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Statistical Design of a One-sided CUSUM Control Chart to Detect a Mean Shift in a FIMAX Model with Underlying Exponential White Noise

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Abstract

Control charts comprise an important statistical technique used to monitor the quality of a process, of which the cumulative sum (CUSUM) chart is effective at detecting small-to-moderate shifts in the parameter of interest of a process such as a fractionally integrated moving average with exogenous variables (FIMAX) model with underlying exponential white noise. When a specific size of the mean shift is assumed, the CUSUM control chart can be optimally designed in terms of the average run length (ARL). Herein, the ARL is derived by using analytical integral equations as explicit formulas with proven existence and uniqueness based on Banach's fixed-point theorem. This approach was compared with the ARL derived by using the numerical integral equation (NIE) method for out-of-control situations. In simulations studies, the precision of the proposed ARL method based on explicit formulas achieved the same accuracy as the NIE method in terms of percentage accuracy for a wide variety of out-of-control situations. Meanwhile, the computational time required for the explicit formulas was far shorter than for the NIE method. Furthermore, we illustrate the practicability of the explicit formulas method by using real data following a FIMAX model with exponential white noise running on a CUSUM control chart.

Keywords: Average Run Length (ARL), integral equation, fractionally integrated moving average process with exogenous variable.

1. Introduction

Control charts comprise a statistical technique that plays an important role in monitoring the quality of processes in the manufacturing and manufacturing sectors by detecting changes in a process. There are three main types of charts. The first is the Shewhart control chart, which is the most popular chart because of its simple derivation and sensitivity to large shifts in a process parameter. However, it is ineffective at detecting small-to-moderate changes in a process parameter, and due to this disadvantage, researchers have developed control charts based on sampling designs. The cumulative sum (CUSUM) control chart introduced by Page (1954) and the exponentially weighted moving average (EWMA) control chart primarily suggested by Roberts (1959) are applicable in these scenarios. Several researchers have reviewed these charts from different perspectives (see for example Brook and Evans 1972, Hawkins 1993, Chatterjee and Qiu 2009). The CUSUM control chart is extensively used for the efficient monitoring of internal quality control parameters. In addition, its use in analytical laboratories has been highlighted by Hibbert (2007). The CUSUM control chart appears to be more suited to the requirements of laboratory control (Kateman and Buydens 1993)) than the Shewhart control chart. Later, Sheng-Shu and Fong-Jung (2013) suggested that the efficiency of a CUSUM control chart in monitoring the quality control of a wafer production process was better than the EWMA control chart. Recently, the CUSUM control chart was considered the best tool for quality control in the healthcare domain (Novoa and Varela 2020). As an ongoing study on the benefits of an upward shift of a process on a CUSUM control chart by focusing on upper-sided CUSUM. Based on these studies, the CUSUM control chart is suitable for use in a variety of practical applications.

In general, an assumption concerning control charts is that the observations generated by the underlying process are independent and normally distributed. Notwithstanding, this assumption is usually violated in practice. Since the observations can follow more complex patterns, a considerable amount of research work concerning CUSUM control charts for autocorrelated processes has been published (e.g., Yashchin 1993, Bohm and Hackl 1996, Lu and Reynolds 2001, Chang and Wu 2011).

Time-series and forecasting data comprise real observations that can include trend, seasonal, and autocorrelation characteristics. A time-series component is often found in observations from real stochastic processes. Especially, autoregressive (AR) and moving average (MA) components can be found in econometric observations in a time-series model. When a model is established, the error (the difference between the actual and approximated values), which is also known as white noise, should be kept as small as possible to maximize the accuracy. The white noise arising from autocorrelated observations is usually normally distributed but sometimes this is not the case. Therefore, time-series models in which the white noise is distributed exponentially (e.g., wind speed, the amount of oxygen in a river, and water flow rate) are particularly interesting. For instance, an ARMA(1,1) process with exponential white noise was considered by Jacob and Lewis (1977). Later, a Bayesian analysis of an AR(1) process with exponential white noise was conducted by Mohamed and Hocine (2003), while exponential white noise was considered by Pereira and Turkman (2004) while conducting a Bayesian analysis on threshold AR models. Recently, parameter estimations for an AR model with an unknown order and exponential white noise were proposed by Suparman (2018).

Time-series analysis and forecasting are demanding issues for statistical modeling. In practice, such as in finance and economy, the present value of variables based on the distant past has been identified as a long-memory process when analyzing time-series data. The traditional models for time-series analysis concerning short-memory processes are AR, MA, ARMA, AR integrated MA (ARIMA), among others. To analyze a long-memory process based on Box-Jenkin's ARIMA model, the AR fractionally integrated MA (ARFIMA or $\text{ARFIMA}(p,d,q)$) model was suggested by Granger and Joyeux (1980). For the $\text{ARFIMA}(p,d,q)$ model, where p is the order of the AR process, d is the fractional integration parameter, and q is the order of the

MA process. The characteristics of the ARFIMA model allow d to take a fractional value in the range of $0 < d < 0.5$ for a stationary process with a long memory, while $d = 0$ and $d < 0$ indicate that the process has short and intermediate dependence, respectively. An in-depth review of long-memory processes and the ARFIMA model was provided by Baillie (1996).

Sometimes, the time-series data differs from the original time series, and information from exogenous variables can be used to explain the behavior of a dependent variable. This characteristic can be incorporated into an existing model to improve its performance and forecasting accuracy, thereby producing the ARFIMAX or ARFIMAX(p,d,q,r) model, in which r is the exogenous variable such as inflation, exchange rate, etc. (see Degiannakis 2008). Ebens (1999) only estimated the ARFIMAX model without AR term, or FIMAX model. This approach has motivated us to only consider part of the ARFIMAX model (i.e., FIMAX(d,q,r) because, according to Charles et al. (2001), this is the most interesting part of creating forecasting models. Focusing on the MA model for forecasting future value smoothing helps to remove random variations that appear as coarseness in a plot of raw time-series data, thereby reducing the noise and emphasizing the signal for identifying trends and cycles. The FIMAX model underlying exponential white noise was used in the present study due to its suitability in a variety of practical applications and use with real-life data related to economic indicators in quality control. In addition, time-series modeling on a CUSUM control chart is based on the concept of extending the classical charts to time-dependent processes by making use of the underlying time-series structure (Yashchin 1993, Knoth and Frisén 2012), which has also been extended to other control charts for specific time-series models such as ARFIMA. For example, Rabyk and Schmid (2016) recently presented the EWMA control chart, which is used to identify changes in the mean of a long-memory process by designing the control chart around the ARFIMA(p,d,q) process. When residual control charts with ARFIMA and ARIMA models were used to monitor Taiwan's air quality (Pan and Chen 2008), the ARFIMA model was shown to be more appropriate than the ARIMA model. Ramjee (2000) discovered that Shewhart and EWMA control charts performed poorly with correlated observations in an ARFIMA model and developed the hyperbolic weighted MA (HWMA) control chart in their place. Following that, Ramjee et al. (2002) developed a HWMA forecast-based control chart that was optimized for use with a non-stationary ARFIMA model.

Evaluation of the CUSUM control chart performance can be measured by using the ARL comprising two characteristics: in-control ARL (ARL_0), which refers to the average number of observations from the in-control process before an out-of-control signal is raised, and out-of-control ARL (ARL_1), which is the average number of observations required to detect a shift in a process variable. Ideally, the value of ARL_0 is large while that of ARL_1 is small for detecting small-to-moderate shifts in the process mean (Bagshaw and Johnson 1975). The goal when designing a control chart is to calculate the ARL as a benchmark. Various methods for calculating the ARL of a CUSUM control chart have been proposed in the literature. For instance, Brook and Evans (1972) adopted the Markov-chain approach, Lucas and Crosier (1982) used the integral equation approach, Siegmund (1985) presented a method based on solving ARL equations, and Areepong and Peerajit (2022) proposed an NIE method. Of particular interest, analytical integral equations based on explicit formulas and the NIE method on a control chart for models with exogenous variables have also been adopted by various researchers (Peerajit et al. 2018, Peerajit et al. 2019, Sunthornwat and Areepong 2020, Silpakob et al. 2021).

At the best of our knowledge, evaluating the ARL based on explicit formulas for a FIMAX model with underlying exponential white noise on a CUSUM control chart has not previously been reported. In this paper, explicit formulas for the ARL based on analytical integral equations are proposed for this scenario, and their performance was compared with the NIE method in terms of the percentage accuracy and computational time.

The rest of this paper is structured as follows. The second section provides a brief description of the general form of a $\text{FIMAX}(d,q,r)$ process with exponential white noise and an introduction to the one-sided CUSUM and EWMA control charts for detecting a shift in the process mean for a FIMAX model. In the third section, the analytical ARL derived by using integral equations is presented and its existence and uniqueness are confirmed by applying Banach's fixed-point theorem. Calculation of the ARL derived from the explicit formulas to compare the performances of ARL with NIE Method, the percentage accuracy, and evaluation of the ARL on a CUSUM control chart are proposed in the fourth section. In the fifth section, we show the practicability of the method on two real-life datasets for processes on a CUSUM control chart compared with a EWMA control chart. Finally, conclusions on the study and recommendations for future research are given in the last section.

