-(1+\eta)e^{\eta/x}x+2(\eta+x+\eta x)\right)(\beta+y+\beta y)\right)}{x^2\left(\eta+x+\eta x-(1+\eta)e^{\eta/x}x\right)\left((1+\eta)(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}xy+\theta(\eta+x+\eta x)(\beta+y+\beta y)\right)}\end{aligned}$$

and similarly $\zeta_2(x,y)$ is the failure rate of Y given $X>x$,

$$\begin{aligned}\zeta_2(x,y) &= \zeta_2(x|Y>y)\\ &= -\frac{\beta^2(1+y)\left((1+\eta)(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}xy+\theta\left(-(1+\eta)e^{\eta/x}x+2(\eta+x+\eta x)\right)(\beta+y+\beta y)\right)}{y^2\left(\beta+y+\beta y-(1+\beta)e^{\beta/y}y\right)\left((1+\eta)(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}xy+\theta(\eta+x+\eta x)(\beta+y+\beta y)\right)}\end{aligned}$$

The vector $(\zeta_1(x,y),\zeta_2(x,y))$ are named as the hazard gradient of a bivariate random vector (X,Y). The conditional failure (hazard) rate function for the BILD, $\zeta(x|Y=y)$ of X given $Y=y$ and $\zeta(y|X=x)$ of Y given $X=x$ are

$$\begin{aligned}\zeta(x|Y=y) &= -\frac{f(x|y)}{S(x|y)}\\ &= \frac{-\eta^2e^{-\frac{\eta}{x}}(1+x)\left(1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right)}{(1+\eta)x^3\left[1-e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\left(1+\theta\left(1-e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right)\right]}\end{aligned}$$

and

$$\begin{aligned}\zeta(y|X=x) &= -\frac{f(y|x)}{S(y|x)}\\ &= \frac{-\beta^2e^{-\frac{\beta}{y}}(1+y)\left(1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-2e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right)}{(1+\beta)y^3\left[1-e^{-\frac{\eta}{y}}\left(1+\frac{\eta}{y+\beta y}\right)\left(1+\theta\left(1-2e^{-\frac{\eta}{x}}\left(1+\frac{\eta}{x+\eta x}\right)\right)\left(1-e^{-\frac{\beta}{y}}\left(1+\frac{\beta}{y+\beta y}\right)\right)\right)\right]}.\end{aligned}$$

In the following subsections, we explore some local dependence measures for the BILD and introduce its essential properties.

4.4. Local dependence function

Holland and Wang (1987) defined a local dependence function $\gamma(x,y)$ for random variables X and Y as

$$\gamma(x,y) = \frac{\partial^2}{\partial x\partial y}lnf(x,y)$$

we are interested in studying the dependence function for a particular cause. It can measure the total positive of order 2 (TP2) property of a bivariate distribution easily. In studying the dependence; (TP2) is consider to be a useful tool; for more details, see Holland and Wang (1987) and Balakrishnan and Lai (2009).

Proposition 1 Let $(X,Y)\sim$ BILD(η,β,θ). Then the local dependence $\gamma(x,y)$ for BILD is

$$\begin{aligned}\gamma_1(x,y) &= \frac{\partial^2}{\partial x\partial y}lnf(x,y)\\ &= \frac{4\eta^2(1+\eta)\beta^2(1+\beta)e^{\frac{\eta}{x}+\frac{\beta}{y}}\theta(1+x)(1+y)}{xy\left((1+\beta)e^{\beta/y}\left((1+\eta)e^{\eta/x}(1+\theta)x-2\theta\mathrm{A}\right)y+2\theta\left(-(1+\eta)e^{\eta/x}x+2\mathrm{A}\right)\mathrm{B}\right)^2}\end{aligned} \quad (6)$$

where $\mathrm{A}=(\eta+x+\eta x)$ and $\mathrm{B}=(\beta+y+\beta y)$.