2. The ARFIMAX Model and the Properties of the Control Charts

2.1. The ARFIMAX model

This section provides a brief explanation of the ARFIMA model with exogenous variables (ARFIMAX) as the basis for the FIMAX model with exponential white noise on an upper-sided CUSUM control chart.

The ARFIMAX model was first suggested by Ebens (1999) by incorporating periodicity as follows:

$$\phi_p(B)(1-B)^d(Y_t-\mu)=\theta_q(B)\varepsilon_t+\sum_{j=1}^{r}\omega_j X_{jt},$$

where μ is the constant process mean, $\phi_p(B)$ and $\theta_q(B)$ are polynomial operators with orders p, q respectively, d and r, are exogenous variables with orders X_{jt} is the fractional integration parameter.

For convenience, the model of interest has been restricted to the $\text{ARFIMAX}(p,d,q,r)$ model with $p=0$, i.e.,

$$(1-B)^d(Y_t-\mu)=\theta_q(B)\varepsilon_t+\sum_{j=1}^{r}{}_j\,\omega X_{jt}, \tag{1}$$

where $\theta_q(B)=1-\sum_{i=1}^{q}\theta_i(B^i)$, B is a backward-shift operator, and ε_t refers to the exponential white noise.

Fractional differencing operator $(1-B)^d$ can be determined naturally by using a binomial series expansion as follows (Granger and Joyeux 1980, Hosking 1981):

$$(1-B)^d:=\sum_{k=0}^{\infty}\binom{d}{k}(-B)^k=1-dB+\frac{1}{2!}d(d-1)B^2-..., \tag{2}$$

for all real d and thus, d becomes crucial to describe the degree of persistence. By taking the fractional differencing operator from (2) and placing it in (1), the general form of a $\text{FIMAX}(d,q,r)$ model with exponential white noise on an upper-sided CUSUM control chart becomes

$$Y_t = \mu + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \sum_{j=1}^{r} \omega_j X_{jt} + \left(dY_{t-1} - \frac{1}{2} d(d-1) Y_{t-2} + \dots \right), \tag{3}$$

where ε_t are i.i.d. observations in an exponential distribution; $\varepsilon_t \sim Exp(\beta)$. $X_{jt}, j = 1,2,...,r$ are the exogenous variables and ω_j are the coefficients corresponding to r, θ_i is an MA coefficient; $|\theta_i| < 1; i = 1,2,...,q$ and fractionally integration parameters are all nonzero. The initial value $Y_{t-1}, Y_{t-2}, \dots, X_{1t}, X_{2t}, ..., X_{rt}$ and ε_t are equal to 1.

2.2. The One-Sided CUSUM and EWMA Control Charts Running a FIMAX Model with Exponential White Noise

Here, we focus on upper one-sided CUSUM and EWMA control charts with an upper control limit (UCL) for detecting shifts in the process mean.

1) The one-sided CUSUM control chart

The CUSUM control chart can be used to monitor and detect small-to-moderate shifts in the process mean. Assume that Y_t, $t = 1,2,...$ is the sequence for the general form of a $\text{FIMAX}(d,q,r)$ with exponential white noise; i.e., $\varepsilon_t \sim Exp(\beta)$, where β_0 is the in-control mean and $\delta = 0$ for the in-control process and β_1 is the out-of-control mean and $\delta > 0$ for the out-of-control process.

C_t, the one-sided upper CUSUM statistic, can be written as

$$C_t = \max\{0, C_{t-1} + Y_t - \eta\}, \; t = 1,2,\dots , \tag{4}$$

where η represents the reference value parameter with $\eta > 0$ and C_0 is starting value (that is $C_0 = \vartheta$ where ϑ is the initial value; $\vartheta \in [0,b]$).

The designed CUSUM control chart performance can be evaluated based on the ARL. In (4), reference value η is usually pre-specified and control limit b of the chart should be carefully calculated to provide the desired in-control ARL (ARL_0) value. The implemented CUSUM control chart minimizes the value of ARL_1 for a mean shift compared to other control charts with the same specified ARL_0 value. Note that the values of the decision interval or boundary are represented by b, and the one-sided CUSUM control chart gives an out-of-control signal when $C_t > b$.

2) The one-sided EWMA control chart

The EWMA control chart (Roberts 1959) is one of the most widely used tools to detect small-to-moderate changes in a process parameter. The EWMA statistic is given as

$$C_t = \lambda Y_t + (1-\lambda) C_{t-1}, \; t = 1,2,\dots , \tag{5}$$

where λ is the smoothing parameter of EWMA statistic and is chosen such that $\lambda \in (0,1]$ and C_{t-1} represents past information with an initial value of $C_0 = Y_0$ equal to the target mean value.

The upper and lower control limits of the EWMA control chart are respectively given by

$$\mathrm{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]},$$

$$\mathrm{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \tag{6}$$

where L is an appropriate control width limit generally selected based on the constraint of the desired in-control ARL. When $C_t \in [\mathrm{LCL}, \mathrm{UCL}],$ the process is considered to be in-control.

Since, an increase or one-sided change in the mean detection is investigated. The following (alarm) stopping time (τ_b) is utilized:

$$\tau_b = \inf\left\{ t > 0 : C_t > b \right\}, \tag{7}$$

where $b > 0$ is the UCL for the CUSUM and EWMA control charts.

3. The Analytical ARL for a CUSUM Control Chart Running a FIMAX Process with Exponential White Noise

In this section, the analytical ARLis derived from integral equations called explicit formulas used to detect shifts in the process mean of the FIMAX model with underlying exponential white noise running on a CUSUM control chart. In addition, the existence and uniqueness of the analytical ARL is proved by applying Banach's fixed-point theorem.

Calculating the ARL through integral equations was initially carried out for a EWMA control chart on which the EWMA statistic monitors upper- and lower-sided shifts simultaneously. Subsequently, integral equations can also be applied to a CUSUM control chart to detect shifts in the process mean. Previously, most research has been concentrated on detecting changes in an upper-sided CUSUM (C_t) chart. Here, we focus on the upper control limits starting at LCL = 0 as the center line (CL) and ending at $b > 0,$ where b = UCL.

Let $L(\vartheta)$ denote the ARL for a one-sided CUSUM running a FIMAX model with initial value ϑ; where $L(\vartheta) = \mathrm{E}_m(\tau_b) < \infty,$ and suppose that the initial value of monitoring statistics $C_0 = \vartheta.$ Hence, analytical integral equations derived from the Fredholm integral equation of the second kind for the analytical ARL of a one-sided CUSUM control chart (C_t) becomes

$$L(\vartheta) = 1 + L(0)F\left(\eta - \vartheta - Y_t\right) + \int_0^b L(s) f(s + \eta - \vartheta - Y_t) ds, \tag{8}$$

where $f(.)$ and $F(.)$ are the probability density function and cumulative distribution function of an exponential distribution. Therefore, (8) can be obtained by solving the following integral equations:

$$L(\vartheta) = 1 + L(0)(1 - e^{\left\{\begin{array}{l} -\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt} \\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+\ldots)) \end{array}\right\}}) + \beta(e^{\left\{\begin{array}{l} \beta(\mu-\eta+\vartheta+\varepsilon_t-\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}+\sum_{j=1}^{r}\omega_j X_{jt} \\ +(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+\ldots.)) \end{array}\right\}})\int_0^b L(s)e^{-\beta s} ds, \tag{9}$$

for a theoretically continuous function. From (9), we proceed to prove the existence and uniqueness of the solutions of ARL through Banach's fixed-point theorem as follows.

Theorem 1 *Banach's fixed point theorem (see Sofonea 2005) Let* (X, d) *be a complete metric space and suppose that* $\Lambda : X \to X$ *be a contraction mapping with contraction constant* $0 \le \lambda < 1$ *that is,*

$$d\left(\Lambda\left(L_1\right),\Lambda\left(L_0\right)\right) \le \lambda d\left(L_1,L_0\right) \text{ for all } L_1,L_0 \in X.$$

Then, there exists a unique $L(.) \in X$ *such that* $\Lambda(L(\vartheta)) = L(\vartheta)$, *such that,* Λ *has a unique fixed point in* X.

Proof: To prove the existence of the solutions of the integral equations, let $\mathbf{C}[0,b]$ be a set of all of the continuous functions of $L(\vartheta)$ on interval $[0,b]$. Define the following recursion formula for $n \ge 0,$ as $L_0(\vartheta) \in \mathbf{C}[0,b],$ i.e. $L_{n+1}(\vartheta) = \Lambda(L_n(\vartheta))$ for some initial points $L(\vartheta)_0 \in \mathbf{C}[0,b],$ where $\mathbf{C}[0,b]$ is complete. Now, $\left\{L_n(\vartheta)\right\}_{n\ge0}$ is a Cauchy sequence of integral equations as follows:

$$d\left(L_2(\vartheta),L_1(\vartheta)\right) = d\left(\Lambda\left(L_1(\vartheta)\right),\Lambda\left(L_0(\vartheta)\right)\right) \le \lambda d\left(L_1(\vartheta),L_0(\vartheta)\right).$$

By applying the triangle inequality for any positive integer $n,$ we obtain

$$\begin{aligned} d\left(L_{n+m}(\vartheta),L_n(\vartheta)\right) &\le d(L_{n+m}(\vartheta),L_{n+m-1}(\vartheta)) + d(L_{n+m-1}(\vartheta),L_{n+m-2}(\vartheta)) + \ldots + d(L_{n+1}(\vartheta),L_n(\vartheta)). \\ &\le \lambda^{n+m-1} d(L_1(\vartheta),L_0(\vartheta)) + \lambda^{n+m-2} d(L_1(\vartheta),L_0(\vartheta)) + \ldots + \lambda^n d(L_1(\vartheta),L_0(\vartheta)). \\ &= \sum_{i=1}^{m} \lambda^{n+i-1} d(L_1(\vartheta),L_0(\vartheta)). \end{aligned}$$

Note that $\sum_{i=1}^{m} \lambda^{n+i-1} d(L_1(\vartheta),L_0(\vartheta)) = \lambda^n \sum_{i=0}^{m-1} \lambda^i = \lambda^n \dfrac{(1-\lambda^i)}{1-\lambda} = \dfrac{\lambda^n}{1-\lambda}.$ Moreover, if $n \to \infty,$ $\lambda^i \to 0$ as $i \to \infty,$ then

$$d\left(L_{n+m}(\vartheta),L_n(\vartheta)\right) \le \frac{\lambda^n}{1-\lambda} d(L_1(\vartheta),L_0(\vartheta)).$$

If $n \to \infty,$ $\dfrac{\lambda^n}{(1-\lambda)} \to 0$ and by letting $\varepsilon > 0$ be arbitrary, we can write

$$d\left(L(\vartheta)_{n+m},L(\vartheta)_n\right) \le \frac{\varepsilon}{1-\lambda} d(L_1(\vartheta),L_0(\vartheta)) = \varepsilon.$$

Therefore, $\left\{L_n(\vartheta)\right\}_{n\ge0}$ is a Cauchy sequence, and thus, some $L(\vartheta) \in \mathbf{C}[0,b]$ exist such that $\lim_{n\to\infty} L_n(\vartheta) = L(\vartheta).$ That is to say, $\Lambda(L(\vartheta)) = L(\vartheta),$ where $L(\vartheta)$ is a fixed point of $\Lambda.$ Thus, the proof of existence is complete.

Theorem 2 *The analytical integral equations* $(L(\vartheta))$ *for a FIMAX model with underlying exponential white noise on a CUSUM control chart exist and are unique.*

Proof: To prove the uniqueness of the solution of the integral equation. Let Λ be a contraction on $\mathbf{C}[0,b]$ and $L_1(\vartheta), L_2(\vartheta) \in \mathbf{C}[0,b].$ It is obvious that the metric space $\left(\mathbf{C}[0,b], \|\,.\,\|_\infty\right)$ is complete. That is to say, a set of continuous functions of the ARL defined on $[0,b]$ and $\mathbf{C}[0,b]$ become norm space if we define

$$\| L \|_\infty = \sup_{\vartheta\in[0,b]} \left| \int_0^b k(\vartheta,z)dz \right|, \text{ for all functions } k(\vartheta,z)\in \mathbf{C}[0,b],$$

where $k(\vartheta,z)$ is a kernel function of the analytical ARL.

$$\| \Lambda(L_1)-\Lambda(L_2)\|_\infty = \sup_{\vartheta\in[0,b]} \left| \int_0^b \beta(e^{\left\{\begin{array}{l}\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}) \left|L_1(z)-L_2(z)\right| dz \right|$$

$$\leq \sup_{\vartheta\in[0,b]} \left| \int_0^b \beta(e^{\left\{\begin{array}{l}\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}) \right| \left|L_1(z)-L_2(z)\right| dz$$

$$\leq \sup_{\vartheta\in[0,b]} \int_0^b \left| \beta(e^{\left\{\begin{array}{l}\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}) \right| dz \left\|L_1(z)-L_2(z)\right\|_\infty .$$

Subsequently, it follows that

$$\left\|\Lambda(L_1)-\Lambda(L_2)\right\|_\infty \leq \upsilon \left\|L_1(z)-L_2(z)\right\|_\infty ,$$

where $0<\upsilon = \sup_{\vartheta\in[0,b]} \int_0^b \left| \beta(e^{\left\{\begin{array}{l}\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}) \right| dz <1$ is a positive constant. This completes the proof.

Therefore, the analytical ARL by using integral equations based on proven explicit formulas exists and is unique. This can be used on a CUSUM control chart to detect a particular shift size for a FIMAX model with underlying exponential white noise.

Henceforth, the analytical integral equations are presented as explicit formulas showing that the calculation starting from (9) has already been proven. By substituting $g=\int_0^b L(s)e^{-\beta s}ds$ into (9), we obtain

$$L(\vartheta) = 1+L(0)(1-e^{\left\{\begin{array}{l}-\beta(\eta-\mu-\vartheta-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}})+\beta(e^{\left\{\begin{array}{l}\beta(\mu-\eta+\vartheta+\varepsilon_t-\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}+\sum_{j=1}^{r}\omega_j X_{jt}\\ +(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+....))\end{array}\right\}})g. \quad (10)$$

Preferably for $\vartheta=0$,

$$L(0) = 1+L(0)(1-e^{\left\{\begin{array}{l}-\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\ -(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}})+\beta(e^{\left\{\begin{array}{l}\beta(\mu-\eta+\varepsilon_t-\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}+\sum_{j=1}^{r}\omega_j X_{jt}\\ +(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+....))\end{array}\right\}})g.$$

Thus,

$$L(0)=e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}+g\beta. \tag{11}$$

Subsequently, substituting $L(0)$ in (11) into (10) results in

$$L(\vartheta)=1+e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}+g\beta-e^{\beta\vartheta}. \tag{12}$$

Constant $g=\int_0^b L(s)e^{-\beta s}ds$ can be obtained as

$$g=\int_0^b(1+e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}+g\beta-e^{\beta s})e^{-\beta s}ds=(1+e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}+g\beta)\int_0^b e^{-\beta s}ds-b.$$

Hence, constant g can be derived as

$$g=\frac{e^{\beta b}}{\beta}(1+e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}})(1-e^{-\beta b})-be^{\beta b}. \tag{13}$$

Finally, constant g derived in (13) can be substituted into (12) to yield

$$L(\vartheta)=e^{\beta b}(1+e^{\left\{\begin{array}{l}\beta(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}-g\beta)-e^{\beta\vartheta};\ \vartheta\geq 0. \tag{14}$$

Thus, the calculation of the ARL derived by using explicit formulas for the FIMAX model with exponential white noise running on a CUSUM control chart is complete.

The process is in-control (ARL_0) if exponential parameter $\beta=\beta_0$. Otherwise, the process is out-of-control (ARL_1) if exponential parameter $\beta=\beta_1$. Thus, the upper-sided CUSUM control chart can be written as

$$L(\vartheta)=\begin{cases}\text{ARL}_0=e^{\beta_0 b}(1+e^{\left\{\begin{array}{l}\beta_0(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}-g\beta_0)-e^{\beta_0\vartheta},\\ \text{ARL}_0=e^{\beta_1 b}(1+e^{\left\{\begin{array}{l}\beta_1(\eta-\mu-\varepsilon_t+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\omega_j X_{jt}\\-(dY_{t-1}-\frac{1}{2}d(d-1)Y_{t-2}+...))\end{array}\right\}}-g\beta_1)-e^{\beta_1\vartheta}.\end{cases} \tag{15}$$

4. Numerical Results and Comparisons

The statistical performance of the proposed ARL method on a CUSUM control chart is assessed by obtaining the ARL value minimized for a specified process mean shift while measuring the rapidity of the control chart response. The ARLs of a CUSUM control chart via analytical integral equations as explicit formulas and the NIE method can be compared through an algorithm developed in the Mathematica program.

4.1. Comparison of the ARLs obtained via the integral equations and NIE methods

The approximated ARL based on numerical quadrature rules (Peerajit 2022) can be calculated by using the integral equations in (9). Hence, the NIE method ($\tilde{L}(\vartheta)$) for the ARL to detect a change in the mean of a FIMAX model running on a CUSUM control chart can be written as

$$\tilde{L}(\vartheta) \approx 1 + L(a_1)F\left(\eta - \vartheta - \mu - \varepsilon_t + \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \sum_{j=1}^{r}\omega_j X_{jt} + \left(dY_{t-1} - \frac{1}{2}d(d-1)Y_{t-2} + ...\right)\right)$$
$$+ w_1 L(a_1) f\left(a_1 + \eta - \vartheta - \mu - \varepsilon_t + \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \sum_{j=1}^{r}\omega_j X_{jt} + \left(dY_{t-1} - \frac{1}{2}d(d-1)Y_{t-2} + ...\right)\right)$$
$$+ \sum_{k=2}^{m} w_k L(a_k) f\left(a_k + \eta - \vartheta - \mu - \varepsilon_t + \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \sum_{j=1}^{r}\omega_j X_{jt} + \left(dY_{t-1} - \frac{1}{2}d(d-1)Y_{t-2} + ...\right)\right), \tag{16}$$

where w_k is a weight define different quadrature rules with $w_k = \frac{b}{m}$, and a_k is a set of point with $a_k = \frac{b}{2m}(2k-1);\ k = 1, 2, \ldots, m.$

4.2. Measurement indices

An efficient CUSUM control chart has a sufficiently large in-control ARL (ARL_0) value and a small out-of-control ARL (ARL_1) value for a shift in the mean of a long-memory FIMAX model. A large ARL_0 value is required to avoid unnecessary false alarms while a small ARL_1 indicates that the chart's efficiency is good. In addition to the computations of the ARLs, we calculated the percentage accuracy to measure the overall effectiveness of the methods for each mean shift δ level in the process as follows:

$$\%\,\text{Accuracy} = \left(1 - \left|\frac{L(\vartheta) - \tilde{L}(\vartheta)}{L(\vartheta)}\right|\right) \times 100\%, \tag{16}$$

where $L(\vartheta)$ and $\tilde{L}(\vartheta)$ are respectively, the ARL derived from the explicit formulas and approximated from NIE method.