If we substitute $\theta = 0$ in Eqn. (6), then $\gamma_1(x, y) = 0$, which results in independence between x and y. According to Holland and Wang (1987), a bivariate density $f(x, y)$ satisfies the TP2 property if and only if $\gamma_1(x, y) \geq 0$. So, we can write down the result as follow:

Proposition 2 Let $(X, Y) \sim$ BILD(η, β, θ). Then for $\theta \geq 0$ the density $f(x, y)$ given in (5) is TP2.

Since the TP2 is a strong dependence, so that if any function is TP2, then the other dependence properties are satisfied like: stochastically increasing (SI), right-tail increasing (RTI), association, and positive quadrant dependence (PQD) (see Nelsen (2006) and Balakrishnan and Lai (2009). The BILD has all these properties of dependence if and only if $0 \leq \theta \leq 1$.

4.5. Clayton-Oakes association measure

Oakes (1989) discussed Clayton-Oakes association measure by using the density function and reliability function then it can be defined as:

$$l(x_1, x_2) = \frac{f(x, y)S(x, y)}{S_1(x, y)S_2(x, y)}$$

where $S_1(x, y) = \frac{\partial}{\partial x}S(x, y)$ and $S_2(x, y) = \frac{\partial}{\partial y}S(x, y)$.

Proposition 3 Let $(x, y) \sim$ BILD (η, β, θ) derived from FGM copula, then

$$l_1(x_1, x_2) = \frac{(1+\eta)^3(1+\beta)^3(1-\mathrm{C})(1-\mathrm{D})e^{\frac{3\eta}{x}+\frac{3\beta}{y}}x^3y^3(1+\mathrm{CD}\theta)[1+(1-2\mathrm{C})(1-2\mathrm{D})\theta]}{\mathrm{F}\left(\mathrm{A}-(1+\eta)e^{\eta/x}x\right)\left(\mathrm{B}-(1+\beta)e^{\beta/y}y\right)\left[2\mathrm{AB}\theta+(1+\eta)e^{\eta/x}x\left((1+\beta)e^{\beta/y}y-\mathrm{B}\theta\right)\right]}.$$

$$\mathrm{A} = (\eta + x + \eta x), \mathrm{B} = (\beta + y + \beta y), \mathrm{C} = e^{-\frac{\eta}{x}}\left(1 + \frac{\eta}{x + \eta x}\right), \mathrm{D} = e^{-\frac{\beta}{y}}\left(1 + \frac{\beta}{y + \beta y}\right)$$

and $\mathrm{F} = \left(2\mathrm{AB}\theta + (1+\beta)e^{\beta/y}\left(-\mathrm{A}\theta + (1+\eta)e^{\eta/x}x\right)y\right)$.

5. Estimation of Bivariate Inverse Lindley Distribution Based on Copula

In this section, we are discussing the maximum likelihood estimator to estimate the parameters of the BILD. To compute the maximum likelihood estimation (MLE), we need to solve a three-dimensional optimization problem simultaneously, using the joint density function $f(x, y; \eta, \beta, \theta)$ from Eqn. (5),
let $a(x, \eta) = \left(1 - 2e^{-\frac{\beta}{x}}\left(1 + \frac{\eta}{x+\eta x}\right)\right)$ and $a(y, \beta) = \left(1 - 2e^{-\frac{\beta}{y}}\left(1 + \frac{\beta}{y+\beta y}\right)\right)$,
then maximize likelihood function can be written as:

$$L = \left(\frac{\eta\beta}{(1+\eta)(1+\beta)}\right)^n \prod_{i=1}^{n}\left(\frac{(1+x_i)(1+y_i)}{x_i^3 y^{3_i}}\right) e^{-\sum_{i=1}^{n}\left(\frac{\eta}{x_i}\right)-\sum_{i=1}^{n}\left(\frac{\beta}{y_i}\right)} \times \prod_{i=1}^{n}\left(1 + \theta\, a(x_i, \eta)\; a(y_i, \beta)\right)$$