4.3. Evaluating the ARL on a CUSUM control chart

For this task, we assumed that for a fixed mean, both the in-control and out-of-control processes are exponentially distributed with process means $\beta_0 = 1$ and $\beta_1 = \beta_0(1+\delta)$, respectively. Thus, mean shift δ is equal to zero when the process is in-control and $\delta > 0$ when the process is out-of-control. In the study, δ was set as 0.01, 0.05, 0.10, 0.20, 0.50, or 2.00 respectively. In this study, the mean shifts are were categorized by as small $(\delta < 1.00)$ and moderate $(1.00 \le \delta < 3)$ (Osei-Aning 2017). The in-control ARL threshold was set as ARL_0 = 370 or 500 with parameter combinations (η, b): η was set as 3.0, 3.5 or 4.0 and b was calculated according to (15) to obtain the desired in-control ARL_0 values. Parameter combination (η, b) values for coefficient parameters $\theta_1 = 0.10,\ \theta_2 = 0.20,$ and $\omega_1 = 0.30$ are reported in Table 1. The

ARL values obtained by using the NIE method are determined for the number of division points $m = 800$, which are extensively reported in 4 tables (Tables 2-5 are for the proposed CUSUM control chart and Table 8 for the practical application).

Table 1 The (η, b) combinations for a CUSUM control chart running a FIMAX model with exponential white noise

Models	Coefficient of models				ARL_0	η		
	d	θ_1	θ_2	ω_1		3.0	3.5	4.0
FIMAX(d, 1, 1)	0.15	0.10	-	0.30	370	3.601757	2.972260	2.415525
					500	3.934211	3.288163	2.724043
	0.30	0.10	-	0.30	370	3.916957	3.225274	2.645690
					500	4.264094	3.546397	2.956727
	0.45	0.10	-	0.30	370	4.221620	3.445651	2.839573
					500	4.590686	3.772771	3.153285
FIMAX(d, 2, 1)	0.15	0.10	0.20	0.30	370	3.336174	2.744130	2.202665
					500	3.660118	3.056455	2.509347
	0.30	0.10	0.20	0.30	370	3.617738	2.985596	2.427825
					500	3.950800	3.301738	2.736463
	0.45	0.10	0.20	0.30	370	3.874266	3.192342	2.616147
					500	4.2190300	3.512697	2.926825

Table 1 reports η for calculating ARL_0. Observe that the UCL (b) is indirectly proportional to η for FIMAX(d, 1, 1) and FIMAX(d, 2, 1) on a CUSUM control chart. For instance, parameter combinations $(\eta = 3.0, b = 3.601757)$, $(\eta = 3.5, b = 2.972260)$, and $(\eta = 4.0, b = 2.415525)$ were chosen to obtain $ARL_0 = 370$ for the FIMAX(0.15, 1, 1) model. In the same manner, combinations parameter $(\eta = 3.0, b = 3.934211)$, $(\eta = 3.5, b = 3.288163)$, and $(\eta = 4.0, b = 2.724043)$. were chosen to obtain ARL_0 = 500. For both FIMAX$(d, 1, 1)$, FIMAX$(d, 2, 1)$, both d and b increased for each value of η when d changed from 0.15 to 0.45.

In the ARL results for the CUSUM control charts summarized in Tables 2-5, we only reported those for ARL_0 = 370 (cf. Tables 2 and 3) since the resulting pattern was the same for ARL_0 = 500. Our major findings are as follows:

1. Both the explicit formulas and NIE ARL methods showed great sensitivity by quickly detecting small-to-moderate shifts in the process mean on a CUSUM control chart.

2. In the out-of-control cases when $\delta > 0$, the values of ARL_1 derived from the explicit formulas and approximated from the NIE method for a FIMAX process running on a CUSUM control chart both decrease rapidly as δ increases.

3. The ARL_1 values produced by the explicit formulas are very similar to those approximated by the NIE method.

4. As can be seen by the highlighted values in bold in Tables 2 and 3, the percentage accuracy results were approximate 99% in all cases, meaning that the proposed explicit formulas method is very accurate.

5. The ARL_1 values from both methods reveal that the CUSUM control chart is very effective at detecting small-to-moderate shifts in the process mean when η is small. However, the methods are more sensitive to small shifts $(\delta < 1.00)$ when the value of η is large.

The values in parentheses represent the computational times when calculating the ARL by using both methods. It was found that the computational time with the explicit formulas method was only a fraction of a second while the NIE method took 32-36 minutes.

Overall, the ARL_1 derived by using the explicit formulas was as accurate as the NIE method but required less computational time, thereby demonstrating the benefit of using the former.

The ARL_1 values derived from the explicit formulas under the FIMAX(d, 1, 1) and FIMAX(d, 2, 1) model in terms of the reference value (η) and the lowest ARL_1 value for d = 0.45, 0.30 and 0.15 for $\text{ARL}_0 = 370$ are presented in Figures 1 and 2. For both models, the methods provided the lowest ARL_1 value when η was 3.0, followed by 3.5 and 4.0.

The lowest ARL_1 values were achieved when η =3 and d = 0.45. For clarification, detection of the change in the process mean are shown in Figure 3(a) for the FIMAX (0.45, 1, 1) model and Figure 3(b) for the FIMAX (0.45, 2, 1) model.

In summary, the results show that the ARL by using analytical integral equations was better than the NIE method because the former provides an exact solution for the ARL. However, when the exact solution is unavailable, the NIE method is a good alternative for approximating the ARL in this situation.

5. Practical Application of the Proposed Analytical ARL Derived by Using Explicit Formulas

Here, two real datasets from FIMAX processes running on a one-sided CUSUM control chart are used to demonstrate the practicability of the proposed analytical ARL derived by using explicit formulas. These data are autocorrelated with underlying exponential white noise. Autocorrelation of the observations was evaluated by using the Box-Jenkins technique to determine the fitting of time-series data. After that, the t-statistic was to ascertain whether the datasets are suitable for a $\text{FIMAX}(d,q,r)$ model. The coefficient values of the $\text{FIMAX}(d,q,r)$ model and testing of the mean of the white noise exponential distribution are reported in Tables 6 and 7, respectively.

Table 2 Out-of-control ARL values of a one-sided CUSUM control chart running a FIMAX(*d*, 1, 1) model when ARL_0 = 370