and the loglikelihood function can be written as:

$$\begin{aligned} \ln L \;=\; & n\left(\ln \eta^2 - \ln(1+\eta)\right) + n\left(\ln \beta^2 - ln(1+\beta)\right) + \sum_{i=1}^{n}\ln\left(\frac{1+x_i}{x_i^3}\right) - \sum_{i=1}^{n}\left(\frac{\eta}{x_i}\right) \\ & + \sum_{i=1}^{n}\ln\left(\frac{1+y_i}{y_i^3}\right) + -\sum_{i=1}^{n}\left(\frac{\beta}{y_i}\right) + \sum_{i=1}^{n}\ln\left(1 + \theta\left(a(x_i, \eta)\right)\left(a(y_i, \beta)\right)\right) \end{aligned} \qquad (7)$$

diffrentiate Eqn. (7) partially with respect to η , β and θ separately, as following

$$\frac{\partial \ln L}{\partial \eta} = \frac{n(2+\eta)}{\eta(1+\eta)} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + \sum_{i=1}^{n}\frac{2\eta e^{\frac{-\eta}{x_i}}\theta(1+\eta+2x_i+\eta x_i)a(y_i,\beta)}{(1+\eta)^2 x_i^2(1+\theta(a(x_i,\eta)a(y_i\beta)))}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n(2+\beta)}{\beta(1+\beta)} - \sum_{i=1}^{n}\left(\frac{1}{y_i}\right) + \sum_{i=1}^{n}\frac{2\beta e^{\frac{-\beta}{x_i}}\theta(1+\beta+2y_i+\beta y_i)a(x_i,\beta)}{(1+\beta)^2 x_i^2(1+\theta(a(x_i,\eta)a(y_i\beta)))}$$

and

$$\frac{\partial \ln L}{\partial \theta} = \sum_{i=1}^{n}\frac{(a(x_i,\eta))(a(y_i,\beta))}{1+\theta(a(x_i,\eta)a(y_i,\beta))}.$$

The MLE $(\hat{\eta},\hat{\beta},\hat{\theta})$ can be obtained by solving likelihood equations simultantaneously, the estimate of $\hat{\eta}$, $\hat{\beta}$ and $\hat{\theta}$ are handled numerically through statistical software,

$$\left.\frac{\partial \ln L}{\partial \eta}\right|_{\eta=\hat{\eta}} = 0, \quad \left.\frac{\partial \ln L}{\partial \beta}\right|_{\beta=\hat{\beta}} = 0, \quad \left.\frac{\partial \ln L}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0,$$

there is no closed form expression for MLE $(\hat{\eta},\hat{\beta},\hat{\theta})$ the estimate of the parameters are handled numerically using a non-linear optimization algorithm.

6. Results

6.1. Simulation study

We are establishing the simulation study for the BILD derived from the FGM copula for MLE estimation. According to Nelsen (2006), we can use the condition distribtion function of copula to generate random variate from the bivariate $f(x,y) = f(x)f(x|y)$. By using the following procedures, we can generate random samples from BILD by using the conditional approach:

Let X and Y be a random variables whose joint distribution function is BILD. The FGM copula $C(u,v) = uv[1+\theta(1-u)(1-v)]$ and so the conditional distribution function is $c_u(v) = \frac{\partial}{\partial u}C(u,v) = v[1+\theta(1-v)(1-2u)]$. Thus, the random numbers (x_i,y_i) can be generated by using the algorithm:

1) From uniform distribution $U(0,1)$ generate two independent sample u and t.
2) Set $t = \frac{\partial}{\partial u}C(u,v)$ and solved for v.
3) Find $X = F^{-1}(u;\eta)$ and $Y = F^{-1}(v;\beta)$; where F^{-1} is the inverse of the CDF ILD.
4) Finally, set $x = X$ and $y = Y$.