η	δ	FIMAX(*d*, 1, 1) model								
		d = 0.15			*d* = 0.30			*d* = 0.45		
		$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**
3.0	0.01	346.983	346.279	**99.80**	346.304	345.567	**99.79**	345.469	344.714	**99.78**
		(0.00)	(36.04)		(0.00)	(32.69)		(0.00)	(32.78)	
	0.05	271.641	271.119	**99.81**	269.126	268.586	**99.80**	266.059	265.516	**99.80**
		(0.00)	(33.24)		(0.00)	(32.72)		(0.00)	(32.91)	
	0.10	205.079	204.710	**99.82**	201.529	201.154	**99.81**	197.254	196.884	**99.81**
		(0.00)	(37.03)		(0.00)	(32.85)		(0.00)	(32.68)	
	0.20	125.431	125.232	**99.84**	121.639	121.442	**99.84**	117.169	116.984	**99.84**
		(0.00)	(34.96)		(0.00)	(32.60)		(0.00)	(32.88)	
	0.50	42.877	42.828	**99.89**	40.717	40.672	**99.89**	38.306	38.267	**99.90**
		(0.00)	(34.76)		(0.00)	(32.77)		(0.00)	(32.64)	
	0.70	26.113	26.089	**99.91**	24.703	24.681	**99.91**	23.173	23.155	**99.92**
		(0.00)	(34.45)		(0.00)	(32.63)		(0.00)	(32.99)	
	0.90	17.781	17.767	**99.92**	16.829	16.817	**99.93**	15.824	15.814	**99.94**
		(0.00)	(34.38)		(0.00)	(32.86)		(0.00)	(32.75)	
	1.50	8.346	8.341	**99.94**	7.987	7.983	**99.95**	7.634	7.631	**99.96**
		(0.00)	(33.62)		(0.00)	(32.60)		(0.00)	(32.89)	
	2.00	5.703	5.700	**99.95**	5.514	5.512	**99.96**	5.339	5.337	**99.96**
		(0.00)	(32.73)		(0.00)	(32.84)		(0.00)	(32.64)	
3.5	0.01	347.910	347.302	**99.83**	347.596	346.946	**99.81**	347.259	346.577	**99.80**
		(0.00)	(32.57)		(0.00)	(32.82)		(0.00)	(32.51)	
	0.05	275.125	274.666	**99.83**	273.937	273.450	**99.82**	272.677	272.168	**99.81**
		(0.00)	(32.73)		(0.00)	(32.54)		(0.00)	(32.72)	
	0.10	210.075	209.744	**99.84**	208.359	208.011	**99.83**	206.555	206.193	**99.82**
		(0.00)	(32.66)		(0.00)	(32.64)		(0.00)	(32.76)	
	0.20	130.925	130.740	**99.86**	129.016	128.823	**99.85**	127.033	126.836	**99.84**
		(0.00)	(32.70)		(0.00)	(32.59)		(0.00)	(32.95)	
	0.50	46.234	46.185	**99.89**	45.032	44.982	**99.89**	43.825	43.776	**99.89**
		(0.00)	(32.67)		(0.00)	(32.77)		(0.00)	(32.62)	
	0.70	28.391	28.365	**99.91**	27.562	27.537	**99.91**	26.745	26.720	**99.91**
		(0.00)	(32.82)		(0.00)	(32.78)		(0.00)	(32.72)	
	0.90	19.370	19.355	**99.92**	18.784	18.770	**99.93**	18.215	18.201	**99.92**
		(0.00)	(32.69)		(0.00)	(32.71)		(0.00)	(32.95)	
	1.50	8.996	8.991	**99.94**	8.749	8.744	**99.94**	8.517	8.512	**99.94**
		(0.00)	(32.76)		(0.00)	(32.86)		(0.00)	(32.89)	
	2.00	6.066	6.063	**99.95**	5.925	5.922	**99.95**	5.796	5.793	**99.95**
		(0.00)	(32.90)		(0.00)	(32.71)		(0.00)	(32.62)	
4.0	0.01	348.388	347.882	**99.85**	348.221	347.672	**99.84**	348.047	347.463	**99.83**
		(0.00)	(32.69)		(0.00)	(32.74)		(0.00)	(32.68)	
	0.05	276.951	276.567	**99.86**	276.311	275.894	**99.85**	275.645	275.207	**99.84**
		(0.00)	(32.95)		(0.00)	(32.85)		(0.00)	(32.73)	
	0.10	212.743	212.463	**99.87**	211.802	211.490	**99.85**	210.837	210.518	**99.85**
		(0.00)	(32.85)		(0.00)	(32.81)		(0.00)	(32.47)	
	0.20	133.959	133.799	**99.88**	132.879	132.709	**99.87**	131.783	131.604	**99.86**
		(0.00)	(32.77)		(0.00)	(32.66)		(0.00)	(32.80)	
	0.50	48.247	48.203	**99.91**	47.514	47.467	**99.90**	46.790	46.742	**99.90**
		(0.00)	(32.56)		(0.00)	(32.69)		(0.00)	(32.70)	
	0.70	29.815	29.792	**99.92**	29.290	29.266	**99.92**	28.779	28.754	**99.91**
		(0.00)	(32.71)		(0.00)	(32.91)		(0.00)	(32.75)	
	0.90	20.401	20.387	**99.93**	20.017	20.003	**99.93**	19.648	19.633	**99.92**
		(0.00)	(32.99)		(0.00)	(32.82)		(0.00)	(33.38)	
	1.50	9.456	9.451	**99.95**	9.281	9.276	**99.95**	9.117	9.112	**99.95**
		(0.00)	(32.72)		(0.00)	(32.57)		(0.00)	(32.78)	
	2.00	6.337	6.335	**99.97**	6.233	6.230	**99.95**	6.136	6.133	**99.95**
		(0.00)	(32.85)		(0.00)	(32.78)		(0.00)	(32.77)	

Table 3 Out-of-control ARL values of a one-sided CUSUM control chart running a FIMAX(d, 2, 1) model when $ARL_0 = 370$

η	δ	FIMAX(d, 2, 1) model								
		$d = 0.15$			$d = 0.30$			$d = 0.45$		
		$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**
3.0	0.01	347.434 (0.00)	346.767 (32.86)	**99.81**	346.952 (0.00)	346.247 (32.75)	**99.80**	346.407 (0.00)	345.673 (32.88)	**99.79**
	0.05	273.331 (0.00)	272.833 (32.66)	**99.82**	271.527 (0.00)	271.004 (32.86)	**99.81**	269.503 (0.00)	268.965 (32.87)	**99.80**
	0.10	207.910 (0.00)	207.135 (32.90)	**99.63**	204.918 (0.00)	204.549 (32.87)	**99.82**	202.059 (0.00)	201.684 (34.74)	**99.81**
	0.20	128.057 (0.00)	127.862 (32.76)	**99.85**	125.257 (0.00)	125.058 (32.84)	**99.84**	122.199 (0.00)	122.002 (32.69)	**99.84**
	0.50	44.444 (0.00)	44.394 (32.89)	**99.89**	42.775 (0.00)	42.727 (32.85)	**99.89**	41.029 (0.00)	40.984 (32.74)	**99.89**
	0.70	27.162 (0.00)	27.137 (32.77)	**99.91**	26.046 (0.00)	26.022 (32.94)	**99.91**	24.904 (0.00)	24.882 (32.78)	**99.91**
	0.90	18.505 (0.00)	18.490 (32.95)	**99.92**	17.735 (0.00)	17.721 (32.84)	**99.92**	16.963 (0.00)	16.951 (32.77)	**99.93**
	1.50	8.634 (0.00)	8.629 (32.77)	**99.94**	8.328 (0.00)	8.323 (33.01)	**99.94**	8.036 (0.00)	8.033 (32.66)	**99.96**
	2.00	5.860 (0.00)	5.858 (32.97)	**99.97**	5.693 (0.00)	5.691 (32.79)	**99.96**	5.540 (0.00)	5.538 (32.92)	**99.96**
3.5	0.01	348.137 (0.00)	347.57 (32.80)	**99.84**	347.895 (0.00)	347.285 (32.67)	**99.82**	347.641 (0.00)	346.996 (32.85)	**99.81**
	0.05	275.989 (0.00)	275.560 (32.79)	**99.84**	275.069 (0.00)	274.608 (32.57)	**99.83**	274.107 (0.00)	273.623 (32.76)	**99.82**
	0.10	211.334 (0.00)	211.022 (32.83)	**99.85**	209.993 (0.00)	209.661 (32.52)	**99.84**	208.604 (0.00)	208.257 (32.84)	**99.83**
	0.20	132.346 (0.00)	132.170 (32.72)	**99.87**	130.833 (0.00)	130.648 (32.85)	**99.86**	129.285 (0.00)	129.094 (32.85)	**99.85**
	0.50	47.159 (0.00)	47.111 (32.77)	**99.90**	46.176 (0.00)	46.127 (32.93)	**99.89**	45.199 (0.00)	45.150 (32.74)	**99.89**
	0.70	29.039 (0.00)	29.014 (32.96)	**99.91**	28.350 (0.00)	28.325 (32.69)	**99.91**	27.677 (0.00)	27.651 (32.62)	**99.91**
	0.90	19.835 (0.00)	19.820 (32.78)	**99.92**	19.341 (0.00)	19.326 (32.61)	**99.92**	18.865 (0.00)	18.850 (32.81)	**99.92**
	1.50	9.199 (0.00)	9.195 (32.68)	**99.96**	8.984 (0.00)	8.979 (32.79)	**99.94**	8.782 (0.00)	8.777 (32.94)	**99.94**
	2.00	6.184 (0.00)	6.182 (33.31)	**99.97**	6.059 (0.00)	6.056 (32.66)	**99.95**	5.943 (0.00)	5.941 (32.84)	**99.97**
4.0	0.01	348.509 (0.00)	348.047 (32.84)	**99.87**	348.379 (0.00)	347.872 (32.61)	**99.85**	348.244 (0.00)	347.701 (33.06)	**99.84**
	0.05	277.423 (0.00)	277.069 (32.68)	**99.87**	276.919 (0.00)	276.534 (32.93)	**99.86**	276.401 (0.00)	275.989 (32.81)	**99.85**
	0.10	213.441 (0.00)	213.183 (32.76)	**99.88**	212.697 (0.00)	212.416 (32.84)	**99.87**	211.935 (0.00)	211.635 (32.82)	**99.86**
	0.20	134.769 (0.00)	134.623 (32.80)	**99.89**	133.907 (0.00)	133.747 (32.87)	**99.88**	133.031 (0.00)	132.861 (32.65)	**99.87**
	0.50	48.812 (0.00)	48.771 (32.66)	**99.92**	48.211 (0.00)	48.166 (32.82)	**99.91**	47.615 (0.00)	47.569 (32.76)	**99.90**
	0.70	30.227 (0.00)	30.205 (32.69)	**99.93**	29.790 (0.00)	29.766 (32.83)	**99.92**	29.363 (0.00)	29.338 (32.92)	**99.91**
	0.90	20.705 (0.00)	20.692 (32.85)	**99.94**	20.382 (0.00)	20.368 (32.94)	**99.93**	20.070 (0.00)	20.055 (32.85)	**99.93**
	1.50	9.598 (0.00)	9.594 (33.10)	**99.96**	9.447 (0.00)	9.442 (32.84)	**99.95**	9.304 (0.00)	9.300 (32.69)	**99.96**
	2.00	6.424 (0.00)	6.422 (32.73)	**99.97**	6.332 (0.00)	6.330 (32.63)	**99.97**	6.247 (0.00)	6.244 (32.82)	**99.95**

Table 4 Out-of-control ARL values of a one-sided CUSUM control chart running a FIMAX(d, 1, 1) model when ARL_0 = 500