A simulation study is carrying out based on the following data generalized from BILD the value of the parameters η and β is choosen with different value of the copula parameter θ and different sizes of sample (n = 15,30,50,100), as shown for the following cases for the random variables generating from FGM-BILD:
case 1:$(\eta = 0.5, \beta = 0.5, \theta = 0.3)$
case 2:$(\eta = 1.5, \beta = 2.5, \theta = 0.7)$
case 3:$(\eta = 3.5, \beta = 3, \theta = -0.3)$
case 4:$(\eta = 4.5, \beta = 4, \theta = -0.7)$.

The estimate of parameters by the MLE estimation for the simulation study based on the 10000 replications are summarized in Tables 1 - 4. Based on the tables, some consequences can be drawn: In the simulation study, if the sample size increases, the value of mean square error and length of confidence interval decreases.

In the simulation tables, when the parameters η, β and θ increase, then the corresponding mean square error also increases for the small sample and after that decreases gradually by increasing the sample size; on the contrary, the simulation table does not affect by the increasing of the dependence parameter θ as observed from the corresponding MSE for different values of the dependence parameter.

In general, the effect of marginal parameters has a negligible impact on estimating the copula parameters, as shown in the tables. We can remark that by increasing the marginal parameters, the MSE increases for the small sample and decreases gradually as the sample size increases. The data analysis and the simulation study were accomplished by the R software (version 3.5.3).

Table 1 (MLE) Simulation study of the parameters of FGM-BILD (case 1)

n	parameter	Estimate	MSE	L.C.I.	U.C.I.
	$\eta(0.5)$	0.5117	0.0214	0.3516	0.7488
15	$\beta(0.5)$	0.5246	0.0233	0.3564	0.7677
	$\theta(0.5)$	0.4562	0.1010	0.0184	0.9613
	$\eta(0.5)$	0.5075	0.0056	0.3936	0.6559
30	$\beta(0.5)$	0.5099	0.0048	0.3945	0.6559
	$\theta(0.3)$	0.4550	0.0945	0.0248	0.9573
	$\eta(0.5)$	0.5076	0.0034	0.4137	0.6249
50	$\beta(0.5)$	0.5051	0.0043	0.4120	0.6168
	$\theta(0.3)$	0.4276	0.0812	0.0255	0.9367
	$\eta(0.5)$	0.5030	0.0013	0.4352	0.5798
100	$\beta(0.5)$	0.5024	0.0013	0.4350	0.5786
	$\theta(0.3)$	0.3805	0.0574	0.0236	0.8576

Table 2 (MLE) Simulation study of the parameters of FGM-BILD (case 2)

n	parameter	Estimate	MSE	L.C.I.	U.C.I.
	$\eta(1.5)$	1.5824	0.1187	1.0595	2.3603
15	$\beta(2.5)$	2.6508	0.3755	1.7548	4.0385
	$\theta(0.7)$	0.5082	0.1128	0.0306	0.9723
	$\eta(1.5)$	1.5430	0.0521	1.1748	2.0417
30	$\beta(2.5)$	2.5787	0.1631	1.9226	3.4857
	$\theta(0.7)$	0.5531	0.0951	0.0396	0.9767
	$\eta(1.5)$	1.5189	0.0296	1.2232	1.8977
50	$\beta(2.5)$	2.5391	0.0943	2.0219	3.2105
	$\theta(0.7)$	0.5848	0.0770	0.0743	0.9767
	$\eta(1.5)$	1.5132	0.0142	1.2973	1.7668
	$\beta(2.5)$	2.5244	0.0440	2.1488	2.9789
	$\theta(0.7)$	0.6395	0.0506	0.1646	0.9746