η	δ	FIMAX(d, 1, 1) model								
		d = 0.15			d = 0.30			d = 0.45		
		$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**
3.0	0.01	467.103	466.065	**99.78**	466.019	464.935	**99.77**	464.631	463.522	**99.76**
		(0.00)	(36.13)		(0.00)	(32.37)		(0.00)	(32.48)	
	0.05	360.367	359.609	**99.79**	356.401	355.623	**99.78**	351.371	350.594	**99.78**
		(0.00)	(34.14)		(0.00)	(32.32)		(0.00)	(32.56)	
	0.10	267.517	266.992	**99.80**	262.002	261.472	**99.80**	255.095	254.581	**99.80**
		(0.00)	(37.09)		(0.00)	(32.49)		(0.00)	(32.39)	
	0.20	158.855	158.581	**99.83**	153.112	152.845	**99.83**	146.087	145.841	**99.83**
		(0.00)	(35.33)		(0.00)	(32.40)		(0.00)	(32.53)	
	0.50	50.945	50.884	**99.88**	47.855	47.800	**99.89**	44.285	44.240	**99.90**
		(0.00)	(35.42)		(0.00)	(32.45)		(0.00)	(32.43)	
	0.70	30.163	30.134	**99.90**	28.198	28.173	**99.91**	25.997	25.977	**99.92**
		(0.00)	(34.66)		(0.00)	(32.32)		(0.00)	(32.58)	
	0.90	20.117	20.101	**99.92**	18.818	18.804	**99.93**	17.403	17.393	**99.94**
		(0.00)	(34.84)		(0.00)	(32.60)		(0.00)	(32.48)	
	1.50	9.103	9.099	**99.96**	8.635	8.631	**99.95**	8.160	8.157	**99.96**
		(0.00)	(34.68)		(0.00)	(32.45)		(0.00)	(32.73)	
	2.00	6.123	6.121	**99.97**	5.883	5.881	**99.97**	5.652	5.651	**99.98**
		(0.00)	(32.33)		(0.00)	(32.55)		(0.00)	(32.43)	
3.5	0.01	468.543	467.636	**99.81**	468.059	467.094	**99.79**	467.538	466.528	**99.78**
		(0.00)	(32.42)		(0.00)	(32.63)		(0.00)	(32.49)	
	0.05	365.716	365.041	**99.82**	363.906	363.193	**99.80**	361.973	361.231	**99.80**
		(0.00)	(32.55)		(0.00)	(32.55)		(0.00)	(32.60)	
	0.10	275.079	274.590	**99.82**	272.502	272.152	**99.87**	269.769	269.253	**99.81**
		(0.00)	(32.43)		(0.00)	(32.41)		(0.00)	(32.60)	
	0.20	166.969	166.708	**99.84**	164.167	163.898	**99.84**	161.239	160.967	**99.83**
		(0.00)	(32.42)		(0.00)	(32.50)		(0.00)	(32.50)	
	0.50	55.643	55.579	**99.88**	53.970	53.906	**99.88**	52.282	52.218	**99.88**
		(0.00)	(32.68)		(0.00)	(32.50)		(0.00)	(32.56)	
	0.70	33.271	33.238	**99.90**	32.146	32.114	**99.90**	31.032	31.001	**99.90**
		(0.00)	(32.43)		(0.00)	(32.67)		(0.00)	(32.44)	
	0.90	22.241	22.223	**99.92**	21.462	21.444	**99.92**	20.702	20.684	**99.91**
		(0.00)	(32.47)		(0.00)	(32.64)		(0.00)	(32.45)	
	1.50	9.938	9.932	**99.94**	9.622	9.616	**99.94**	9.324	9.319	**99.95**
		(0.00)	(32.52)		(0.00)	(32.58)		(0.00)	(32.60)	
	2.00	6.580	6.576	**99.94**	6.403	6.400	**99.95**	6.241	6.238	**99.95**
		(0.00)	(32.59)		(0.00)	(32.45)		(0.00)	(32.44)	
4.0	0.01	469.279	468.512	**99.84**	469.023	468.195	**99.82**	468.756	467.877	**99.81**
		(0.00)	(32.60)		(0.00)	(32.38)		(0.00)	(32.49)	
	0.05	368.496	367.918	**99.84**	367.519	366.588	**99.75**	366.515	365.861	**99.82**
		(0.00)	(32.55)		(0.00)	(32.61)		(0.00)	(32.44)	
	0.10	279.085	278.671	**99.85**	277.671	277.228	**99.84**	276.224	275.758	**99.83**
		(0.00)	(32.51)		(0.00)	(32.40)		(0.00)	(32.47)	
	0.20	171.418	171.189	**99.87**	169.832	169.588	**99.86**	168.227	167.972	**99.85**
		(0.00)	(32.48)		(0.00)	(32.53)		(0.00)	(32.61)	
	0.50	58.444	58.385	**99.90**	57.422	57.360	**99.89**	56.415	56.351	**99.89**
		(0.00)	(32.45)		(0.00)	(32.46)		(0.00)	(32.41)	
	0.70	35.206	35.175	**99.91**	34.491	34.459	**99.91**	33.797	33.764	**99.90**
		(0.00)	(32.43)		(0.00)	(32.57)		(0.00)	(32.58)	
	0.90	23.615	23.597	**99.92**	23.102	23.084	**99.92**	22.611	22.592	**99.92**
		(0.00)	(32.56)		(0.00)	(32.61)		(0.00)	(32.48)	
	1.50	10.527	10.521	**99.94**	10.302	10.296	**99.94**	10.092	10.086	**99.94**
		(0.00)	(32.41)		(0.00)	(32.49)		(0.00)	(32.49)	
	2.00	6.920	6.917	**99.96**	6.788	6.785	**99.96**	6.667	6.603	**99.04**
		(0.00)	(32.56)		(0.00)	(32.55)		(0.00)	(32.49)	

Table 5 Out-of-control ARL values of a one-sided CUSUM control chart running a FIMAX(d, 2, 1) model when $ARL_0 = 500$

η	δ	FIMAX(d, 2, 1) model								
		$d = 0.15$			$d = 0.30$			$d = 0.45$		
		$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**	$L(\vartheta)$	$\tilde{L}(\vartheta)$	**% Acc**
3.0	0.01	467.809 (0.00)	466.822 (32.48)	**99.79**	467.055 (0.00)	466.014 (32.56)	**99.78**	466.185 (0.00)	465.105 (32.61)	**99.77**
	0.05	362.979 (0.00)	362.251 (32.44)	**99.80**	360.191 (0.00)	359.432 (32.57)	**99.79**	357.003 (0.00)	356.226 (32.68)	**99.78**
	0.10	271.189 (0.00)	270.678 (32.46)	**99.81**	267.269 (0.00)	266.744 (32.62)	**99.80**	262.834 (0.00)	262.304 (35.20)	**99.80**
	0.20	162.755 (0.00)	162.483 (32.38)	**99.83**	158.594 (0.00)	158.321 (32.45)	**99.83**	153.969 (0.00)	153.702 (32.70)	**99.83**
	0.50	53.149 (0.00)	53.084 (32.44)	**99.88**	50.802 (0.00)	50.740 (32.54)	**99.88**	48.307 (0.00)	48.250 (32.54)	**99.88**
	0.70	31.601 (0.00)	31.570 (32.34)	**99.90**	30.071 (0.00)	30.041 (32.64)	**99.90**	28.482 (0.00)	28.456 (32.82)	**99.91**
	0.90	21.089 (0.00)	21.071 (32.68)	**99.91**	20.055 (0.00)	20.039 (32.60)	**99.92**	19.004 (0.00)	18.989 (32.47)	**99.92**
	1.50	9.474 (0.00)	9.469 (32.56)	**99.95**	9.079 (0.00)	9.075 (32.71)	**99.96**	8.700 (0.00)	8.696 (32.45)	**99.95**
	2.00	6.322 (0.00)	6.319 (32.65)	**99.95**	6.111 (0.00)	6.109 (32.49)	**99.97**	5.915 (0.00)	5.913 (32.46)	**99.97**
3.5	0.01	468.894 (0.00)	468.041 (32.46)	**99.82**	468.521 (0.00)	467.609 (32.82)	**99.81**	468.129 (0.00)	467.171 (32.39)	**99.80**
	0.05	367.032 (0.00)	366.395 (32.65)	**99.83**	365.631 (0.00)	364.953 (32.44)	**99.81**	364.165 (0.00)	363.457 (32.38)	**99.81**
	0.10	276.968 (0.00)	276.513 (32.45)	**99.84**	274.957 (0.00)	274.477 (32.46)	**99.83**	272.869 (0.00)	272.369 (32.39)	**99.82**
	0.20	169.049 (0.00)	168.801 (32.47)	**99.85**	166.835 (0.00)	166.574 (32.39)	**99.84**	164.564 (0.00)	164.295 (32.40)	**99.84**
	0.50	56.928 (0.00)	56.865 (32.40)	**99.89**	55.562 (0.00)	55.497 (32.38)	**99.88**	54.204 (0.00)	54.139 (32.54)	**99.88**
	0.70	34.149 (0.00)	34.117 (32.56)	**99.91**	33.212 (0.00)	33.183 (32.66)	**99.91**	32.302 (0.00)	32.269 (32.53)	**99.90**
	0.90	22.860 (0.00)	22.841 (32.53)	**99.92**	22.203 (0.00)	22.184 (32.49)	**99.91**	21.569 (0.00)	21.551 (32.54)	**99.92**
	1.50	10.198 (0.00)	10.192 (32.49)	**99.94**	9.922 (0.00)	9.916 (32.37)	**99.94**	9.664 (0.00)	9.659 (32.49)	**99.95**
	2.00	6.278 (0.00)	6.725 (32.48)	**92.88**	6.570 (0.00)	6.567 (32.41)	**99.95**	6.426 (0.00)	6.423 (32.56)	**99.95**
4.0	0.01	469.469 (0.00)	468.758 (32.55)	**99.85**	469.268 (0.00)	468.496 (32.56)	**99.84**	469.059 (0.00)	468.239 (32.42)	**99.83**
	0.05	369.219 (0.00)	368.683 (32.61)	**99.85**	368.449 (0.00)	367.869 (32.72)	**99.84**	367.657 (0.00)	367.043 (32.39)	**99.83**
	0.10	280.141 (0.00)	279.755 (32.56)	**99.86**	279.017 (0.00)	278.601 (32.69)	**99.85**	277.869 (0.00)	277.429 (32.56)	**99.84**
	0.20	172.616 (0.00)	172.401 (32.72)	**99.88**	171.342 (0.00)	171.111 (32.52)	**99.87**	170.054 (0.00)	169.812 (32.71)	**99.86**
	0.50	59.238 (0.00)	59.181 (32.51)	**99.90**	58.394 (0.00)	58.334 (32.51)	**99.90**	57.564 (0.00)	57.501 (32.26)	**99.89**
	0.70	35.769 (0.00)	35.759 (32.45)	**99.97**	35.170 (0.00)	35.140 (32.38)	**99.91**	34.589 (0.00)	34.557 (32.57)	**99.91**
	0.90	24.024 (0.00)	24.006 (32.73)	**99.93**	23.590 (0.00)	23.572 (32.56)	**99.92**	23.172 (0.00)	23.154 (32.55)	**99.92**
	1.50	10.711 (0.00)	10.706 (32.67)	**99.95**	10.516 (0.00)	10.510 (32.58)	**99.94**	10.332 (0.00)	10.327 (32.52)	**99.95**
	2.00	7.030 (0.00)	7.027 (32.83)	**99.96**	6.913 (0.00)	6.910 (32.72)	**99.96**	6.806 (0.00)	6.803 (32.30)	**99.96**