Table 3 (MLE) Simulation study of the parameters of FGM-BILD (case 3)

n	parameter	Estimate	MSE	L.C.I.	U.C.I.
15	$\eta(3.5)$	3.6907	0.7633	2.4008	5.7022
	$\beta(3)$	3.1538	0.5498	2.0773	4.8963
	$\theta(-0.3)$	-0.4593	0.0999	-0.9602	-0.0210
30	$\eta(3.5)$	3.5855	0.3335	2.6585	4.8945
	$\beta(3)$	3.0783	0.2436	2.2894	4.1390
	$\theta(-0.3)$	-0.4476	0.0945	-0.9547	-0.0204
50	$\eta(3.5)$	3.5529	0.1824	2.8305	4.4891
	$\beta(3)$	3.0403	0.1312	2.4162	3.7930
	$\theta(-0.3)$	-0.4246	0.0812	-0.9271	-0.0260
100	$\eta(3.5)$	3.5277	0.0897	2.9892	4.1734
	$\beta(3)$	3.0139	0.0634	2.5721	3.5476
	$\theta(-0.3)$	-0.3837	0.0588	-0.8583	-0.0247

Table 4 (MLE) Simulation study of the parameters of FGM-BILD (case 4)

n	parameter	Estimate	MSE	L.C.I.	U.C.I.
15	$\eta(4.5)$	4.7357	1.3513	3.0197	7.4656
	$\beta(4)$	4.2469	1.0661	2.7547	6.6520
	$\theta(-0.7)$	-0.5153	0.1115	-0.9718	-0.0283
30	$\eta(4.5)$	4.6313	0.5956	3.4050	6.3344
	$\beta(4)$	4.1149	0.4584	3.0288	5.6529
	$\theta(-0.7)$	-0.5550	0.0913	-0.9748	-0.0462
50	$\eta(4.5)$	4.5594	0.3316	3.5884	5.8246
	$\beta(4)$	4.0632	0.2556	3.2034	5.1671
	$\theta(-0.7)$	-0.5958	0.0747	-0.9776	-0.0789
100	$\eta(4.5)$	4.5372	0.1585	3.8526	5.3894
	$\beta(4)$	4.0301	0.1229	3.4200	4.7935
	$\theta(-0.7)$	-0.6381	0.0523	-0.9776	-0.1541

7. Application of Real Data and Discussion

To study the proposed model BILD and elucidate the MLE estimation procedure, We consider the drought data for (Panhandle) climate division of Nebraska state; the real drought data set is demonstrated for the 83 drought events in climate division (Panhandle), we got the data from Nadarajah (2009). The data comprises the monthly modified Palmer Drought Severity Index (PDSI) from January 1895 to December 2004. The PDSI is often used to measure droughts depend on recent precipitation and temperature; see Alley (1984) for details; when the PDSI is less than zero, then drought is said to have been occurring; see Yevjevich (1967). The bivariate data sets x and y represent the duration of drought and non-drought, respectively. The BILD was determined by fitting the model to the observed values.

To verify that IL distribution fits the data set x (drought) and y (non-drought) accurately, we calculate Kolmogrove-Smirnov (K-S) goodness of fit test statistics and maximize log-likelihood (LL) for the marginal of IL distribution. For the drought data set, the K-S is 0.1927 and -LL is 215.5371; similarly, for the non-drought data set, the K-S is 0.2530 and -LL is 242.2437. Therefore, the IL Distribution can be used to fit the data as shown from the previous result, Figures 6 and 7.

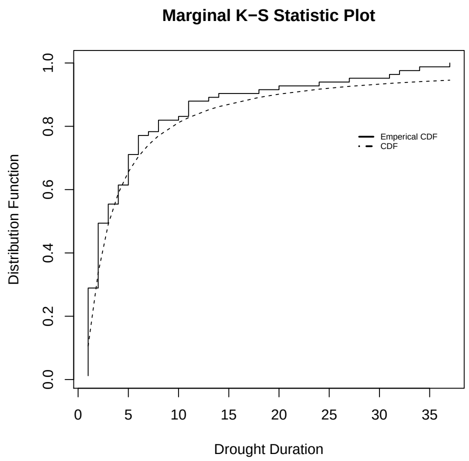

Figure 6 K-S plot for dataset x

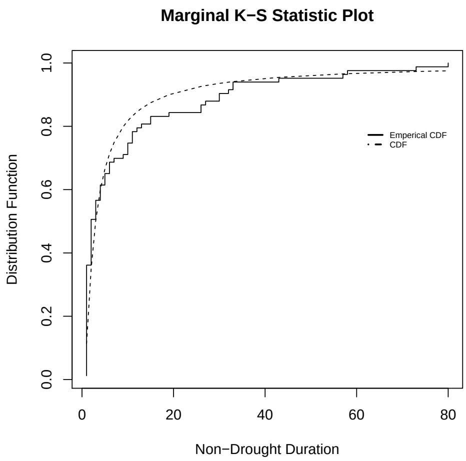

Figure 7 K-S plot for dataset y

The parameters estimation are displayed in Table 5 for the proposed BILD model derived from FGM copula using the MLE estimate.

Table 5 MLE estimate for BIL disribution derived from FGM copula

Copula	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	-LL	AIC
FGM	2.6859	2.5898	0.2855	457.2872	920.5744

In the following, for the comparison reason, we have fitted the Bivariate Generalized Exponential distribution based in Clayton copula (ClaytonBGED); which was discussed by Kundu and Gupta (2011), FGM Bivariate Generalized Exponential (FGMBGED); which was introduced by Elaal and Jarwan (2017) and FGM Bivariate Weibull distribution (FGMBWD); which was discussed by Almetwally et al. (2020). Also, we compare our model with other models that are not coming from the copula approach Nadarajah (2007, 2009) fitted the drought and non-drought data by using the bivariate gamma distribution (BGD) and bivariate Pareto distribution (BPD). Table 6 demonstrate that BILD derived from FGM copula has the smallest value for the Akaike information criterion (AIC) among different models.

Table 6 The MLEs of the parameters, copula parameter estimate, the log-likelihood values and AIC values

Copula	Model	MLE	-LL	AIC
FGM	BILD	$\hat{\eta}$= 2.6859 ,$\hat{\beta}$= 2.5898,$\hat{\theta}$= 0.2855	457.2872	920.5744
	BGD	$\hat{\theta}_1$= 0.337 ,$\hat{\theta}_2$= 0.252	550.6	1105.2
	BPD	$\hat{\theta}_1$=2.131 ,$\hat{\theta}_2$= 11.034,$\hat{\theta}$= 8.127	491.863	989.726
Clayton	BGED	$\hat{\lambda}_1$=0.1639 ,$\hat{\lambda}_2$= 0.9763 ,$\hat{\theta}$=0.7543 $\hat{\lambda}_3$= 0.0718 $\hat{\lambda}_4$= 0.6219	495.1963	1000.393
FGM	BGED	$\hat{\lambda}_1$= 0.1639 ,$\hat{\lambda}_2$= 0.9763,$\hat{\theta}$=0.4503 $\hat{\lambda}_3$= 0.0718 $\hat{\lambda}_4$=0.6219	497.0947	1004.189
FGM	BWD	$\hat{\lambda}_1$=0.2112 ,$\hat{\lambda}_2$= 0.8990,$\hat{\theta}$=0.4177 $\hat{\lambda}_3$= 0.2389 $\hat{\lambda}_4$= 0.7094	491.9845	993.969

8. Conclusions

In this paper, we introduced the BILD derived from FGM copula as an alternative distribution to analyze bivariate lifetime data. Several properties of the BILD have been discussed, like the conditional distribution, moment generating function and some concepts regarding the bivariate failure rate and the hazard gradient function. We have debated TP2 property via the local dependence function and indicated its relation with it. Parameters were estimated for the application of real data by utilizing MLE estimation. It seems from the application of real data that BILD derived from FGM copula is successfully fitting the data as well as for the simulation study, and the model FGM-BILD provides better results than other models.

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