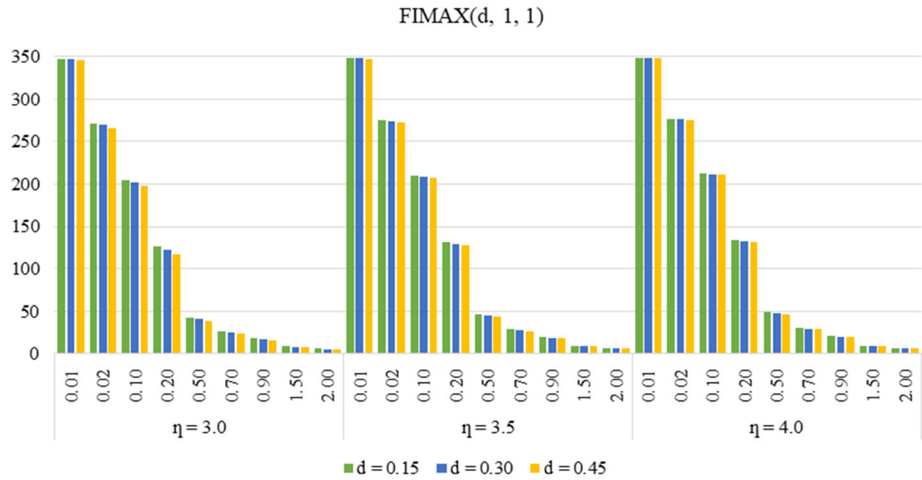


Figure 1 Out-of-control ARL values for a one-sided CUSUM control chart running a FIMAX (d, 1, 1) model for ARL_0 = 370

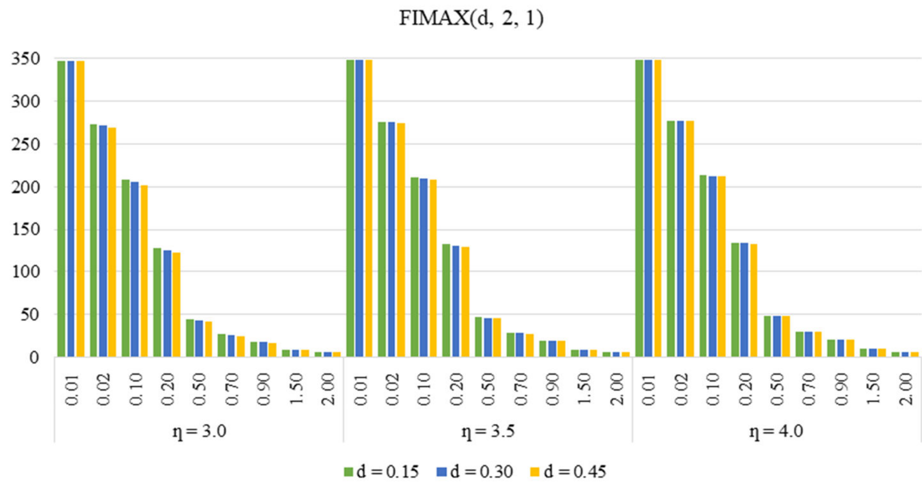
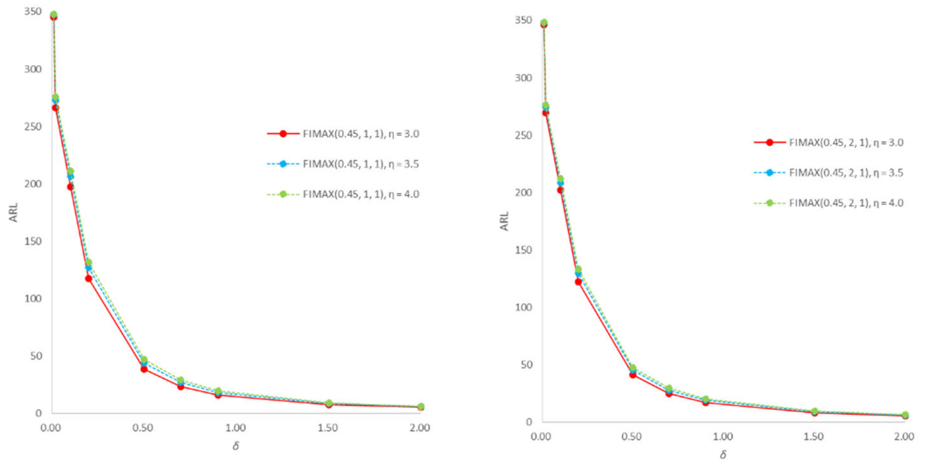


Figure 2 Out-of-control ARL values for a one-sided CUSUM control chart running a FIMAX (d, 2, 1) model for ARL_0 = 370



(a) FIMAX (0.45, 1, 1) model **(b)** FIMAX (0.45, 2, 1) model

Figure 3 Out-of-control ARL values for mean shift in FIMAX models on a one-sided CUSUM control chart for $ARL_0 = 370$

Table 6 The coefficients for the $\text{FIMAX}(d,q,r)$ models using the real datasets

Parameter	Coefficient	Std. Error	t-Statistic	p-value
Application 1: FIMAX(0.5, 1, 2) model				
USD/THB	0.115290	0.049857	2.312399	0.0247*
Dubai Crude Oil	0.020341	0.006610	3.077152	0.0033*
d	0.500000	8.49E-06	58907.84	0.0000*
MA(1)	0.696887	0.103482	6.734358	0.0000*
Application 2: FIMAX(0.5, 2, 1) model				
USD_THB	–9.53709	1.954063	–4.88065	0.0000*
d	0.500000	4.74E-06	105435.9	0.0000*
MA(1)	0.671338	0.068698	9.772358	0.0000*
MA(2)	0.468445	0.06933	6.756739	0.0000*

This output was obtained by using the Eviews 10 statistical software package for filtering the model and estimating model parameters.
*A significance level of 0.05.

Table 7 One-sample Kolmogorov-Smirnov test results for the real datasets

Residual of Application	Exponential parameter	Kolmogorov-Smirnov Z	p-value
1	0.1483	0.4500*	0.9870*
2	1.0303	0.8860*	0.4120*

This output was obtained by using the SPSS software package for test.
*A significance level of 0.05.

Application 1: Monthly data from the consumer price index (CPI) of Thailand in 2019 as the base year (source: http://www.indexpr.moc.go.th) with the two exogenous variables, X_{1t} as the USD/THB exchange rate (source: https://th.investing.com/currencies/usd-thb) and X_{1t} as the Dubai crude oil (Platts) financial futures stock price (source: https://th.investing.com/commodities/dubai-crude-oil-platts-futures-historical-data) collected from January 2017 to August 2021 (56 observations). The dataset was tested for its suitability for a FIMAX(0.5, 1, 2) process. The p-values of the parameters: $X_{1t} = 0.0247, X_{2t} = 0.0033$ and $\theta_1 = 0.0000$ at a significance level $\alpha = 0.05$ were obtained by using the t-test statistic (Table 6). It was found that the p-values of variables X_{1t} and X_{2t} at 0.05 were significant, which indicates that the exchange rate and stock prices do impact the monthly CPI of Thailand. As evidenced by using Kolmogorov-Smirnov tests, the three coefficients of the FIMAX model were non-zero: θ_1 = 0.696887, $\omega_1 = 0.115290,$ and $\omega_2 = 0.020341$ in the form

$$(1-B)^{0.5} Y_t = (1-\theta_1 B)\varepsilon_t + \omega_1 X_{1t} + \omega_2 X_{2t},$$

while the white noise was significantly exponentially distributed with mean 0.1483 (Table 7). Hence, the FIMAX($d = 0.5,\ q = 1, r = 2$) model takes the form

$$Y_t = \varepsilon_t - 0.696887\varepsilon_{t-1} + 0.115290X_{1t} + 0.020341X_{2t} + 0.5000Y_{t-1} + 0.1260Y_{t-2} + 0.0625Y_{t-3},$$

where $\varepsilon_t \sim Exp(\beta_0 = 0.1483).$ Therefore, the in-control process was 0.1483.

Application 2: Daily data on the crude oil futures price (source: https://th.investing.com/commodities/
crudeoil) with the exogenous variable being the USD/THB exchange rate (source: https://th.investing.com/currencies/usd-thb) collected 5 days per week for the period of 1 August 2014 to 26 March 2015 and consisting of 170 observations. The dataset was tested for its suitability for a FIMAX(0.5, 2, 1) process. The p-value for variable X_{1t} at 0.05 (p-values = 0.0000) was significant, thereby indicating that the exchange rate does impact the crude oil futures price with coefficient $\omega_1 = -9.53709.$ Moreover, the coefficients of the MA(1) and MA(2) model were $\theta_1 = 0.671338$ and $\theta_2 = 0.468445,$ respectively.

From Table 7, the residual of model was confirmed as having exponential white noise with mean $\beta_0 = 1.0303.$ Hence, the general form of the FIMAX($d = 0.5,\ q = 2, r = 1$) model can be written as

$$Y_t = \varepsilon_t - 0.671338\varepsilon_{t-1} - 0.468445\varepsilon_{t-1} - 9.53709X_{1t} + 0.5000Y_{t-1} + 0.1260Y_{t-2} + 0.0625Y_{t-3},$$

where $\varepsilon_t \sim Exp(1.0303).$ Moreover, we used 1.0303 for the in-control process.

5.1. Performance comparison of the ARL based on explicit formulas using real data processes on CUSUM and EWMA control charts

The performance of the ARL derived by using explicit formulas on a CUSUM control chart and a EWMA control chart (as defined by Sunthornwat et al. (2020)) with both real datasets is reported in Table 8 $ARL_0 = 370$. For both the first and second datasets, ($\eta = 2.0,\ b = 0.1810229$) and ($\eta = 3.5,\ b = 1.89795$) for the CUSUM control chart via (15) and ($\lambda = 0.5, L = 0.01511975$)

and ($\lambda = 0.28$, $L = 0.649933$) for the EWMA control chart were computed by using the Mathematica program. For a small-to-moderate shift in the process mean (from 0.01 to 1.50), the out-of-control ARL decreased in both control charts. These results are in accordance with the calculated ARL results in Tables 2-5. Moreover, ARL_1 running on the CUSUM control chart tended to decrease more rapidly than on the EWMA control chart as the shift in the process mean increased.

Table 8 Comparison of the results of ARL via explicit formulas on one-sided CUSUM and EWMA control charts with real data processes for ARL_0=370

δ	Application 1		%Acc	Application 2		%Acc
	CUSUM $\eta = 2.0$, $b = 0.1810229$	EWMA $\lambda = 0.5$, $L = 0.01511975$		CUSUM $\eta = 3.5$, $b = 1.89795$	EWMA $\lambda = 0.28$, $L = 0.649933$	
0.01	223.669	**216.207**	96.66	**349.094**	353.958	98.61
0.05	48.478	**42.623**	87.92	**279.555**	297.553	93.56
0.10	14.132	**11.742**	83.09	**216.390**	241.456	88.42
0.20	3.756	**3.117**	82.99	**137.787**	163.176	81.57
0.50	1.357	**1.260**	92.82	**50.256**	61.175	78.27
0.70	1.186	**1.133**	95.47	**31.021**	36.618	81.96
0.90	1.120	**1.084**	96.77	**21.124**	23.963	86.56
1.50	1.054	**1.036**	98.28	**9.584**	9.737	98.40

For the first dataset, ARL_1 running on the EWMA control chart was more sensitive than on the CUSUM control chart at detecting varying shifts in the process mean. However, since the percentage accuracy was greater than 75% for various mean shifts in the FIMAX(0.5, 1, 2) process, the CUSUM control chart is an efficient alternative to the EWMA control chart for detecting small-to-moderate process mean shifts in this situation. The values of the CUSUM statistic (C_t) computed from the CPI dataset with the two exogenous variables were a good fit for a FIMAX(0.5, 1, 2) process (Figure 5). The control limits for the CUSUM control chart were LCL = 0, CL = 0.1486 and UCL = 0.1810229. The results show that the C_t value of the CUSUM control chart exceeded the threshold at the 7^{th} point, thereby indicating that the process was out-of-control in November 2018. Hence, it can be concluded that the CUSUM control chart is applicable for monitoring the CPI of Thailand under the influence of the THB/USD exchange rate and the Dubai crude oil futures stock price.

For the second dataset, it is noteworthy that for every process mean shift, the ARL_1 values were smaller on the CUSUM control chart than on the EWMA control chart, while the percentage accuracy was greater than 75%. Hence, the CUSUM control chart was quicker at detecting process mean shifts in this scenario. For the crude oil futures price with an exogenous variable fitted to a FIMAX(0.5, 2, 1) process, the residuals of the model are plotted in Figure 6; the values of the control limits for the CUSUM control chart are LCL = 0, CL = 0.10303 and UCL = 1.89795. The results show that the C_t values of the CUSUM control chart exceeded the bound for the first time at the 5^{th} observation, and thus the process was out-of-control on Aug 15, 2014.

6. Conclusions and Recommendations

Besides the well-established ARL measure, more performance indicators for control charts are needed. We developed and derived the ARL for FIMAX processes running on a CUSUM control chart by using analytical integral equations as explicit formulas and compared it with the NIE method. This one-step ARL calculation using explicit formulas made calculating the ARL easy and was extremely rapid in terms of computational time. It was also proven that the explicit formulas existed and were unique.

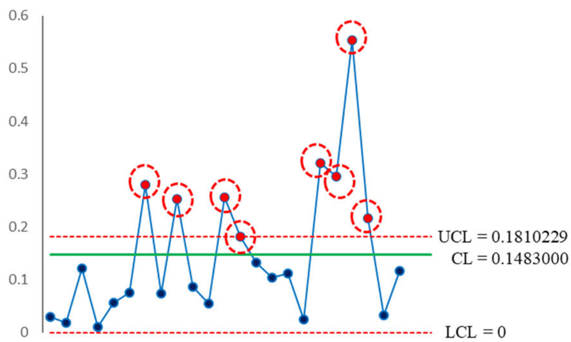


Figure 4 The proposed ARL method for a CUSUM control chart running a FIMAX(0.5, 1, 2) process with the first real dataset

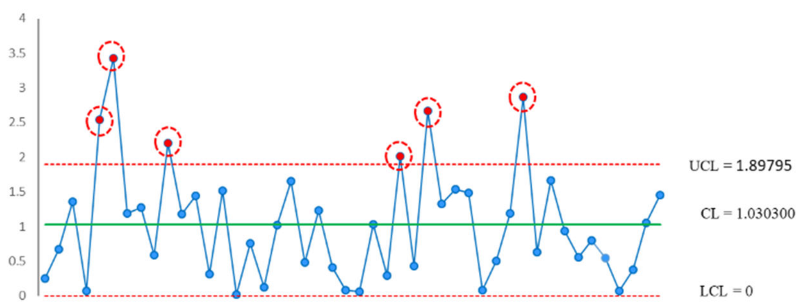


Figure 5 The proposed ARL method for a CUSUM control chart running a FIMAX(0.5, 2, 1) process with the second real dataset

We focused on FIMAX processes with one or more exogenous variables that influence an econometric model when forecasting as this approach is usually more accurate than those without them. The in-control and out-of-control ARL settings were established for different parameter settings and levels of process mean shift by using explicit formulas. For the out-of-control situation, we compared detecting the out-of-control (ARL_1) process by using the explicit formula and the NIE method, which revealed similar accuracy. These results are consistent with those from previous research (Peerajit et al. 2019, Sunthornwat et al. 2020, Phanthuna and Areepong

2021). In addition, we investigated the practicability of our method using FIMAX models based on real data and obtained similar results as with the simulated data results.

The current research was focused on monitoring shifts in the process mean of a FIMAX process with exponential white. The methodology can also be extended to monitor changes in the process variance and for white noise following other exponential family distributions. Moreover, we established the optimal control limit for the CUSUM control chart. We will also expand our approach to commonly used control charts in various fields (such as economics, industry, etc.), as well as to new control charts being developed for various processes and for data coming from a complex process or an unstable environment. For instance, investigating the ARL for a CUSUM control chart derived by using the neutrosophic statistical method.

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